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A DYNAMIC DOMINATION PROBLEM IN TREES

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ABSTRACT. We consider a dynamic domination problem for graphs in which an infinite sequence of attacks occur at vertices with guards and the guard at the attacked vertex is required to vacate the vertex by moving to a neighboring vertex with no guard. Other guards are allowed to move at the same time, and before and after each attack and the resulting guard movements, the vertices containing guards form a dominating set of the graph. The minimum number of guards that can successfully defend the graph against such an arbitrary sequence of attacks is the m-eviction number. This parameter lies between the domination and independence numbers of the graph.

We characterize the classes of trees for which the m-eviction number equals the domination number and the independence number, respectively.

1. Introduction

We consider a dynamic domination problem for simple, finite, undirected graphs and denote such a graph by $G = (V, E)$. A set $D \subseteq V$ is a *dominating set* if each $v \in V - D$ is adjacent to a vertex in D . The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ -*set*. The maximum cardinality of an independent set $X \subseteq V$ (i.e., no two vertices in X are adjacent) is the *independence number* $\alpha(G)$. An independent set of cardinality $\alpha(G)$ is called a α -*set*.

For each $i \geq 1$, let each $D_i \subseteq V$ be a dominating set of vertices with one “moveable entity” located at each vertex of D_i . Following the occurrence of an “event” at a vertex, the moveable entity relocates along an edge, either to or away from the site of the event. The nature of the moveable entity and

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the event depends on the specific problem being considered. For example, in the *eternal dominating set problem*, each entity can be considered to be a guard stationed at the vertex, and an event is considered to be an attack at another vertex. Following an attack at a vertex $r_i \in V - D_i$, one or several guards (depending on the exact nature of the model) move along edges to adjacent vertices, with one guard moving to r_i , to defend the graph, thus occupying the vertices in the set D_{i+1} . The minimum number of guards required to protect the graph is called the *eternal domination number* $\gamma^\infty(G)$ (if only one guard is allowed to move at a time) or the *m-eternal domination number* $\gamma_m^\infty(G)$ (if any number of guards can move). The eternal and m-eternal domination problems were introduced in [2] and [5], respectively. The problems are instances of a general class of problems known as *graph protection* problems. For other work on eternal domination and related protection models we refer the reader to [1, 6, 7, 9, 10, 11, 12, 13, 14].

In the *eternal dominating set eviction problem*, or simply the *eternal eviction problem*, the vertices of the graph can be considered to be servers in a computer network, and an entity located at a vertex can be considered to be files stored on the server. When a server has to be disabled for upgrades or maintenance, the files stored on that machine have to be migrated to another in the network to maintain good access for all users of the network. For consistency in terminology, we use the term “guard” synonymously with “file”, i.e., we talk about moving guards around the network, instead of files. Thus, in this case, an attack occurs at a vertex $r_i \in D_i$, and one or several guards (again depending on the nature of the model) move along edges to adjacent vertices, with the guard at r_i moving away and no guard moving to r_i at the same time, to occupy the vertices in D_{i+1} . We emphasize that each D_i is a dominating set and that $r_i \in D_i - D_{i+1}$. The minimum number of guards that can protect the graph according to this model is the *eternal eviction number* $e^\infty(G)$ (if only the guard on r_i moves) or the *m-eternal eviction number* $e_m^\infty(G)$ (if the guard on r_i and any other guards may move). Each set D_i is called an *m-eternal eviction set* of G .

The eternal eviction problem was introduced in [8], where, amongst other results, the following bounds were established.

Theorem 1. [8] *For any graph G , $\gamma(G) \leq e_m^\infty(G) \leq \alpha(G)$.*

Proof. The proof is from [8]. The lower bound is obvious. For the upper bound, let I be a maximum independent set of G . Let D denote the configuration of guards in G , initially $D = I$. Suppose an attack occurs at vertex $v \in D$. If v has an external private neighbor u , then move the guard from v to u and observe that the new configuration of guards is a maximum independent set (and thus a dominating set). Next assume v has no external private neighbor. Let w be a neighbor of v . Move the guard from v to w ; this configuration of guards is still a dominating set. When the next attack occurs, move the guard from w back to v . Thus there is at most one guard outside a maximum independent set at any time and the configuration of guards is a dominating set at all times. \square

For a simple example, consider P_n , the path on $n \geq 2$ vertices. As stated in [8], $e_m^\infty(P_n) = \lceil \frac{n+1}{3} \rceil$, while it is well known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. Hence $e_m^\infty(P_n) = \gamma(P_n)$ if and only if $n \not\equiv 0 \pmod{3}$, and $e_m^\infty(P_n) = \alpha(P_n)$ if and only if $n \in \{2, 3, 4, 6\}$.

The purpose of this paper is to characterize the classes of trees T such that $e_m^\infty(T) = \gamma(T)$ and $e_m^\infty(T) = \alpha(T)$, respectively.

2. Definitions and statements of main results

The tree $K_{1,m}$, $m \geq 1$, is called a *star*; the graph K_1 is called the *trivial star*. The *central vertex* of a star with $m > 1$ is the vertex of degree m . A *star partition* Π of a nontrivial tree T is a partition of $V(T)$ into parts such that (i) each part of Π induces a star, (ii) no two K_1 parts are adjacent, (iii) each vertex of T adjacent to exactly one leaf forms a K_2 part of Π with this leaf, and (iv) any K_1 part is adjacent to at least two other parts. A star S of order $k \geq 1$ in a star partition is assigned *weight* $k - 1$. The *weight* $\omega(\Pi)$ of a star partition Π is the sum of the weights of its parts. Let $s(T)$ denote the minimum weight of any star partition of T . As an example, note that P_5 has a star partition of weight 2, consisting of two K_2 's and one K_1 .

As explained in [8], for each k and each part of size k of a star partition Π of the tree T , it is possible to place a guard on each of $k - 1$ vertices of the part to form an m -eternal eviction set of T . After attacks, the guards may move around in the same part, never leaving the part where they were originally placed. This observation leads to the following theorem.

Theorem 2. [8] *For any nontrivial tree T , $e_m^\infty(T) = s(T)$.*

Let Π be a star partition of a tree T . The K_2 parts H_i , $i = 1, \dots, t$, of Π are *colinear* if all their vertices lie on a common path. Colinear K_2 parts $H_i = \{v_{i,1}, v_{i,2}\}$, $i = 1, \dots, t$, are *consecutive* if $v_{i,2}v_{i+1,1} \in E(T)$ for each $i = 1, \dots, t - 1$. Three consecutive colinear K_2 parts are called a *K_2 -triple*. The colinear K_2 parts H_i , $i = 1, \dots, t$, $t \geq 4$, form a *(2,2)-chain* if H_1 and H_2 are consecutive, as are H_{t-1} and H_t , and all parts of Π , other than H_1, \dots, H_t , on the path from H_1 to H_t are K_1 parts. Thus in a (2,2)-chain with more than four parts there may be any number, including zero, additional consecutive K_2 parts between the first and last pairs; other K_2 parts may be separated by K_1 parts (but, of course, no two K_1 parts are adjacent). The tree formed by adding two K_2 's to $P_{11} = v_1, \dots, v_{11}$, joining a leaf of one to v_4 and a leaf of the other to v_8 , is an example of a tree with a minimum weight star partition that has a (2,2)-chain but no minimum weight star partition that has a K_2 -triple. A *large part* of a star partition of T is any part other than K_1 . A star partition that has only K_1 and K_2 parts is called a *(1,2)-partition*.

A *stem* of a tree T is a vertex of degree at least two that is adjacent to a leaf. A stem is a *weak stem* if it is adjacent to exactly one leaf, and a *strong stem* otherwise.

We use Theorem 2 to prove the following results.

Theorem 3. *For any nontrivial tree T , $e_m^\infty(T) = \gamma(T)$ if and only if*

- (i) T has no strong stems,
- (ii) T has no minimum weight star partition containing a K_2 -triple, and
- (iii) T has no minimum weight star partition containing a (2,2)-chain.

