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THE THEOREMS OF SCHUR AND BAER: A SURVEY

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Abstract. This paper gives a short survey of some of the known results generalizing the theorem, credited to I. Schur, that if the central factor group is finite then the derived subgroup is also finite.

1. Introduction

I. Schur [33] was one of the first mathematicians to study the connection between the central factor group, $G/Z(G)$, of a group G and its derived subgroup, G'. He is credited with the following well-known result which is now known as Schur's Theorem:

Theorem 1.1. *Let G be a group and let $C \leq Z(G)$, the centre of G. Suppose that G/C is finite. Then G', the derived subgroup of G, is finite.*

There is a natural related question that can be posed:

- How does the order of $G/Z(G)$ affect the order of G'?

Schur[33] introduced $M(G)$, the multiplicator (or multiplier) of a finite group G that now bears his name. He proved that if G is a finite group, then $Z(G) \cap G'$ is a homomorphic image of $M(G/Z(G))$. This observation appears to be crucial in obtaining bounds for $|G'|$ in terms of $|G/Z(G)|$, once one observes, using a result of P. Hall [14], that attention can be restricted to finite groups for such calculations. Later, the concept of the Schur multiplier was extended to infinite groups. A close inspection of [33] reveals that Schur's paper is concerned only with finite groups, which makes the genesis of the name "Schur's Theorem" interesting.

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Theorem 1.1 appears in 1957 in the book [18] of I. Kaplansky,who attributes the theorem, independently, to R. Baer, B. H. Neumann and E. Witt, and provides a proof due to D. Ornstein. In his 1951 paper, B. H. Neumann [28, Theorem 5.3] proves Theorem 1.1, stating that it is one of the main theorems of his paper. Interestingly though, he adds a note saying that Baer had drawn his attention to [2, Section 6, Theorem 4], which yields Theorem 1.1 as a consequence and which is much more general. Further investigation shows that Baer [2, Section 3, Theorem 3] proved that if G is a group and N is a normal subgroup of G of finite index n then $G' \cap N)/[N, G]$ is a finite group whose elements have order dividing n. Certainly, when $N \leq Z(G)$, we obtain Theorem 1.1 as a consequence, and [2, Section 6, Theorem 4] is a generalization of [2, Section 3, Theorem 3]. It is not until his later paper [3] that Baer gives an explicit proof of Theorem 1.1, without recourse to the paper [2]. Baer, of course, makes the connection between his result and Schur's paper, but does not explicitly mention Theorem 1.1 as being due to Schur.

Neumann gives an upper bound for the order of G'. He states that it can be shown, using the results of Schur [33], that $|G'| \leq |G : Z|^{|G:Z|^2+1}$. Thus here he also already uses the results of the classical article of Schur.

Baer's paper [2] also contains some numerical results. In [2, Section 6, Corollary 3] he showed that if G is a group for which $|G/Z(G)| \leq t$ then G' has exponent t^2. The bound given by Neumann for $|G'|$ has been considerably improved. A better bound was later established by J. Wiegold [34], which is best possible when $t = p^m$, in a result which runs as follows.

Theorem 1.2. *Let G be a group. Suppose that the order of $G/Z(G)$ is t. Then the order of G' is at most t^m, where $m = (1/2)(\log_p t - 1)$, and p is the least prime dividing the order of $G/Z(G)$.*

Interestingly, Wiegold in his paper asserts that "it has long been well-known" that every group with $G/Z(G)$ finite also has G' finite (no reference is given) and suggests that the simplest proof (of the "many in existence")is perhaps due to Rosenlicht [31]. Wiegold's proof of Theorem 1.2 quickly reduces to the finite case and then uses the results of Schur concerning the multiplier.

However the name "Schur's theorem" appears in print for the first time, evidently, in the famous lectures of P. Hall [15], who gave his own proof of Theorem 1.1, and called it Schur's Theorem , again without providing a reference. Thus the name "Schur's Theorem" seems to originate with Hall and that name has stuck.

