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A CLASSIFICATION OF FINITE GROUPS WITH INTEGRAL BI-CAYLEY GRAPHS

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Abstract. The bi-Cayley graph of a finite group G with respect to a subset $S \subseteq G$, which is denoted by $\mathrm{BCay}(G, S)$, is the graph with vertex set $G \times \{1, 2\}$ and edge set $\{\{(x, 1), (sx, 2)\} \mid x \in G,\ s \in S\}$. A finite group G is called a *bi-Cayley integral group* if for any subset S of G, $\mathrm{BCay}(G, S)$ is a graph with integer eigenvalues. In this paper we prove that a finite group G is a bi-Cayley integral group if and only if G is isomorphic to one of the groups $\mathbb{Z}_2^k$, for some k, $\mathbb{Z}_3$ or S_3.

1. **Introduction**

Throughout the paper, groups are finite and graphs are undirected, finite, and without loops and multiple edges. The bi-Cayley graph of a finite group G with respect to a subset $S \subseteq G$, which is denoted by $\mathrm{BCay}(G, S)$, is the graph with vertex set $G \times \{1, 2\}$ and edge set $\{\{(x, 1), (sx, 2)\} \mid x \in G,\ s \in S\}$. A graph Γ is called *integral* if all eigenvalues of the adjacency matrix of Γ are integers. The concept of integral graphs was first defined by Harary and Schwenk [9]. During the last fourty years many mathematicians tried to construct and classify integral graphs, for a survey on integral graphs up to 2002, see [6]. Integral graphs are very rare, indeed the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for sufficiently large n, see [3]. Known characterizations of integral graphs are restricted to special classes of graphs including Cayley graphs, see for example [1, 2, 11]. Klotz and Sander [11] called a group G Cayley integral group whenever all undirected Cayley graphs over G are integral. They showed that finite abelian Cayley integral groups are $\mathbb{Z}_2^n \times \mathbb{Z}_3^m$ and $\mathbb{Z}_2^n \times \mathbb{Z}_4^m$, where $\mathbb{Z}_n$ is the cyclic group of order n. Recently, the classification of finite Cayley integral

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groups completed in [2] (also independently in [4]) by proving that finite non-abelian Cayley integral groups are the symmetric group S_3 of degree 3, Dic_{12} and $Q_8 \times \mathbb{Z}_2^n$ for some integer $n \geq 0$, where Dic_{12} is the dicyclic group of order 12 and Q_8 is the quaternion group of order 8. In this paper we consider the bi-Cayley graphs and classify groups G with the property that all bi-Cayley graphs of G are integral. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [14, 7], respectively.

2. **Eigenvalues of bi-Cayley graphs**

Let G be a finite group and $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G, d_k, $\varrho^{(k)}$, and χ_k be the degree, unitary matrix representations and the corresponding irreducible character of ρ_k, $k = 1, \ldots, m$, respectively. For a subset X of G and $l = 1, \ldots, m$, we set $\varrho^{(l)}(X) := \sum_{x\in X} \varrho^{(l)}(x)$. Note that if G is a finite abelian group, every unitary irreducible matrix representation is an irreducible character, $m = |G|$ and $d_l = 1$ for each $l = 1, \ldots, m$. Also it is well-known that for each irreducible character χ of G and each $g \in G$, $\chi(g^{-1})$ is the complex conjugate of $\chi(g)$. We use these notations in this section.

For a positive integer n, a graph Γ is called n-Cayley graph over a group G if the full automorphism group of Γ has a semiregular subgroup isomorphic to G with n orbits. Recently, the present authors determined the eigenvalues of n-Cayley graphs in [5]. By [5, Lemma 2], an n-Cayley graph is characterized by n^2 subsets T_{ij}, $1 \leq i, j \leq n$, of G (some subsets maybe empty). An n-Cayley graph over a group G can be identified by a graph $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leq i, j \leq n)$, where T_{ij}'s are subsets of G, $V(\Gamma) = G \times \{1, \ldots, n\}$ and (x, i) is adjacent to (y, j) if and only if $yx^{-1} \in T_{ij}$ (see [5]). Note that 2-Cayley graphs are called by some authors semi-Cayley [13, 8] and by some authors bi-Cayley graphs [12]. The concept of bi-Cayley graphs, studied in this paper, first defined in [15] for a special case of the bi-Cayley graphs which defined later in [12]. Hence we follow [15] and call it the bi-Cayley graph of G with respect to S.

It follows from the definition of bi-Cayley graphs that $\mathrm{BCay}(G, S) \cong \mathrm{Cay}(G, T_{11}, T_{22}, T_{12}, T_{21})$, where $T_{11} = T_{22} = \emptyset$, $T_{12} = S$ and $T_{21} = S^{-1}$. Now, the following theorem is a direct consequence of the main theorem of [5].