Theorem 4. *For any nontrivial tree T , $e_m^\infty(T) = \alpha(T)$ if and only if T has a minimum weight star partition containing no K_1 parts.*

We conclude this section with some additional definitions. In Section 3 we prove some lemmas required for the proofs of our main results. Theorem 3 is proved in Section 4, while the proof of Theorem 4 can be found in Section 5. Open problems are discussed in Section 6.

Denote the open and closed neighborhoods of $X \subseteq V$ by $N(X)$ and $N[X]$, respectively, and abbreviate $N(\{x\})$ and $N[\{x\}]$ to $N(x)$ and $N[x]$. The *private neighborhood* $\text{pn}(x, X)$ of $x \in X$ relative to X is defined by $\text{pn}(x, X) = N[x] - N[X - \{x\}]$. The *external private neighborhood* $\text{epn}(x, X)$ of $x \in X$ relative to X is defined by $\text{epn}(x, X) = \text{pn}(x, X) - \{x\}$.

An *internal vertex* of a tree is any vertex that is not a leaf, and a *branch vertex* is a vertex of degree three or more. Denote the set of internal vertices of T by $I(T)$. For a vertex v and a leaf ℓ of a tree T , a $v - \ell$ *endpath* is a path from v to ℓ , all of whose internal vertices (if they exist) have degree two in T . If the leaf ℓ is unimportant we refer to a $v - \ell$ endpath simply as a v -*endpath*. Let $B_l(T)$ denote the set of all branch vertices v of T for which there exists a v -endpath. A vertex $v \in B_l(T)$ is an *end branch vertex* if it has at most one neighbor that does not lie on a v -endpath. Let $B_e(T)$ denote the set of all end branch vertices of T . Note that if T has exactly one branch vertex v , then $B_e(T) = B_l(T) = \{v\}$, and if T has at least two branch vertices, then $|B_e(T)| \geq 2$ – take any two branch vertices at maximum distance apart. A *spider* $S(a_1, \dots, a_t)$, where $t \geq 3$, is a tree with exactly one branch vertex v together with v -endpaths P_i , $i = 1, \dots, t$, called *legs*, such that P_i has length a_i .

3. Lemmas

We begin with some general lemmas required in the proofs of our main results.

Lemma 5. *If the tree T has no strong stems, then T has a $(1, 2)$ -partition that is a minimum weight star partition.*

Proof. Among all minimum weight star partitions of T , let Π be one with the smallest number of parts $K_{1,m}$, $m \geq 2$. Suppose $H \cong K_{1,m}$, $m \geq 2$, is a part of Π . Since T has no strong stems, at most one leaf of H is a leaf of T . Let X be a set of $m - 1$ leaves of H that are not leaves of T . For each $x \in X$, let Y_x be the set of neighbors of x in $T - H$. Then $Y_x \cap Y_{x'} = \emptyset$ whenever $x \neq x'$. For each $x \in X$, if some $y \in Y_x$ belongs to a K_1 part of Π , move x from H to form the part $\{x, y\}$; if no vertex in Y_x belongs to a K_1 part of Π , move x from H to form the part $\{x\}$. The resulting partition is a star partition of T with weight no more than that of Π but with fewer parts of type $K_{1,m}$, $m \geq 2$, contrary to the choice of Π . \square

Lemma 6. *All minimum weight $(1, 2)$ -partitions of a tree T have the same number of K_2 parts. Furthermore, if one such partition has a K_1 part, then all minimum weight star partitions have K_1 parts.*

Proof. Let Π and Π' be minimum weight $(1, 2)$ -partitions of T with t_i and t'_i parts, respectively, isomorphic to K_i , $i = 1, 2$. Then $t_2 = \omega(\Pi) = \omega(\Pi') = t'_2$ and $|V(T)| = t_1 + 2t_2$. Suppose $t_1 \geq 1$. Let Φ be any minimum weight star partition of T and say Φ contains r K_1 parts and k_m $K_{1,m}$ parts, $m \geq 1$. Then $\omega(\Phi) = t_2 = \sum_{m \geq 1} mk_m \geq \sum_{m \geq 1} k_m$ and thus

$$t_1 + 2t_2 = V(T) = r + \sum_{m \geq 1} (1 + m)k_m = r + \sum_{m \geq 1} mk_m + \sum_{m \geq 1} k_m = r + t_2 + \sum_{m \geq 1} k_m \leq r + 2t_2,$$

from which it follows that $r \geq t_1 \geq 1$. □

The next three lemmas deal with the behavior of minimum weight $(1, 2)$ -partitions on endpaths.

Lemma 7. *Let T be a tree that satisfies Theorem 3(i) and (ii), with v an internal vertex of T that has a v -endpath $P = v, v_1, \dots, v_k$ of length $k \equiv 1 \pmod{3}$. For any minimum weight $(1, 2)$ -partition Π of T , either $\{v, v_1\}$ or $\{v\}$ is a part of Π . In the latter case there exists a minimum weight $(1, 2)$ -partition of T containing the part $\{v, v_1\}$ that agrees with Π everywhere except on P .*

Proof. Let $w \neq v_1$ be a neighbor of v and suppose $\{w, v\}$ is a part of Π . Then $k \geq 4$. Let $P' = P - v$ and Π' be the restriction of Π to P' . Say Π' has m_i K_i parts, $i = 1, 2$. We first show that $m_2 = m_1 + 2$. Let the last part of Π' on P' be the part that contains v_k . Since $\{v_k, v_{k-1}\}$ is a part of any $(1, 2)$ -partition of T , $\{v_k, v_{k-1}\}$ is the last part of Π' on P' . This and the fact that K_1 parts are not adjacent imply that $m_2 \geq m_1$. However, $m_1 = m_2$ would imply that $k \equiv 0 \pmod{3}$, while $m_1 + 1 = m_2$ would imply that $k \equiv 2 \pmod{3}$, neither of which is the case. Therefore $m_2 \geq m_1 + 2$. Since a minimum weight $(1, 2)$ -partition of T is obtained by maximizing the number of K_1 parts, and since a $(1, 2)$ -partition of P' with two more K_2 parts than K_1 parts is clearly possible, we conclude that $m_2 = m_1 + 2$. Define the partition Θ' of P' by

$$\begin{aligned} \Theta' = & \{ \{v_1, v_2\} \} \cup \{ \{v_i, v_{i+1}\} : i \equiv 0 \pmod{3}, 3 \leq i \leq k-1 \} \cup \\ & \{ \{v_i\} : i \equiv 2 \pmod{3}, 5 \leq i \leq k-2 \}. \end{aligned}$$

Then Θ' and Π' contain the same number of K_2 parts, so that $(\Pi - \Pi') \cup \Theta'$ is a minimum weight $(1, 2)$ -partition of T . But then $\{w, v\}$, $\{v_1, v_2\}$, $\{v_3, v_4\}$ form a K_2 -triple, contradicting Theorem 3(ii).