Several generalizations of Schur's Theorem follow virtually immediately and the subject lay dormant for some time. This theme has been taken up again more recently and it is these investigations that we wish to discuss here.

2. Schur Classes

There are a number of ways in which we can try to extend Schur's theorem and the following idea seems very natural. Our terminology is due to Franciosi, de Giovanni and Kurdachenko [13].

Definition 2.1. *A class $\mathfrak{X}$ of groups is called a* Schur Class *if*

$$G/Z(G) \in \mathfrak{X} \textit{ implies } G' \in \mathfrak{X}, \textit{ for all groups } G.$$

Thus the class of finite groups is a Schur class and the following natural question arises.

- Which classes of groups are Schur classes?

It is easy to see from the proof of Schur's Theorem that in fact, if $C \leq Z(G)$ and if G/C is a finite π-group, for some set of primes π, then G' is a finite π-group. It is also easy to deduce that, with the same hypothesis on C, if G/C is a locally finite π-group then G' is a locally finite π-group. The extension to the case when $G/Z(G)$ is polycyclic-by-finite is also quite straighforward: in this case G' is likewise polycyclic-by-finite. The dual situation is less trivial and is due to A. Schlette [32, Proposition 3.2] who proved that if $G/Z(G)$ is a Chernikov group then G' is also Chernikov.

The following classes are therefore Schur classes.

- Finite π-groups for all sets of primes π;
- Locally finite π-groups;
- Polycyclic-by-finite groups;
- Chernikov groups.

This seems to be the extent of the "easy theorems" that can be obtained generalizing Schur's theorem. On the other hand there are some natural classes of groups that are not Schur classes.

- (S. I. Adian [1]) There is a torsion-free group G such that $G/Z(G)$ is an infinite non-locally finite p-group of finite exponent;
- (A. Olshanskii [29]) There is a group G such that $G = G'$, $Z(G)$ is free abelian of countable rank, and $G/Z(G)$ is a Tarski monster. Thus $G/Z(G)$ is in this case an infinite p-group whose proper subgroups have order the prime p.

A short word of explanation is perhaps required for the second item here. In [29, Chapter 27] a certain group $G(\infty)$ is constructed, as a limit of a sequence, $\{G(k)\}$, of groups. All proper subgroups of $G(\infty)$ have prime order, by [29, Theorem 28.1]. The combination of Lemma 27.2, Lemma 25.1 and Theorem 31.1(2) of [29] guarantees that for the aspherical corepresentation $G(\infty) = F/N$ the group $N/[N, F]$ is free abelian of countable rank. Now arguing as in the proof of [29, Corollary 31.2] we deduce that the Schur multiplier of $G(\infty)$ is a free abelian group of countable rank. The proof also uses the fact that the group $G(\infty)$ does not coincide with $G(k)$ for any $k \in \mathbb{N}$. This can be shown by arguing as in the proof of [29, Theorem 19.3] and replacing the reference to Theorem 19.1 by that to Theorem 26.2.

The Tarski monsters of course are examples of groups which have the minimum condition and also the maximum condition. These examples of Adian and Olshanskii therefore show that

- The class of periodic groups is not a Schur class;
- The class of groups with the minimum (or maximum) condition is not a Schur class.

The class of Chernikov groups and the class of polycyclic-by-finite groups admit several other natural, common, generalizations. We recall that a group G is called minimax if it has a finite series

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$$

in which the factors G_{i+1}/G_i either have the maximum condition or the minimum condition. Furthermore, a group G is of finite rank r if every finitely generated subgroup of G is at most r-generator. The Tarski monsters have rank 2 and are also minimax. Hence, the examples given above show

- The class of minimax groups is not a Schur class;
- The class of groups of finite rank is not a Schur class.

There are however a number of positive results for these classes of groups, the first of which is due to L. A. Kurdachenko [19].

Theorem 2.2. *Let G be a group and suppose that $G/Z(G)$ is a soluble-by-finite minimax group. Then G' is also a soluble-by-finite minimax group.*

Thus

- The class of soluble-by-finite minimax groups is a Schur class.