Theorem 2.1. Let $\Gamma = \mathrm{BCay}(G, S)$ with adjacency matrix A. Let

$$A_l = \begin{bmatrix} 0_{d_l} & \varrho^{(l)}(S^{-1}) \\ \varrho^{(l)}(S) & 0_{d_l} \end{bmatrix},$$

where 0_{d_l} is the $d_l \times d_l$ matrix with all entries 0. Then $p_A(\lambda) = \prod_{l=1}^{m} p_{A_l}(\lambda)^{d_l}$, where $p_X(\lambda)$ is the characteristic polynomial of matrix X.

In particular, if G is abelian then eigenvalues of $\mathrm{BCay}(G, S)$ are $\pm|\sum_{s\in S} \chi_i(s)|$, $i = 1, \ldots, |G|$.

3. **Bi-Cayley integral groups**

Recall that a finite group G is said to be a *bi-Cayley integral group* if for any subset S of G, $\mathrm{BCay}(G,S)$ is a graph with integer eigenvalues (an integral graph). In this section we characterize all bi-Cayley integral groups. Let us denote the cycle with n vertices, the complete graph with n vertices and the complete bipartite graph with partition sizes m,n with C_n, K_n and $K_{m,n}$, respectively. By section 1.5 of [7], the eigenvalues of C_n are $2\cos(2\pi j/n)$, $j=0,\dots,n-1$, the eigenvalues of K_n are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$ and the eigenvalues of $K_{m,n}$ are $\pm\sqrt{mn}$ and 0 with multiplicity $m+n-2$. It is well-known that C_n is integral if and only if $n\in\{3,4,6\}$. If λ is an eigenvalue with multiplicity n, for the convenience, we write $\lambda^{[n]}$.

Since $\mathrm{BCay}(G,\emptyset)\cong 2|G|K_1$, $\mathrm{BCay}(G,G)\cong K_{|G|,|G|}$ and for every element a of G, $\mathrm{BCay}(G,\{a\})\cong |G|K_2$, $\mathrm{BCay}(G,S)$ is integral whenever $S=\emptyset$, $S=G$ or $|S|=1$.

The tensor product of two graphs Γ_1 and Γ_2, $\Gamma_1\otimes\Gamma_2$ has $V(\Gamma_1)\times V(\Gamma_2)$ as its vertex set with(u_1,u_2) is adjacent to (u_2,v_2) whenever u_1,u_2 are adjacent in Γ_1 and u_2,v_2 are adjacent in Γ_2.

In the following lemma, we recall the eigenvalues of the tensor product of two graphs.

Lemma 3.1. (see [7, Section 1.5.7]) If $\lambda_1,\dots,\lambda_n$ are the eigenvalues of Γ_1, and $\mu_1,\dots,\mu_m$ are the eigenvalues of Γ_2, then the eigenvalues of $\Gamma_1\otimes\Gamma_2$ are $\lambda_i\mu_j$ for $i=1,\dots,n$ and $j=1,\dots,m$.

For a subset S of a group G, the Cayley graph of G over S, denoted by $\mathrm{Cay}(G,S)$, is the graph with vertex set G and (x,y) is an edge if $yx^{-1}\in S$. If S is inverse-closed then $\mathrm{Cay}(G,S)$ is undirected. Also if $1\notin S$ then $\mathrm{Cay}(G,S)$ is loop-free. The following lemma is obvious from the definition of a bi-Cayley graph.

Lemma 3.2. Let S be an inverse-closed subset of G. Then $\mathrm{BCay}(G,S)\cong\mathrm{Cay}(G,S)\otimes K_2$. In particular, $\mathrm{BCay}(G,S)$ is integral if and only if $\mathrm{Cay}(G,S)$ is integral. Therefore every bi-Cayley integral group is a Cayley integral group.

Remark 3.3. Let G be a bi-Cayley integral group. Then by Lemma 3.2 and [4, Theorem 4.2] (or [2, Theorem 1.2]), G is isomorphic to one of the groups

$$\mathbb{Z}_2^n\times\mathbb{Z}_3^m, \mathbb{Z}_2^n\times\mathbb{Z}_4^m, Q_8\times\mathbb{Z}_2^n, S_3, \mathrm{Dic}_{12}.$$

In what follows, we examine the above groups to classify bi-Cayley integral groups. Let us start with cyclic bi-Cayley integral groups.

Lemma 3.4. *G is a cyclic bi-Cayley integral group if and only if $G\cong\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. First suppose that $G\cong\mathbb{Z}_2$ and S be a subset of G. Then $S=\emptyset$ or $S=G$ or $|S|=1$. Hence $\mathrm{BCay}(G,S)$ is integral.