Suppose $\{v\}$ is a part of Π . As above, Π' contains two more K_2 parts than K_1 parts. Let $v_0 = v$ and define the partition Φ of P by

$$\begin{aligned} \Phi = & \{ \{v_i, v_{i+1}\} : i \equiv 0 \pmod{3}, 3 \leq i \leq k-1 \} \cup \\ & \{ \{v_i\} : i \equiv 2 \pmod{3}, 2 \leq i \leq k-2 \}. \end{aligned}$$

Then Φ and Π' have the same number of K_2 parts, so that $\Theta = (\Pi - \Pi' - \{v\}) \cup \Phi$ is a minimum weight $(1, 2)$ -partition of T that agrees with Π everywhere except on P , as required. □

Lemma 8. *Let T be a tree that satisfies Theorem 3(i) and (ii), with v an internal vertex of T that has a v -endpath $P = v, v_1, \dots, v_k$ of length $k \equiv 0 \pmod{3}$. Then T has no minimum weight $(1, 2)$ -partition containing the part $\{v\}$. For any minimum weight $(1, 2)$ -partition Π of T containing the part $\{v, v_1\}$ and any neighbor $w \neq v_1$ of v , there exists a minimum weight $(1, 2)$ -partition of T containing the part $\{v, w\}$ that agrees with Π everywhere except on $P \cup \{w\}$.*

Proof. Suppose Φ is a minimum weight $(1, 2)$ -partition containing the part $\{v\}$ and let Φ' be its restriction to P . Then $\{v_1, v_2\}$ is a part of Φ , hence $k \neq 3$ and so $k \geq 6$. Since $\{v_k, v_{k-1}\}$ is also a part and $k \equiv 0 \pmod{3}$, it follows, as in the proof of Lemma 7, that Φ' contains three more K_2 than K_1 parts. Let

$$\begin{aligned} \Phi'' &= \{\{v_i, v_{i+1}\} : i \equiv 1 \pmod{3}, 1 \leq i \leq k-5\} \cup \\ &\quad \{\{v_i\} : i \equiv 0 \pmod{3}, 3 \leq i \leq k-6\} \cup \{\{v_{k-3}, v_{k-2}\}, \{v_{k-1}, v_k\}\}. \end{aligned}$$

Then Φ' and Φ'' have the same number of K_2 parts, so that $(\Phi - \Phi') \cup \Phi''$ is a minimum weight $(1, 2)$ -partition of T . But then $\{v_{k-5}, v_{k-4}\}, \{v_{k-3}, v_{k-2}\}, \{v_{k-1}, v_k\}$ form a K_2 -triple, contradicting Theorem 3(ii).

Suppose Π contains the part $\{v, v_1\}$ and let Π' be its restriction to P . As before, Π' contains two more K_2 than K_1 parts. Let

$$\begin{aligned} \Pi'' &= \{\{v_0, v_1\}\} \cup \{\{v_i, v_{i+1}\} : i \equiv 2 \pmod{3}, 2 \leq i \leq k-1\} \cup \\ &\quad \{\{v_i\} : i \equiv 1 \pmod{3}, 4 \leq i \leq k-2\}. \end{aligned}$$

Then Π' and Π'' have the same number of K_2 parts, so that $\Theta' = (\Pi - \Pi') \cup \Pi''$ is a minimum weight $(1, 2)$ -partition of T . Since $\{v_0, v_1\}, \{v_2, v_3\}$ are consecutive, Theorem 3(ii) implies that $\{w\}$ is a part of Π for each $w \in N(v) - \{v_1\}$. Therefore $\Theta = (\Theta' - \{w\} - \{v, v_1\}) \cup \{\{w, v\}, \{v_1\}\}$ is a minimum weight $(1, 2)$ -partition of T that agrees with Π everywhere except on $P \cup \{w\}$. \square

The proof of the following lemma is similar to the proofs of Lemmas 7 and 8 and is omitted.

Lemma 9. *Let T be a tree that satisfies Theorem 3(i) and (ii), with v an internal vertex of T that has a v -endpath $P = v, v_1, \dots, v_k$ of length $k \equiv 2 \pmod{3}$. Then T has no minimum weight $(1, 2)$ -partition containing the part $\{v, v_1\}$.*

The following two results can be proved by simple inductive arguments; the proofs are omitted.

Lemma 10. *Suppose a tree T has a star partition in which each part is a K_2 . Then for each $v \in V(T)$ there exists a maximum independent set of T that contains v .*

Lemma 11. *Any $(1, 2)$ -partition of a tree T containing no consecutive K_2 parts is a minimum weight star partition of T .*

The next lemma shows that for a certain type of star partition of a tree T , all leaves of parts of the form $K_{1,m}$, $m \geq 2$, are also leaves of T .

Lemma 12. *If Π is a minimum weight star partition of a tree T with the maximum number of K_2 parts, and $H \cong K_{1,m}$, $m \geq 2$, is a part of Π , then each leaf of H is a leaf of T .*

Proof. Suppose a leaf ℓ of H is adjacent to the vertex x of a K_1 part and let $H' = H - \ell$. Then $(\Pi - H - \{x\}) \cup H' \cup \{\{\ell, x\}\}$ is a minimum weight star partition of T with more K_2 parts than Π , a contradiction. Now, if ℓ is not a leaf of T , let $H' = H - \ell$ and note that $\Pi' = (\Pi - H) \cup H' \cup \{\{\ell\}\}$ is a star partition of T such that $\omega(\Pi') = \omega(\Pi) - 1$, also a contradiction. \square

In the next two subsections we show that Theorem 3 holds for paths and spiders, respectively.

3.1. Paths.

Lemma 13. *The path P_n , $n \geq 2$, satisfies Theorem 3(i) – (iii) if and only if $n \not\equiv 0 \pmod{3}$, in which case $e_m^\infty(P_n) = \gamma(P_n)$. If $n \equiv 0 \pmod{3}$, then $e_m^\infty(P_n) > \gamma(P_n)$.*

Proof. If $n \in \{2, 3\}$, the lemma follows immediately, so assume $n \geq 4$. Let $P_n = v_1, \dots, v_n$. If $n \equiv 1 \pmod{3}$, let

$$\begin{aligned} \Pi_1 = & \{\{v_1, v_2\}\} \cup \{\{v_i, v_{i+1}\} : i \equiv 0 \pmod{3}, 3 \leq i \leq n-1\} \cup \\ & \{\{v_i\} : i \equiv 2 \pmod{3}, 5 \leq i \leq n-2\}; \end{aligned}$$

if $n \equiv 2 \pmod{3}$, let

$$\Pi_2 = \{\{v_i, v_{i+1}\} : i \equiv 1 \pmod{3}, 1 \leq i \leq n-1\} \cup \{\{v_i\} : i \equiv 0 \pmod{3}, 3 \leq i \leq n-2\},$$

and if $n \equiv 0 \pmod{3}$, $n > 3$, let

$$\begin{aligned} \Pi_3 = & \{\{v_1, v_2\}, \{v_3, v_4\}\} \cup \{\{v_i, v_{i+1}\} : i \equiv 2 \pmod{3}, 5 \leq i \leq n-1\} \cup \\ & \{\{v_i\} : i \equiv 1 \pmod{3}, 7 \leq i \leq n-2\}. \end{aligned}$$

Each Π_i has $\lceil \frac{n+1}{3} \rceil$ K_2 components. As shown in [8], $e_m^\infty(P_n) = \lceil \frac{n+1}{3} \rceil$, thus Π_i is a minimum weight star partition. Note that Π_3 has a K_2 -triple. Thus P_n , $n \equiv 0 \pmod{3}$, does not satisfy Theorem 3(ii). Also, $\gamma(P_n) = \frac{n}{3} < \lceil \frac{n+1}{3} \rceil$.

Now suppose $n \not\equiv 0 \pmod{3}$. Since Π_1 has only one pair of consecutive K_2 components, while Π_2 has none, and no star partition has consecutive K_1 parts, it is easy to see that P_n , $n \equiv 1$ or $2 \pmod{3}$, has no minimum weight star partition with a K_2 -triple, nor one with a $(2, 2)$ -chain. Also, (i) obviously holds, and $\gamma(P_n) = \lceil \frac{n}{3} \rceil = \lceil \frac{n+1}{3} \rceil = e_m^\infty(P_n)$. □

3.2. Spiders.

Lemma 14. *The spider $T = S(a_1, \dots, a_t)$ satisfies Theorem 3(i) – (iii) if and only if $a_i \equiv 1 \pmod{3}$ for exactly one i , or $a_i \equiv 0 \pmod{3}$ for all i , or $a_i \equiv 2 \pmod{3}$ for all i , in which cases $e_m^\infty(T) = \gamma(T)$. In all other cases $e_m^\infty(T) > \gamma(T)$.*