For the class of groups of finite rank there are also positive results and since there is a numerical invariant concerned with this class of groups, we can obtain bounds concerning this invariant. The motivation here again lies in finite groups. A finite groupof course has finite rank. When G is finite and $G/Z(G)$ is of rank r then G' has finite rank bounded by a function of r, a fact that was first observed by Lubotzky and Mann [25]. Their result can be stated as follows, and involves the theory of powerful p-groups.

Theorem 2.3. *Let G be a finite group such that $G/Z(G)$ is of rank r. Then G' is of rank at most $\binom{r}{2} + r^2 \log_2 r + r^2 + r$.*

For infinite groups, the first result appears in [12] where the authors prove that if $G/Z(G)$ is a soluble group of finite rank then G' has finite rank. The authors do obtain a bound here but it is also dependent upon the derived length of $G/Z(G)$, as well as its rank.

Thus the class of soluble groups of finite rank is a Schur class. Next we recall that a group G is called generalized radical if it has an ascending series whose factors are either locally nilpotent or locally finite. Thus a generalized radical group either contains an ascendant locally nilpotent subgroup, in which case the Hirsch-Plotkin radical is non-trivial, or it contains an ascendant locally finite subgroup, in which case the locally finite radical of G is non-trivial. Thus a generalized radical group has an ascending normal series each of whose factors is either locally nilpotent or locally finite.

The most general positive result that has been obtained so far for groups of finite rank is the following recent result, which builds on Theorem 2.3 and the results of [12]. This more general result is due to Kurdachenko and Shumyatsky [23] and its proof depends upon Theorem 2.3. Indeed Kurdachenko and Shumyatsky were unaware of Theorem 2.3 and obtained an independent proof of that result, using the theory of powerful p-groups and the classification of finite simple groups.

Theorem 2.4. *Let G be a locally generalized radical group and suppose that $C \leq Z(G)$ is such that G/C has finite rank at most r. Then G' has finite rank at most $\rho(r)$, for some function ρ of r only.*

In the proof of this result essential knowledge is required concerning the structure of a locally generalized radical group of finite rank. This structure is given in [8]. Indeed if G is a locally generalized radical group of finite rank then there is a series of normal subgroups

$$T \leq L \leq K \leq S \leq G$$

such that T is locally finite, L/T is a torsion-free nilpotent group, K/L is a finitely generated torsion-free abelian group, G/K is finite and S/T is soluble. This fact is also used in the proofs of some of the later results given below. We make a further remark concerning Theorem 2.4 and that is that it applies to locally residually finite groups. For, if G is a residually finite group then so is $G/Z(G)$ and it is known, again by a result of Lubotzky and Mann [26], that a finitely generated residually finite group of finite rank is almost soluble, and hence generalized radical.

There is one further generalization in this direction. For this we need the concept of section p-rank.

Definition 2.5. *Let p be a prime. The group G has finite section p-rank r if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section of G precisely of order p^r.*

The result for groups of finite section p-rank is due to A. Ballester-Bolinches, S. Camp-Mora, L. Kurdachenko and J. Otal [4], and is as follows.

Theorem 2.6. *Let G be locally generalized radical and suppose that $G/Z(G)$ has section p-rank at most s, for the prime p. Then G' has section p-rank at most $f(s)$, for some function $f(s)$ of s only.*

We end this section of our survey with a further result extending Theorem 1.1 in a different direction. It is related to the example of Adian given above and is due to A. Mann [27].

Theorem 2.7. *Let G be a group and let $C \leq Z(G)$ be such that G/C is locally finite of exponent e. Then G' is a locally finite group of exponent at most $m(e)$, where m is a function of e only.*

This summarizes some of the recent activity connected with generalizations of Schur's Theorem in one direction. However there are other directions in which we can go, based on Baer's generalization [2, Section 6, Theorem 4], mentioned in Section 1. It is to these investigations that we now turn.

3. Other Generalizations of Schur's Theorem

In order to discuss generalizations of Schur's Theorem from another context we need some further notation.