Now, suppose that $G\cong\mathbb{Z}_3$ and S be a subset of G. If $S=\emptyset$ or $|S|=1$ or $|S|=3$ then $\mathrm{BCay}(G,S)$ is integral. Let $|S|=2$. Then $\mathrm{BCay}(G,S)\cong C_6$ which is integral.

Finally suppose that $G=\langle y\rangle\cong\mathbb{Z}_n$, $n\geq 4$ is a bi-Cayley integral group. Set $S:=\{1,y\}$. Since G is a bi-Cayley integral group, $\mathrm{BCay}(G,S)\cong C_{2n}$ must be integral. Hence $n\in\{2,3\}$, a contradiction. This completes the proof. □

Lemma 3.5. Let $S\subseteq H\leq G$. Then $\mathrm{BCay}(G,S)\cong\ |G:H|\ \mathrm{BCay}(H,S)$.

Proof. Let $\Gamma=\mathrm{BCay}(G,S)$. If $G=H$ then there is nothing to prove. So we may assume that $H<G$, $|G:H|=m>1$. Let $T=\{t_1=1,t_2,\ldots,t_m\}$ be a right transversal to H in G. For each $i\in\{1,\ldots,m\}$, we define a graph Γ_i with $V(\Gamma_i)=Ht_i\times\{1,2\}$ and $E(\Gamma_i)=\{\{(h_1t_i,1),(h_2t_i,2)\}\mid h_2h_1^{-1}\in S\}$. We claim that Γ is $\Sigma:=\Gamma_1+\cdots+\Gamma_m$, the disjoint union of $\Gamma_1,\ldots,\Gamma_m$. Clearly

$$\begin{aligned}
V(\Gamma_1+\cdots+\Gamma_m) &= \bigcup_{i=1}^{m} V(\Gamma_i)\\
&= \bigcup_{i=1}^{m} Ht_i\times\{1,2\}\\
&= \left(\bigcup_{i=1}^{m} Ht_i\right)\times\{1,2\}\\
&= G\times\{1,2\}\\
&= V(\Gamma).
\end{aligned}$$

Now, let $x,y\in G$. Then there exist unique $i,j\in\{1,\ldots,m\}$ and $a,b\in H$ such that $x=at_i$ and $y=bt_j$. Let $\{(x,1),(y,2)\}\in E(\Gamma)$. Then $yx^{-1}\in S$. So $yx^{-1}=bt_jt_i^{-1}a^{-1}\in H$ which implies that $t_jt_i^{-1}\in H$. Hence $Ht_i=Ht_j$ which means that $t_i=t_j$ and $i=j$. This shows that $ba^{-1}=yx^{-1}\in S$ and so $\{(x,1),(y,2)\}\in E(\Gamma_i)\subseteq E(\Sigma)$. Hence $E(\Gamma)\subseteq E(\Sigma)$. The inverse inclusion is obvious. This proves our claim. Now, for each $i\in\{1,\ldots,m\}$, the map

$$\begin{aligned}
\varphi_i: V(\Gamma_i) &\to V(\Gamma_1)\\
(ht_i,1) &\mapsto (h,1),\\
(ht_i,2) &\mapsto (h,2)
\end{aligned}$$

is a graph isomorphism. On the other hand $\Gamma_1=\mathrm{BCay}(H,S)$. This completes the proof. □

Corollary 3.6. Let G be a bi-Cayley integral group and $H\leq G$. Then H is also a bi-Cayley integral group. In particular, the order of each element of G is 2 or 3.

Proof. Let $S\subseteq H$. Then, by Lemma 3.5, $\mathrm{BCay}(G,S)\cong|G:H|\mathrm{BCay}(H,S)$. Let λ be an eigenvalue of $\mathrm{BCay}(H,S)$. Then λ is an eigenvalue of $\mathrm{BCay}(G,S)$ with multiplicity $|G:H|$. Since G is a bi-Cayley integral group, λ is an integer. This shows that H is also a bi-Cayley integral group.

Now, the last statement follows from the fact that $\langle g\rangle$, where $g\in G$, is a bi-Cayley integral group and Lemma 3.4. □

Lemma 3.7. S_3 is a bi-Cayley integral group.

Proof. Note that, by [10, Lemma 2.1], $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G, gS^{\alpha})$, where $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. Also for any $S \subseteq G$ with $|S| \in \{0, 1, |G|\}$, $\mathrm{BCay}(G,S)$ is integral. Therefore to determine the integrality of bi-Cayley graphs over S_3, it is enough to consider $\mathrm{BCay}(S_3, S)$, where S is one of the following sets:

$$\{(),(1,2)\}, \quad \{(),(1,2,3)\}, \{(),(1,2),(1,3)\}, \quad \{(),(1,2,3),(1,3,2)\},$$
$$\{(),(1,2),(1,3),(2,3)\}, \quad \{(),(1,2,3),(1,2),(2,3)\}, \quad \{(),(1,2),(1,3),(2,3),(1,2,3)\}.$$