Proof. Let v be the branch vertex of T and label the leg P_i as $v, v_{i,1}, \dots, v_{i,a_i}$, $i = 1, \dots, t$. Suppose (without loss of generality) that $a_1 \equiv 1 \pmod{3}$. Define Π and D as follows. Let

$$\begin{aligned} \Pi_1 = & \{\{v, v_{1,1}\}\} \cup \{\{v_{1,j}, v_{1,j+1}\} : j \equiv 0 \pmod{3}, 3 \leq j \leq a_1-1\} \cup \\ & \{\{v_{1,j}\} : j \equiv 2 \pmod{3}, 2 \leq j \leq a_1-2\}, \end{aligned}$$

$$D_1 = \{v, v_{1,3}, v_{1,6}, \dots, v_{1,a_1-1}\};$$

if $a_i \equiv 0 \pmod{3}$, let

$$\begin{aligned} \Pi_i = & \{\{v_{i,j}, v_{i,j+1}\} : j \equiv 2 \pmod{3}, 2 \leq j \leq a_i-1\} \cup \\ & \{\{v_{i,j}\} : j \equiv 1 \pmod{3}, 1 \leq j \leq a_i-2\}, \end{aligned}$$

$$D_i = \{v_{i,2}, v_{i,5}, \dots, v_{i,a_i-1}\};$$

if $a_i \equiv 1 \pmod{3}$ and $i > 1$, let

$$\begin{aligned} \Pi_i &= \{\{v_{i,1}, v_{i,2}\}\} \cup \{\{v_{i,j}, v_{i,j+1}\} : j \equiv 0 \pmod{3}, 3 \leq j \leq a_i - 1\} \cup \\ &\quad \{\{v_{i,j}\} : j \equiv 2 \pmod{3}, 5 \leq j \leq a_i - 2\}, \\ D_i &= \{v_{i,3}, v_{i,6}, \dots, v_{i,a_i-1}\}; \end{aligned}$$

and if $a_i \equiv 2 \pmod{3}$, let

$$\begin{aligned} \Pi_i &= \{\{v_{i,j}, v_{i,j+1}\} : j \equiv 1 \pmod{3}, 1 \leq j \leq a_i - 1\} \cup \\ &\quad \{\{v_{i,j}\} : j \equiv 0 \pmod{3}, 3 \leq j \leq a_i - 2\}, \\ D_i &= \{v_{i,1}, v_{i,4}, \dots, v_{i,a_i-1}\}. \end{aligned}$$

Then $\Pi = \bigcup_{i=1}^t \Pi_i$ is a star partition of T and $D = \bigcup_{i=1}^t D_i$ is a dominating set of T such that each K_2 part of Π , except $\{v_{i,1}, v_{i,2}\}$ if $a_i \equiv 1 \pmod{3}$ and $i > 1$, contains a vertex of D . Let $k = |\{i > 1 : a_i \equiv 1 \pmod{3}\}|$. It is elementary to verify that D is a γ -set and Π is a minimum weight star partition of T . Hence $\gamma(T) = |D| = s(\Pi) - k = e_m^\infty(T) - k$; that is, $\gamma(T) = e_m^\infty(T)$ if and only if $k = 0$. Moreover, Π has consecutive K_2 parts if and only if (a) $k > 0$, in which case it has a K_2 -triple, or (b) $a_i \equiv 2 \pmod{3}$ for at least one i . If (b) holds but not (a), then Π has no K_2 -triples nor any $(2, 2)$ -chains. If $k = 0$, then the K_2 parts of Π cannot be rearranged to form a K_2 -triple or a $(2, 2)$ -chain. Hence T satisfies Theorem 3(ii) and (iii). Clearly, if $k = 0$ then T has no strong stems, thus Theorem 3(i) holds. This proves the lemma if $a_i \equiv 1 \pmod{3}$ for at least one i .

Suppose $a_i \equiv 0 \pmod{3}$ for each i . Define Π and D by

$$(3.1) \quad \begin{aligned} \Pi_1 &= \{\{v, v_{1,1}\}\} \cup \{\{v_{1,j}, v_{1,j+1}\} : j \equiv 2 \pmod{3}, 2 \leq j \leq a_1 - 1\} \cup \\ &\quad \{\{v_{1,j}\} : j \equiv 1 \pmod{3}, 4 \leq j \leq a_1 - 2\}; \end{aligned}$$

for $i > 1$,

$$(3.2) \quad \begin{aligned} \Pi_i &= \{\{v_{i,j}, v_{i,j+1}\} : j \equiv 2 \pmod{3}, 2 \leq j \leq a_i - 1\} \cup \\ &\quad \{\{v_{i,j}\} : j \equiv 1 \pmod{3}, 1 \leq j \leq a_i - 2\}, \end{aligned}$$

$$D = \{v\} \cup \bigcup_{i=1}^t \{v_{i,2}, v_{i,5}, \dots, v_{i,a_i-1}\}.$$

Then $\Pi = \bigcup_{i=1}^t \Pi_i$ is a star partition of T and D is a dominating set of T such that each K_2 part of Π contains a vertex of D . Again it is easy to see that D is a γ -set and Π is a minimum weight star partition of T . Hence $\gamma(T) = e_m^\infty(T)$. Also, the only pair of consecutive K_2 parts of Π are the parts $\{v, v_{1,1}\}$ and $\{v_{1,2}, v_{1,3}\}$, and the K_2 parts cannot be rearranged to form a K_2 -triple or a $(2, 2)$ -chain. The lemma follows for this case as well.

Suppose $a_i \equiv 2 \pmod{3}$ for each i . Define Π and D by

$$(3.3) \quad \begin{aligned} \Pi_i &= \{\{v_{i,j}, v_{i,j+1}\} : j \equiv 1 \pmod{3}, 1 \leq j \leq a_i - 1\} \cup \\ &\quad \{\{v_{i,j}\} : j \equiv 0 \pmod{3}, 3 \leq j \leq a_i - 2\} \cup \{v\}, \end{aligned}$$

$$D = \bigcup_{i=1}^t \{v_{i,1}, v_{i,4}, \dots, v_{i,a_i-1}\}.$$

Once again $\Pi = \bigcup_{i=1}^t \Pi_i$ is a minimum weight star partition and D is a γ -set of T such that each K_2 part of Π contains a vertex of D , and Π contains no consecutive K_2 parts. Thus Theorem 3(i) – (iii) hold and $\gamma(T) = e_m^\infty(T)$.

Finally, assume that there exists an integer r , $1 \leq r < t$, such that $a_i \equiv 0 \pmod{3}$ for $i = 1, \dots, r$ and $a_i \equiv 2 \pmod{3}$ for $i = r + 1, \dots, t$. For $i = 1, \dots, r$, define Π_i as in (3.1) and (3.2), and for $i = r + 1, \dots, t$, define Π_i as in (3.3). Then $\Pi = \bigcup_{i=1}^t \Pi_i$ is a minimum weight star partition of T that contains a K_2 -triple, namely $\{v, v_{1,1}\}$, $\{v_{1,2}, v_{1,3}\}$ and $\{v_{t,1}, v_{t,2}\}$. Thus (ii) does not hold. Let

$$D = \bigcup_{i=1}^r \{v_{i,2}, v_{i,5}, \dots, v_{i,a_i-1}\} \cup \bigcup_{i=r+1}^t \{v_{i,1}, v_{i,4}, \dots, v_{i,a_i-1}\}.$$

Then D is a γ -set of T such that each K_2 part of Π except $\{v, v_{1,1}\}$ contains a vertex of D , hence $\gamma(T) < e_m^\infty(T)$. The proof of the lemma is now complete. \square

4. Proof of Theorem 3

The proof of Theorem 3 is divided into four lemmas.