Let

$$1 = Z_0(G) \leq Z_1(G) = Z(G) \leq Z_2(G) \leq \cdots \leq Z_\alpha(G) \leq \ldots$$

be the upper central series of G and let

$$G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \cdots \geq \gamma_\alpha(G) \ldots$$

be the lower central series of G. Thus, as usual $Z_1(G) = Z(G)$ and

$$Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G)), \text{ for all ordinals } \alpha \text{ and}$$
$$Z_\lambda(G) = \bigcup_{\mu<\lambda} Z_\mu(G) \text{ for all limit ordinals } \lambda.$$

We let $Z_\infty(G)$ denote the last term of this upper central series and as usual call $Z_\infty(G)$ the upper hypercentre of G. The first ordinal μ such that $Z_\infty(G) = Z_\mu(G)$ is called the central length of G and is denoted by $\mathrm{zl}(G)$. Furthermore $\gamma_2(G) = G'$, $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$, for all ordinals α and $\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G)$, for all limit ordinals λ.

As we menitoned in Section 1, R. Baer [2, Section 6, Theorem 4] proved a result which essentially generalized Theorem 1.1 (and this generalization is naturally called Baer's Theorem in the sequel) as follows. This result was also proved by Baer in [3] in the form given below. Of course Schur's Theorem is the case $k = 1$.

Theorem 3.1. *If $G/Z_k(G)$ is finite, for some natural number k, then $\gamma_{k+1}(G)$ is finite.*

We let $G^{\mathbf{L}\mathfrak{N}}$ denote the locally nilpotent residual of a group G. Then, by definition, the locally nilpotent residual of G is

$$G^{\mathbf{L}\mathfrak{N}} = \bigcap\{N \triangleleft G | G/N \text{ is locally nilpotent }\},$$

and we observe that, in general, $G/G^{\mathbf{L}\mathfrak{N}}$ need not be locally nilpotent. However Baer's Theorem implies that if $G/Z_k(G)$ is finite then $G^{\mathbf{L}\mathfrak{N}}$ is finite, since $G/\gamma_{k+1}(G)$ is nilpotent, and in this case $G^{\mathbf{L}\mathfrak{N}} = G^{\mathfrak{N}}$, the nilpotent residual of G.

In their paper [24] Kurdachenko and Subbotin gave numerical bounds for the size of $|\gamma_{k+1}(G)|$ in Baer's Theorem. Other such bounds have been obtained in [10] and [5] in the cases when $G/Z_k(G)$ is a p-group or nilpotent, respectively.

Theorem 3.2. *Let G be a group and let Z be the upper hypercentre of G. Suppose that $zl(G) = k$ is finite and that G/Z is finite of order at most t.*

(i) *There is a function $\beta_1(t, k)$ such that $|\gamma_{k+1}(G)| \le \beta_1(t, k)$;*
(ii) *There is a function $\beta_2(t)$ such that $|G^{\mathbf{L}\mathfrak{N}}| \le \beta_2(t)$.*

It can be seen from the proofs that the functions β_1, β_2 can be defined by

$$\beta_1(t, 1) = w(t), \beta_1(t, k) = w(\beta_1(t, k-1)) + t\beta(t, k-1)$$
$$\beta_2(t) = t^d, \text{ where } d = (1/2)(\log_p t + 1)$$

Here p is the least prime dividing the order of t and $w(t) = t^m$, where $m = (1/2)(\log_p t - 1)$, the function obtained by Wiegold in Theorem 1.2. It is particularly interesting to note that the order of $G^{\mathbf{L}\mathfrak{N}}$ is dependent only upon the order of G/Z.

A very natural question then arises.

- Suppose $\mathrm{zl}(G) = k$ is finite and Z is the upper hypercentre of G. If $G/Z \in \mathfrak{X}$, for some class of groups $\mathfrak{X}$, then is $\gamma_{k+1}(G) \in \mathfrak{X}$?

The answer to this question for many classes of groups can be deduced from the following far-reaching generalization of Baer's Theorem that can be obtained for certain classes of groups. This generalization appears as Theorem 4.21 of [30] and we re-state a special case of this result here.