On the other hand, S_3 is a Cayley integral group by [4, Theorem 4.2] (or [2, Theorem 1.1]). Thus, by Lemma 3.2, it is enough to consider subsets S which are not inverse-closed:

$$\{(),(1,2,3)\}, \quad \{(),(1,2,3),(1,2),(2,3)\}, \quad \{(),(1,2),(1,3),(2,3),(1,2,3)\}.$$

Computing the the spectrum $\mathrm{BCay}(S_3, S)$ is easy from Theorem 2.1 using irreducible representations of S_3. For example, we compute the spectrum of $\mathrm{BCay}(S_3, S)$ whenever $S = \{(),(1,2),(1,3),(2,3),(1,2,3)\}$. First note that $S_3 = \langle a, b\rangle$, where $a = (1,2,3)$, $b = (1,2)$, and the irreducible representations of S_3 are

$$\begin{aligned}
&\rho_1 : b^i a^j \mapsto 1,\\
&\rho_2 : b^i a^j \mapsto (-1)^i,\\
&\rho_3 : a^j \mapsto \begin{bmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{bmatrix}, \quad ba^j \mapsto \begin{bmatrix} 0 & \omega^{-j} \\ \omega^j & 0 \end{bmatrix},
\end{aligned}$$

where $0 \le i \le 1$, $0 \le j \le 2$ and $\omega = \exp(2\pi i/3)$. If $S = \{(),(1,2),(1,3),(2,3),(1,2,3)\}$ and $\Gamma = \mathrm{BCay}(S_3, S)$, then

$$A_1 = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega^2 \\ -\omega^2 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \end{bmatrix}.$$

Since the eigenvalues of A_1, A_2 and A_3 are the multi-sets $\{(\pm 5)^{[1]}\}$, $\{(\pm 1)^{[1]}\}$ and $\{(\pm 1)^{[2]}\}$, respectively, Theorem 2.1 implies that the eigenvalues of Γ are $(\pm 5)^{[1]}, (\pm 1)^{[5]}$.

We can easily compute eigenvalues of the remaining bi-Cayley graphs:

$$(\pm 2)^{[2]}, (\pm 1)^{[4]}, \quad (\pm 4)^{[1]}, (\pm 2)^{[2]}, (0)^{[6]}.$$

Hence all bi-Cayley graphs of S_3 are integral. □

Lemma 3.8. $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not a bi-Cayley integral group.

Proof. Let $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $G = \langle a, b \mid a^3 = b^3 = 1, ab = ba\rangle$ is a presentation of G. Put $S := \{1, a, b\}$. Define $\chi_1 : \langle a\rangle \to \mathbb{C}$, and $\chi_2 : \langle b\rangle \to \mathbb{C}$, where $\chi_1(a) = \chi_2(a) = \exp(2\pi i/3)$. Then $\chi := \chi_1 \times \chi_2$ is an irreducible character of G, see [14, section 3.2], and $|\sum_{s\in S} \chi(s)| = |1 + 2\exp(2\pi i/3)| = \sqrt{3}$ is an eigenvalue of $\mathrm{BCay}(G,S)$, by Theorem 2.1. This shows that G is not a bi-Cayley integral group. □

Lemma 3.9. $\mathbb{Z}_2^k$, $k \ge 1$ is a bi-Cayley integral group.

Proof. Let S be a subset of $G \cong \mathbb{Z}_2^k$. Then S is inverse-closed. Since $\mathbb{Z}_2^k$ is a Cayley integral group, Lemma 3.2 implies that it is a bi-Cayley integral group. □

Now, we are ready to prove the main result of the paper.

Theorem 3.10. Let G be a finite group. Then G is a bi-Cayley integral group if and only if G is isomorphic to one of the groups $\mathbb{Z}_2^k$, for some integer k, $\mathbb{Z}_3$ or S_3.

Proof. By Lemmas 3.4, 3.7 and 3.9, $\mathbb{Z}_2^k$, for some integer k, $\mathbb{Z}_3$ and S_3 are bi-Cayley integral groups. Conversely, suppose that G is a bi-Cayley integral group. Then by Remark 3.3, G is isomorphic to one of the groups $\mathbb{Z}_2^n \times \mathbb{Z}_3^m$, $\mathbb{Z}_2^n \times \mathbb{Z}_4^m$, $Q_8 \times \mathbb{Z}_2^n$, S_3, or Dic_{12}, for some integers $m, n \geq 0$. By Corollary 3.6 and Lemma 3.8, the only abelian bi-Cayley integral groups are $\mathbb{Z}_2^n$, for some integer n and $\mathbb{Z}_3$. Since Q_8 and Dic_{12} has elements of order 4, by Lemma 3.7 and Corollary 3.6, the only non-abelian bi-Cayley integral group is S_3. This completes the proof. □

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