Lemma 15. *If T has a strong stem, then $e_m^\infty(T) > \gamma(T)$.*

Proof. Let T be a tree with a strong stem u , where ℓ_1, \dots, ℓ_k , $k \geq 2$, are the leaves adjacent to u . Let D be any γ -set of T . Then D contains u and $\gamma(T) \leq e_m^\infty(T)$ by Theorem 1. By the minimality of D , $D \cap \{u, \ell_1, \dots, \ell_k\} = \{u\}$. Place guards on each vertex in D . After an attack on u , the guard g_u on u must move, and no other guard can move to u in the same step. Also, no guard apart from g_u can move to ℓ_1, \dots, ℓ_k . Hence at least one of ℓ_1, \dots, ℓ_k is not dominated after the move. Therefore $e_m^\infty(T) > \gamma(T)$. \square

Lemma 16. *If T has a minimum weight star partition Π containing a K_2 -triple, then $e_m^\infty(T) > \gamma(T)$.*

Proof. Say $\{x, y\}$, $\{u, v\}$ and $\{w, z\}$ are K_2 parts of Π such that $ux, vw \in E(T)$. For each k and each part of Π of order k , place a guard on each of $k - 1$ vertices of the part to form an m-eternal eviction set D of T . Let T_u and T_v be the subtrees of $T - uv$ that contain u and v , respectively.

Assume first that the guard for $\{u, v\}$ is on u . If the guard on $\{w, z\}$ is on z , attack z , thus precipitating a move to w . This move to form a new m-eternal eviction set of T can be accomplished without moving any guards on T_u . Hence we may assume the guard for $\{w, z\}$ is on w while the guard for $\{u, v\}$ is on u . Let D_w be the set of vertices of T that now contain guards, and let $D_{v,w} = D_w \cap V(T_v)$. Note that

$$\text{epn}(u, D_w) \cap V(T_v) = \emptyset$$

and $D_{v,w}$ dominates T_v . An attack on u now causes the guard there to move to v . If the guard for $\{x, y\}$ is on y , an attack on y precipitates a move of the guard to x . As above, this can be accomplished without moving guards on T_v . Let D_x be the set of vertices of T that now contain guards, and let $D_{u,x} = D_x \cap V(T_u)$. Now

$$\text{epn}(v, D_x) \cap V(T_u) = \emptyset$$

and $D_{u,x}$ dominates T_u . It follows that $D_{v,w} \cup D_{u,x}$ is a dominating set of T such that $|D_{v,w} \cup D_{u,x}| = |D| - 1$. Hence $\gamma(T) < e_m^\infty(T)$. \square

Lemma 17. *If T has a minimum weight star partition Π containing a $(2, 2)$ -chain, then $e_m^\infty(T) > \gamma(T)$.*

Proof. Assume without loss of generality that the parts $H_i = \{v_{i,1}, v_{i,2}\}$ of Π , $i = 1, \dots, t$, $t \geq 4$, form a minimal $(2, 2)$ -chain, i.e., it contains no shorter $(2, 2)$ -chain. Let P be the $v_{1,1} - v_{t,2}$ path. Also assume that $v_{2,2}$ is adjacent to a K_1 part on P , otherwise the result follows from Lemma 16. For each $i = 2, \dots, t - 2$, let $v_{i,2}$ and $v_{i+1,1}$ be adjacent to the vertex u_i , where $\{u_i\}$ is a part of Π . Root T at u_2 and let $T_{i,j}$ be the subtree of T formed by $v_{i,j}$ and all its descendants.

For each k and each part of Π of order k , place a guard on each of $k - 1$ vertices of the part to form an m-eternal eviction set D_0 of T . Let g_i be the guard on H_i . We may assume that g_1 is on $v_{1,2}$ and g_t is on $v_{t,1}$. If g_2 is on $v_{2,2}$, attack this vertex, moving g_2 to $v_{2,1}$. While g_2 is on $v_{2,1}$, each vertex in $N(v_{2,2}) - P$ is guarded by a vertex (possibly itself) in $T_{2,2} - T_{2,1}$ with a guard. Attack $v_{2,1}$. Then g_2 moves to $v_{2,2}$, and this move can be executed by moving only guards stationed on $T_{2,1} - T_{1,2}$. Let D_1 be the set of vertices that now contain guards and note that

$$\text{pn}(v_{2,2}, D_1) \subseteq \{v_{2,2}, u_2\}.$$

If g_{t-1} is on $v_{t-1,1}$, attack this vertex, thus sending g_{t-1} to $v_{t-1,2}$. While g_{t-1} is on $v_{t-1,2}$, each vertex in $N(v_{t-1,1}) - P$ is guarded by a vertex (possibly itself) in $T_{t-1,1} - T_{t-1,2}$ with a guard. An attack on $v_{t-1,2}$ sends g_{t-1} to $v_{t-1,1}$, and this move is possible by moving only guards on $T_{t-1,2} - T_{t,1}$. Let D_2 be the set of vertices that now contain guards and note that

$$\text{pn}(v_{2,2}, D_2) \subseteq \{v_{2,2}, u_2\} \text{ and } \text{pn}(v_{t-1,1}, D_2) \subseteq \{v_{t-1,1}, u_{t-2}\}.$$

Similarly, considering H_{t-2}, \dots, H_3 in this order, we form an m-eternal eviction set D' such that $v_{t,1}, v_{t-1,1}, \dots, v_{3,1}, v_{1,2}, v_{2,2} \in D'$ and

$$\text{pn}(v_{2,2}, D') \subseteq \{v_{2,2}, u_2\}, \text{pn}(v_{t-1,1}, D') \subseteq \{v_{t-1,1}, u_{t-2}\}, \text{dots}, \text{pn}(v_{3,1}, D') \subseteq \{v_{3,1}, u_2\}.$$

Therefore $B = (D' - \{v_{t-1,1}\}) \cup \{u_{t-2}\}$ is a dominating set of T such that $|B| = |D| = e_m^\infty(T)$ and

$$\text{pn}(v_{2,2}, B) \subseteq \{v_{2,2}, u_2\}, \text{pn}(v_{t-2,1}, B) \subseteq \{v_{t-2,1}, u_{t-3}\}, \dots, \text{pn}(v_{3,1}, B) \subseteq \{v_{3,1}, u_2\}.$$

Since $\text{pn}(v_{t-2,1}, B) \subseteq \{v_{t-2,1}, u_{t-3}\}$, $B_1 = (B - \{v_{t-2,1}\}) \cup \{u_{t-3}\}$ is a dominating set of T such that $|B_1| = e_m^\infty(T)$. Continuing in this way, we eventually form a dominating set B' such that $\{v_{1,2}, v_{2,2}, u_2, \dots, u_{t-2}, v_{t,1}\} \subseteq B'$ and $|B'| = e_m^\infty(T)$, and in which $v_{2,1}$ is dominated by $v_{1,2}$ while $v_{2,2}$ and u_2 are dominated by u_2 . Hence $\text{pn}(v_{2,2}, B') = \emptyset$ and therefore $B' - \{v_{2,2}\}$ is a dominating set of T . Hence $\gamma(T) < e_m^\infty(T)$. \square

The necessity of conditions (i) – (iii) in the statement of Theorem 3 now follows from Lemmas 15 – 17. We proceed to show their sufficiency in the next lemma.

Lemma 18. *If T is a tree such that Theorem 3(i) – (iii) hold, then $e_m^\infty(T) = \gamma(T)$.*

Proof. The result is easily verified for trees of order four or less. By Lemmas 13 and 14, it also holds for paths and spiders. Assume it holds for trees of order less than n , where $n \geq 5$, and let T be a tree of order n with at least two branch vertices such that (i) – (iii) hold. Then T has no strong stems.