Theorem 3.3. *Let $\mathfrak{X}$ be a class of groups which is closed under taking subnormal subgroups, homomorphic images and extensions. Assume also that the tensor product of two abelian $\mathfrak{X}$-groups is an $\mathfrak{X}$-group and that if $G/Z_1(G) \in \mathfrak{X}$ then $G' \in \mathfrak{X}$. If c is a natural number and if $G/Z_c(G) \in \mathfrak{X}$ then $\gamma_{c+1}(G) \in \mathfrak{X}$.*

In particular, the tensor product of two abelian Chernikov groups is a Chernikov group and hence, by [32], if $G/Z_c(G)$ is a Chernikov group then $\gamma_{c+1}(G)$ is also Chernikov. Recently there has been a flurry of activity generalizing Baer's theorem in other similar ways, but because of the examples of Adian and Ol'shanskii mentioned above, the situation is more complicated. This is true even for groups of finite rank. However we have the following result due to Kurdachenko and Otal [20].

Theorem 3.4. *Let G be a group and let Z denote the upper hypercentre of G. Suppose that $zl(G) = k$ is finite.*

(i) *If G is a locally generalized radical group and G/Z has rank at most r then $\gamma_{k+1}(G)$ has finite rank and there is a function $\tau_1(r,k)$ such that $r(\gamma_{k+1}(G)) \leq \tau_1(r,k)$;*
(ii) *If G/Z is a locally finite group of finite rank at most r, then the hypercentral residual L of G is a locally finite group of finite rank at most $\kappa(r) + r$ and G/L is hypercentral, for some function κ of r only.*

In a forthcoming paper [6] the authors prove the following result.

Theorem 3.5. *Let G be a locally generalized radical group and let p be a prime. Suppose that $G/Z_k(G)$ has finite section p-rank r. Then $\gamma_{k+1}(G)$ has finite section p-rank and there is a function τ such that $r_p(G) \leq \tau(r,k)$.*

We refer the reader to [6] for some of the consequences of this result. A recent result [21] of Kurdachenko, Pypka and Otal provides a generalization of Theorem 2.7 as follows.

Theorem 3.6. *Let G be a group and let Z be the upper hypercenter of G. Suppose that $zl(G) = k$ is finite and that G/Z is a locally finite group of exponent e. Then $\gamma_{k+1}(G)$ is of finite exponent at most $\beta_3(e,k)$ where β_3 is function of e,k only.*

There are a number of gaps in our knowledge at this point. But we pose next the very natural question suggested by the previous ideas.

- More generally, if Z is the upper hypercentre of G and if $G/Z \in \mathfrak{X}$ then is $G^{\mathbf{L}\mathfrak{N}} \in \mathfrak{X}$?

When the upper central length, zl(G) is infinite the situation becomes murky. First there is the following very natural extension of Baer's Theorem, due to M. De Falco, F. de Giovanni, C. Musella, Ya. P. Sysak [11], which perhaps has motivated much of the current research.

Theorem 3.7. *Let G be a group and let Z be the upper hypercentre of G. If G/Z is finite then G is finite-by-hypercentral. In particular, $G^{\mathbf{L}\mathfrak{N}}$ is finite.*

The proof of this very nice result was simplified in [22] using an interesting result of Hekster [17] which is worth observing.

Lemma 3.8. *Let G be a group and let $K \leq G$. Suppose that $G = KZ_n(G)$ for some natural number n. Then $\gamma_{n+1}(G) = \gamma_{n+1}(K)$.*

With this result, Kurdachenko, Otal and Subbotin were able to prove a quantitative version of Theorem 3.7.

Theorem 3.9. *Let G be a group and let Z be the upper hypercentre of G. Suppose that G/Z has finite order at most t. Then G contains a finite normal subgroup L of order at most t^d such that G/L is hypercentral, where $d = (1/2)(\log_p t + 1)$ and p is the least prime divisor of t. Thus $|G^{\mathbf{L}\mathfrak{N}}| \leq t^d$.*

The forthcoming paper [6] sheds more light on this situation and we quote a couple more of the results that will appear there.