Suppose $v \in B_l(T)$ has a v -endpath $P = v, v_1, \dots, v_k$ of length $k \equiv 1 \pmod{3}$. Let $N(v) = \{v_1, w_1, \dots, w_t\}$, $t \geq 2$. For $i = 1, \dots, t$, let T'_i be the component of $T - v$ that contains w_i , and let T_i be the tree obtained by joining P to T'_i via the edge vw_i . By Lemma 7 there exists a minimum weight (1, 2)-partition Π of T containing the part $\{v, v_1\}$. Let Π_i be the restriction of Π to T_i , and Π_P the restriction of Π to P . Note that Π_P is a minimum weight star partition of P . Say Π_P contains m K_2 parts. Then Π_i contains m K_2 parts on P . If Π_i is not a minimum weight star partition of T_i , then by Lemma 5 there exists a (1, 2)-partition Π'_i of T_i such that $\omega(\Pi'_i) < \omega(\Pi_i)$. By Lemma 7 we may assume that $\{v, v_1\}$ is a part of Π'_i . But then $\Pi' = (\Pi - \Pi_i) \cup \Pi'_i$ is a star partition of T such that $\omega(\Pi') < \omega(\Pi)$, which is impossible. Hence Π_i is a minimum weight (1, 2)-partition of T_i for each i . Since the sum $\sum_{i=1}^t e_m^\infty(T_i)$ counts each K_2 part on P t times, it follows that $e_m^\infty(T) = \sum_{i=1}^t e_m^\infty(T_i) - m(t - 1)$.

Since $k \equiv 1 \pmod{3}$, there exists a γ -set D of T such that $v \in D$ and $v_1 \in \text{pn}(v, D)$. Let D_i be the restriction of D to T_i , and D_P the restriction of D to P . Note that D_P is a γ -set of P and D_i is a γ -set of T_i . By Lemma 13, $|D_P| = m$. Hence $\gamma(T) = \sum_{i=1}^t \gamma(T_i) - m(t - 1)$. Since Theorem 3(i) – (iii) hold for T , these statements also hold for each T_i . By the induction hypothesis, $\gamma(T_i) = e_m^\infty(T_i)$, hence $\gamma(T) = e_m^\infty(T)$. Therefore we assume that

A1: for each $v \in B_l(T)$, all v -endpaths have length congruent to 0 or 2 (mod 3).

Suppose there exists a vertex $v \in B_l(T)$ that has two v -endpaths $P = v, u, \ell$ and $P' = v, u', \ell'$ of length two. Let $T' = T - \{u', \ell'\}$. Obviously Theorem 3(i) – (iii) hold for T' . If Π is any minimum weight star partition of T , then $\{u, \ell\}$ and $\{u', \ell'\}$ are K_2 parts of Π . If Θ is any star partition of T' , then $\Theta \cup \{u', \ell'\}$ is a star partition of T . Hence the restriction Π' of Π to T' is a minimum weight star partition of T' , otherwise there exists a star partition of T with weight less than that of Π . Therefore $e_m^\infty(T) = e_m^\infty(T') + 1$. If $\gamma(T) = \gamma(T')$, then there exists a γ -set D' of T' such that $v \in D'$ and $\text{pn}(v, D') = \{v\}$, so that $D = (D' - \{v\}) \cup \{u\}$ dominates T . But $|D' \cap \{u, \ell\}| = 1$ to dominate ℓ , hence $D'' = (D' - \{\ell\}) \cup \{u\}$ is a dominating set of T' such that $|D''| = |D'|$, $v \in D''$ and $\text{pn}(v, D'') = \emptyset$. Hence $D'' - \{v\}$ is a dominating set of T' , contradicting $\gamma(T') = |D'|$. This shows that $\gamma(T) = \gamma(T') + 1$. By the induction hypothesis, $e_m^\infty(T') = \gamma(T')$ and thus $e_m^\infty(T) = \gamma(T)$. We assume therefore that

A2: each end branch vertex has at most one endpath of length two.

Suppose T has a minimum weight (1, 2)-partition Π_0 such that each $v \in B_e(T)$ has a longest v -endpath that

- (a) contains at least two consecutive K_2 parts of Π_0 , or
- (b) has length three, while another v -endpath has length two.

Choose any $v \in B_e(T)$ and construct the path Q and the minimum weight (1, 2)-partitions Π_j , $j \geq 1$, as follows. If (a) holds, then the first four vertices q_1, \dots, q_4 of Q are the vertices of the consecutive K_2 parts. If (b) holds, let the endpaths be v, a, b, c and v, x, y . Then $\{b, c\}$ and $\{x, y\}$ are parts of

Π_0 , hence $\{v, a\}$ is not a part of Π_0 . By Lemma 8, $\{v\}$ is also not a part of Π_0 . Thus there exists $w \in N(v) - \{a, x\}$ such that $\{v, w\}$ is a part of Π_0 for some $w \notin \{a, b, c, x, y\}$. Also by Lemma 8, we may assume that w does not lie on any v -endpath. Now the first four vertices q_1, \dots, q_4 of Q are x, y, v, w in this order. Note that $q_4 \in I(T)$.

Since Theorem 3(ii) holds, each neighbor of q_4 other than q_3 is a K_1 part. Thus T does not have a q_4 -endpath of length two. If $q_4 \in B_e(T)$, choose q_5 to be a neighbor of q_4 on a q_4 -endpath for which (a) holds. Then q_1, \dots, q_5 can be extended to a $(2, 2)$ -chain, contrary to Theorem 3(iii). Hence $q_4 \notin B_e(T)$.

Thus $q_4 \in I(T) - B_e(T)$, and there exists at least one vertex in $N(q_4) - \{q_3\}$ that does not lie on a q_4 -endpath; let q_5 be such a vertex. Since $\{q_5\}$ is a part of Π_0 , Lemma 8 implies that T does not have a q_5 -endpath of length three. If $q_5 \in B_e(T)$, choose q_6 to be a neighbor of q_5 on a q_5 -endpath for which (a) holds. Then q_1, \dots, q_6 can be extended to a $(2, 2)$ -chain, contrary to Theorem 3(iii). Hence $q_5 \notin B_e(T)$. We pause to state this as a general fact.

F1: For any $i \geq 5$ and $j \geq 0$, if $\{q_i\}$ is a part of Π_j , then $q_i \notin B_e(T)$ and q_i does not lie on an endpath.

Thus $q_5 \in I(T) - B_e(T)$, and there exists at least one vertex in $N(q_5) - \{q_4\}$ that does not lie on a q_5 -endpath; let q_6 be such a vertex. Then there is a vertex z such that $\{q_6, z\}$ is a part of Π_0 . Suppose $q_6 \in B_l(T)$ and z lies on a q_6 -endpath P . By A1 and Lemma 9, P has length congruent to 0 (mod 3).

- If $q_6 \in B_e(T)$, let w be the neighbor of q_6 on a longest endpath that satisfies (a) or (b). If $w = z$, let $\Pi_1 = \Pi_0$. If $w \neq z$, apply Lemma 8 to w with z as v_1 and q_6 as v to obtain a minimum weight $(1, 2)$ -partition Π_1 containing the part $\{q_6, w\}$ that agrees with Π_0 everywhere except on $P \cup \{w\}$. Let $q_7 = w$. In both cases (a) and (b), q_1, \dots, q_7 can be extended to a $(2, 2)$ -chain of Π_1 , contrary to Theorem 3(iii).
- If $q_6 \in B_l(T) - B_e(T)$ (and z lies on P), then there exists $w \in N(q_6) - \{q_5\}$ that does not lie on a q_6 -endpath. Apply Lemma 8 to obtain Π_1 as above.

Thus we may assume that z does not lie on an endpath of Π_1 . Let $q_7 = z$. Since Theorem 3(iii) holds, all vertices in $N(q_7) - \{q_6\}$ form K_1 parts. Thus T does not have a q_7 -endpath of length two. If $q_7 \in B_e(T)$ and (a) holds for a q_7 -endpath P' , then q_1, \dots, q_7 can be extended along P' to a $(2, 2)$ -chain of Π_1 , also a contradiction. We also state this as a general fact.

F2: For any $i \geq 6$ and $j \geq 0$, if $\{q_{i-1}, q_i\}$ is a part of Π_j , then $q_i \notin B_e(T)$ and q_i does not lie on an endpath.

We now choose q_8 similar to q_5 , q_9 similar to q_6 , and q_{10} similar to q_7 . In general, using F1 and F2, we choose q_i similar to q_5 (q_6 , q_7 , respectively) if $i \equiv 2$ (mod 3) (0 (mod 3), 1 (mod 3), respectively), thus constructing an infinite path Q . This is impossible since T is finite.

Now let Π be a minimum weight $(1, 2)$ -partition such that some $v \in B_e(T)$ has no longest v -endpath for which (a) or (b) holds. Let P be a longest v -endpath and consider the leaf ℓ of P and its adjacent stem u . By A1, $\deg_T u = 2$. Let $N(u) - \{\ell\} = \{w\}$. If $\deg_T w > 2$, then $w = v$ and there exists another v -endpath, which, by the choice of P , has length at most two, contradicting A1 or A2. Hence

$\deg_T w = 2$. Let $N(w) - \{u\} = \{x\}$. Note that $\{\ell, u\}$ is a part of Π . Since (a) does not hold, $\{w\}$ is a part of Π . Then x is in a K_2 part of Π , say $\{x, y\}$. Let $T' = T - \{\ell, u, w\}$ and let Π' be the restriction of Π to T' .

Suppose there exists a minimum weight $(1, 2)$ -partition Θ' of T' such that $\{x\}$ is a part of Θ' . Then x is neither a leaf nor a stem of T' and so $\deg_{T'} x \geq 3$, that is, $x = v$. If y is on a v -endpath P' , then, by the choice of P , P' has length at most three. Since (b) does not hold and $v = x$ is not a stem, P' has length three. Let y' be the neighbor of v that does not lie on a v -endpath and apply Lemma 8 to obtain a minimum weight $(1, 2)$ -partition of T , containing the part $\{v, y'\}$, that agrees with Π except that the roles of y and y' are interchanged. Hence, for simplicity, we assume that y is not on a v -endpath. Now let $z \in N(v) - \{w, y\}$ and note that z does lie on a v -endpath, say R . As in the case of P' , R has length three. Thus in T' , $\{v\}$ is a part of Θ' . But this contradicts Lemma 8. Hence

F3: x occurs in a K_2 part of each minimum weight $(1, 2)$ -partition of T' .

Therefore, if Φ' is a minimum weight $(1, 2)$ -partition of T' such that $\omega(\Phi') < \omega(\Pi')$, then $\Pi^* = (\Pi - \Pi') \cup \Phi' = \Phi' \cup \{\{w\}, \{\ell, u\}\}$ is a minimum weight $(1, 2)$ -partition of T such that $\omega(\Pi^*) < \omega(\Pi)$, a contradiction. Thus Π' is a minimum weight star partition of T' . Clearly, $e_m^\infty(T) = e_m^\infty(T') + 1$. Moreover, F3 also implies that T' satisfies Theorem 3(ii) and (iii) (otherwise neither does T), while condition (i) follows because y is not a stem.

Let D be any γ -set of T and note that $|\{\ell, u\} \cap D| = 1$. Neither ℓ nor u dominates x , hence $D' = D \cap V(T')$ is a dominating set, indeed, a γ -set of T' such that $|D| = |D'| + 1$, that is, $\gamma(T) = \gamma(T') + 1$. We may now apply the induction hypothesis to T' to obtain that $\gamma(T') = e_m^\infty(T')$ and thus $\gamma(T) = e_m^\infty(T)$. □

Theorem 3 now follows from Lemmas 15 – 18.

5. Proof of Theorem 4

The next two lemmas establish Theorem 4.

Lemma 19. *If a tree T has a minimum weight star partition containing no K_1 parts, then $e_m^\infty(T) = \alpha(T)$.*

Proof. The lemma is clearly true for trees of order four or less having minimum weight star partitions containing only large parts, namely P_2, P_3, P_4 and $K_{1,3}$. Suppose it is true for trees of order less than n , where $n \geq 5$, and assume T is a tree of order n that has a minimum weight star partition Π containing only large parts. If Π has only one part H , then $T \cong H \cong K_{1,n-1}$ and $e_m^\infty(T) = \alpha(T) = n - 1$.

Hence assume Π has large parts H_1 and H_2 such that the vertex v_1 of H_1 is adjacent to the vertex v_2 of H_2 . For $i = 1, 2$, let T_i be the component of $T - v_1v_2$ that contains v_i and let Π_i be the restriction of Π to T_i . Then Π_i is a star partition of T_i . If T_1 has a star partition Π'_1 such that $\omega(\Pi'_1) < \omega(\Pi_1)$, then, since Π_2 has no K_1 part, $\Pi'_1 \cup \Pi_2$ is a star partition of T such that $\omega(\Pi'_1 \cup \Pi_2) < \omega(\Pi)$, which is impossible. Hence Π_1 and, similarly, Π_2 are minimum weight star partitions of T_1 and T_2 ,

respectively, that contain only large parts. By the induction hypothesis, $e_m^\infty(T_i) = \alpha(T_i)$. Therefore $\alpha(T) \leq \alpha(T_1) + \alpha(T_2) = e_m^\infty(T_1) + e_m^\infty(T_2) = e_m^\infty(T) \leq \alpha(T)$, from which the result follows. \square

Lemma 20. *If all minimum weight star partitions of the tree T contain K_1 parts, then $e_m^\infty(T) < \alpha(T)$.*

Proof. The only nontrivial tree of order five or less whose minimum weight star partitions all have a K_1 part is P_5 , and $e_m^\infty(P_5) = 2 < \alpha(P_5) = 3$. Assume the result to be true for trees of order less than n , where $n \geq 6$, and let T be a tree of order n , all of whose minimum weight star partitions have K_1 parts. Among all such partitions, let Π be one with the maximum number of K_2 parts. By Lemma 12,

$$(5.1) \quad \text{all leaves of } K_{1,m} \text{ parts of } \Pi, \text{ where } m \geq 2, \text{ are also leaves of } T.$$

First suppose T has a strong stem u adjacent to the leaves $\ell_1, \ell_2, \dots, \ell_d, d \geq 2$. Then the star H with center u and leaves $\ell_1, \ell_2, \dots, \ell_d$ is a part of Π . Let $T' = T - \ell_d, H' = H - \ell_d$, and $\Pi' = (\Pi - H) \cup H'$. Then Π' is a minimum weight star partition of T' containing K_1 parts. Moreover, any minimum weight star partition of T' that contains no K_1 parts would correspond to a minimum weight star partition of T with no K_1 parts, contrary to the choice of T . By the induction hypothesis, $e_m^\infty(T') < \alpha(T')$, and then clearly $e_m^\infty(T) = e_m^\infty(T') + 1 < \alpha(T') + 1 = \alpha(T)$. Thus we assume henceforth that T has no strong stems. By (5.1),

$$(5.2) \quad \Pi \text{ is a } (1, 2)\text{-partition.}$$

Suppose next that Π has two K_2 parts $H_i, i = 1, 2$, with vertices $v_i \in V(H_i)$ such that $v_1 v_2 \in E(T)$. For $i = 1, 2$, let T_i be the component of $T - v_1 v_2$ that contains v_i , and let Π_i be the restriction of Π to T_i . Then Π_i is a $(1, 2)$ -partition of T_i .