Theorem 3.10. *Let G be a group and let p be a prime. Suppose that the upper hypercentre of G contains a G-invariant subgroup Z such that G/Z is locally finite and has finite section p-rank at most r. Then the locally nilpotent residual L of G is locally finite, $\Pi(L) \subseteq \Pi(G/Z)$, and there is a function τ_2 of r only such that $r_p(L) \leq \tau_2(r)$.*

Again this result has a number of consequences which are elucidated more fully in [6]. We note however that the hypothesis that G/Z be locally finite is essential here. The appropriate example is constructed in [6]. The final main result of [6] also generalizes Theorem 2.7 and runs as follows.

Theorem 3.11. *Let G be a group and suppose that the upper hypercentre of G contains a G-invariant subgroup Z such that G/Z is a locally finite group of exponent e. Then there is a function β_4 of e only such that the locally nilpotent residual L of G is a locally finite group of exponent at most $\beta_4(e)$.*

4. **Linear groups and automorphism groups**

These important theorems of Schur and Baer have also been extended in other directions. In this final section we briefly outline the results obtained in these directions and Baer [3] himself gave some such results concerning automorphism groups. More recently and among these is the following variation concerning automorphisms due to P. Hegarty [16], which we now describe.

For the group G, let $\operatorname{Aut} G$ denote the automorphism group of G. Hegarty's result can be stated as follows.

Theorem 4.1. *If $G/C_G(AutG)$ is finite then $[G, AutG]$ is finite. In this case $AutG$ is also finite.*

In the analogy with Schur's theorem we naturally are replacing $Z(G)$ by $C_G(\operatorname{Aut} G)$ and G' by $[G, \operatorname{Aut} G]$, so the question arises as to what happens when we replace $\operatorname{Aut} G$ by an arbitrary subgroup.

Consequently, we let $A \leq \operatorname{Aut} G$. We also let

$$C_G(A) = \{g \in G | \alpha(g) = g \text{ for all } \alpha \in A\} \text{ and}$$
$$[G, A] = \langle g^{-1}\alpha(g) | g \in G, \alpha \in A \rangle.$$

The first problem that arises here is that in general $C_G(A)$ need not be normal in G, although this is the case when $\mathrm{Inn}(G) \leq A$ since then $C_G(A) \leq C_G(\mathrm{Inn}(G)) = \zeta(G)$. Clearly $C_G(A)$ is always A-invariant. On the other hand, the subgroup $[G, A]$ is normal for every subgroup A of $\mathrm{Aut}(G)$, as is easily seen. It therefore makes sense to first restrict our discussion to subgroups A of $\mathrm{Aut}(G)$ such that $\mathrm{Inn}(G) \leq A$. It is then possible to obtain an analogue of Schur's theorem in the case when $A/\mathrm{Inn}(G)$ is finite. The following result appears in [9].

Theorem 4.2. *Let G be a group and let $\mathit{Inn}\, G \leq A \leq \mathit{Aut}\, G$. Suppose that $|A/\mathit{Inn}\, G| \leq k$ and $|G/C_G(A)| \leq t$. Then $|[G, A]| \leq kt^d$, where $d = \frac{1}{2}(\log_p t + 1)$, and p is the least prime dividing t.*

A very quick sketch proof of this result runs as follows: Note that $C_G(A) \leq Z(G)$ so $|G/Z(G)| \leq t$. By Theorems 1.1 and 1.2 we have that $|G'| \leq t^m$ where $m = \frac{1}{2}(\log_p t - 1)$, with p the least prime dividing t. Next we note that if $\alpha \in A$ then α induces $\bar{\alpha} : G_{\mathrm{ab}} \longrightarrow G_{\mathrm{ab}}$ in a natural way, where G_{ab} is the abelianization of G. It is then necessary to show that $|[G_{\mathrm{ab}}, \bar{\alpha}]| \leq t$ and that if $\{\alpha_1, \dots, \alpha_k\}$ is a transversal to $\mathrm{Inn}\, G$ in A then

$$|[G, A]G'/G'| = |\sum_{1 \leq j \leq k} [G_{ab}, \bar{\alpha}_j]| \leq kt.$$

It then follows that $|[G, A]| \leq tk \cdot t^m$, which completes the proof.