If Θ_1 is a star partition of T_1 such that $\omega(\Theta_1) < \omega(\Pi_1)$, then $\Theta_1 \cup \Pi_2$ is a star partition of T such that $\omega(\Theta_1 \cup \Pi_2) < \omega(\Pi)$, which is impossible. Hence Π_1 and, similarly, Π_2 are minimum weight star partitions of T_1 and T_2 , respectively. Therefore

$$e_m^\infty(T) = \omega(\Pi) = \omega(\Pi_1) + \omega(\Pi_2) = e_m^\infty(T_1) + e_m^\infty(T_2).$$

For $i = 1, 2$, let X_i be a α -set of T_i . If $v_i \in X_i$ for each $i = 1, 2$, then $(X_1 - \{v_1\}) \cup X_2$ and $X_1 \cup (X_2 - \{v_2\})$ are independent sets of T , and

$$(5.3) \quad \alpha(T_1) + \alpha(T_2) - 1 \leq \alpha(T) \leq \alpha(T_1) + \alpha(T_2).$$

If $v_i \notin X_i$ for at least one i , then $X_1 \cup X_2$ is a α -set of T and

$$(5.4) \quad \alpha(T) = \alpha(T_1) + \alpha(T_2).$$

From the statement of the lemma and since $\Pi = \Pi_1 \cup \Pi_2$, at least one Π_i contains K_1 parts. Suppose both Π_1 and Π_2 contain K_1 parts. By Lemma 6 all minimum weight star partitions of T_1 and T_2 contain K_1 parts. By the induction hypothesis, $e_m^\infty(T_i) \leq \alpha(T_i) - 1$ for $i = 1, 2$. From (5.3) and (5.4),

$$e_m^\infty(T) = e_m^\infty(T_1) + e_m^\infty(T_2) \leq \alpha(T_1) + \alpha(T_2) - 2 < \alpha(T)$$

as required. Hence, assume without loss of generality that Π_1 (and all minimum weight partitions of T_1) has K_1 parts and Π_2 does not. By (5.2), Π_2 has only K_2 parts. Let w be the neighbor of v_2 in H_2 . By Lemma 10 there exists a α -set X_2 of T_2 that contains w , hence $v_2 \notin X_2$. By (5.4), $\alpha(T) = \alpha(T_1) + \alpha(T_2)$. Now by the induction hypothesis and Theorem 1, $e_m^\infty(T_1) < \alpha(T_1)$ and $e_m^\infty(T_2) \leq \alpha(T_2)$, so that

$$e_m^\infty(T) = e_m^\infty(T_1) + e_m^\infty(T_2) < \alpha(T_1) + \alpha(T_2) = \alpha(T).$$

We may now assume that Π has no adjacent K_2 parts. Let P be a diametrical path of T such that ℓ is a leaf of T and u is the stem adjacent to ℓ . Then $\{\ell, u\}$ is a K_2 part of Π and $\deg_T u = 2$. Let v be the other neighbor of u and note that $\{v\}$ is a K_1 part of Π . Let $N(v) - \{u\} = \{w_1, \dots, w_r\}$, let T_i be the subtree of $T - \{\ell, u, v\}$ that contains w_i , and let Π_i be the restriction of Π to T_i . Since $\{v\}$ is a K_1 part of Π , each w_i belongs to a K_2 part $\{w_i, x_i\}$ of Π_i . Thus each Π_i is a $(1, 2)$ -partition of T_i with no consecutive K_2 parts. By Lemma 11 each Π_i is a minimum weight partition of T_i .

If $T_i \cong K_2$ for each i , then $\{\ell, v, x_1, \dots, x_r\}$ is a α -set of T of cardinality $r + 2$, while $e_m^\infty(T) = r + 1 < \alpha(T)$. Therefore we assume without loss of generality that $T_1 \not\cong K_2$. Then Π_1 and (by Lemma 6) all minimum weight star partitions of T_1 contain at least one K_1 part. By the induction hypothesis, $e_m^\infty(T_1) < \alpha(T_1)$. Also, if X_i is a α -set of T_i , then $\{\ell\} \cup \bigcup_{i=1}^r X_i$ is an independent set of T . Therefore

$$e_m^\infty(T) = 1 + \sum_{i=1}^r e_m^\infty(T_i) < 1 + \sum_{i=1}^r \alpha(T_i) \leq \alpha(T)$$

as required. □

6. Open Problems

The set $\mathcal{A}(T)$ of vertices that are contained in every minimum dominating set of a tree T is characterized in [15]. In particular, if v is a strong stem of T , then $v \in \mathcal{A}(T)$.

Suppose $v \in \mathcal{A}(T)$ and let D be any γ -set of T . Then $v \in D$. Place guards on each vertex in D . When v is attacked, the guard there must move, and no guard can move to v at the same time. Hence the resulting configuration of guards does not form a dominating set, so $e_m^\infty(T) > \gamma(T)$. Therefore $\mathcal{A}(T) = \emptyset$ is a necessary condition for a tree to satisfy $e_m^\infty(T) = \gamma(T)$, and it is also a stronger condition than Theorem 3(i). However, it does not characterize trees with $e_m^\infty(T) = \gamma(T)$. Consider two disjoint copies of P_6 with vertex sequences $v_{i,1}, \dots, v_{i,6}$ for $i = 1, 2$ and let T be the tree obtained by joining a new vertex u to $v_{1,3}$ and $v_{2,3}$. Then $D_1 = \{v_{1,2}, v_{2,2}, u, v_{1,5}, v_{2,5}\}$, $D_2 = \{v_{1,2}, v_{2,1}, v_{2,3}, v_{1,5}, v_{2,6}\}$ and $D_3 = \{v_{1,1}, v_{2,2}, v_{1,3}, v_{2,5}, v_{1,6}\}$ are γ -sets of T such that $D_1 \cap D_2 \cap D_3 = \emptyset$, hence $\mathcal{A}(T) = \emptyset$. But $e_m^\infty(T) = 6 > 5 = \gamma(T)$.

Problem 1. Find a property \mathbf{P} of a tree T such that if \mathbf{P} holds and $\mathcal{A}(T) = \emptyset$, then $e_m^\infty(T) = \gamma(T)$.

Problem 2. Conversely, is it possible to characterize $\mathcal{A}(T)$ in terms of star partitions of T ?

Constructive characterizations of trees with equal domination-type parameters are given in [3, 4]. That is, recursive constructions that generate precisely the classes of trees having two specific domination parameters equal are described.

Problem 3. *Find constructive characterizations of trees T such that $e_m^\infty(T) = \gamma(T)$ and $e_m^\infty(T) = \alpha(T)$.*

Problem 4. *Find structural properties of a tree T that has a minimum weight star partition containing no K_1 parts. That is, characterize trees T such that $e_m^\infty(T) = \alpha(T)$ in terms of the structure of T without referring to star partitions.*

As defined in [5], a *neo-colonization* of a graph G is a partition $\{V_1, V_2, \dots, V_t\}$ of $V(G)$ such that the subgraph $G[V_i]$ induced by each V_i is connected. The *weight* of V_i is one if $G[V_i] \cong K_n$, and is one plus the connected domination number of $G[V_i]$ otherwise. Any graph without isolated vertices has a minimum weight neo-colonization without K_1 parts. It was proved in [10] that the m-eternal domination number $\gamma_m^\infty(T)$ of a tree T is equal to the minimum weight of a neo-colonization. A minimum weight star partition is a neo-colonization (which may have K_1 parts), or, if K_1 parts are not desired, can be transformed into a K_1 -free neo-colonization by absorbing each K_1 part into a neighboring part. A neo-colonization obtained in either of these ways is called a *basic neo-colonization*. It is easy to prove that all basic neo-colonizations have the same weight. However, a basic neo-colonization need not be a minimum weight neo-colonization: two P_3 's with their middle vertices joined by an edge is one example.

Problem 5. *Characterize trees whose basic neo-colonizations are minimum weight neo-colonizations.*

We note that $e_m^\infty(G)$ is, in general, not comparable to $\gamma_m^\infty(G)$, even in trees. The star $T = K_{1,3}$ is an example where $e_m^\infty(T) > \gamma_m^\infty(T)$, while the spider $T = S(3, 3, 3)$ satisfies $4 = e_m^\infty(T) < \gamma_m^\infty(T) = 5$. Given an arbitrary tree T , it is easy to see that $e_m^\infty(T) < \gamma_m^\infty(T)$ if (i) some minimum weight star partition has more K_1 parts than leaves in all the $K_{1,m}$ parts having $m > 1$, and (ii) the corresponding basic neo-colonization is a minimum weight neo-colonization. We do not know whether the converse is also true.

Problem 6. *Characterize the classes of trees T having, respectively, $e_m^\infty(T) = \gamma_m^\infty(T)$, $e_m^\infty(T) < \gamma_m^\infty(T)$ or $e_m^\infty(T) > \gamma_m^\infty(T)$.*

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