Building on this and starting with $C_G(A)$ and $[G, A]$, where $\mathrm{Inn}\, G \leq A$, we can define the upper and lower A-central series of G. First we let $Z_1(G, A) = C_G(A)$, a normal A-invariant subgroup of G. The Upper A-central series is

$$1 = Z_0(G, A) \leq Z_1(G, A) \leq Z_2(G, A) \leq \cdots \leq Z_\alpha(G, A) \leq \dots,$$

where

$$Z_{\nu+1}(G, A)/Z_\nu(G, A) = Z_1(G/Z_\nu(G, A), A/C_A(Z_\nu(G, A)).$$

We often abuse notation and write this latter group as $Z_1(G/Z_\nu(G, A), A)$. As usual, if ν is a limit ordinal, then we let $Z_\nu(G, A) = \bigcup_{\mu<\nu} Z_\mu(G, A)$. The last term $Z_\infty(G, A) = Z_\gamma(G, A)$ of this series is called the upper A-hypercentre of G and the ordinal γ is called the A-upper central length of G, which we denote by $\mathrm{zl}(G, A)$.

Likewise, the lower A-central series of G is the descending, normal A-invariant series

$$G = \gamma_1(G, A) \geq \gamma_2(G, A) \geq \cdots \geq \gamma_\nu(G, A) \geq \gamma_{\nu+1}(G, A) \geq \dots$$

defined by $\gamma_2(G, A) = [G, A]$ and for each ordinal ν we have $\gamma_{\nu+1}(G, A) = [\gamma_\nu(G, A), A]$, again abusing the notation slightly. As usual, for the limit ordinal λ, we define $\gamma_\lambda(G, A) = \bigcap_{\mu<\lambda} \gamma_\mu(G, A)$. The last term $\gamma_\delta(G, A) = \gamma_\infty(G, A)$ is called the lower A-hypocentre of G. Our result in this direction is as follows, and occurs in [9].

Theorem 4.3. *Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $|A : Inn(G)| = k$ is finite. Let $Z_\alpha(G, A) = Z$ be the upper A-hypercentre of G. Suppose that $\alpha = m$ is finite and that G/Z is finite of order t. Then*

(i) $|\gamma_{m+1}(G, A)| \leq \delta(k, m, t)$, *for some function* δ;
(ii) $|\gamma_\infty(G, A)| \leq \delta_1(k, t)$, *for some function* δ_1.

Finally we discuss some recent results concerned with linear groups over infinite dimensional vector spaces. To this end, let G be a group, F a field , and A a right FG–module. We need a linear analogue of the centre and derived subgroup. Thus we define the FG-centre of A to be

$$Z_{FG}(A) = \{a \in A \mid a(g-1) = 0, g \in G\} = C_A(G).$$

Clearly $Z_{FG}(A)$ is an FG–submodule of A called *the FG–centre of A*. On the other hand, if ωFG is the augmentation ideal of the group ring FG, the two-sided ideal generated by the elements $g-1$, where $g \in G$, then the submodule $A(\omega FG)$ is called *the derived submodule of A*. We let $GL(F, A)$ denote the group of all non-singular F-linear transformations of A under composition. The following question arises:

- Let G be a subgroup of $GL(F, A)$. Suppose that $Z_{FG}(A)$ has finite codimension. For which groups G does $A(\omega FG)$ have finite dimension?

The following simple example shows that the situation is complicated.

Let A have countably infinite dimension over F and let $\{a_n \mid n \geq 1\}$ be a basis of A. For $k \geq 1$ define an F-automorphism g_k of A by

$$a_n g_k = \begin{cases} a_1 + a_k, & \text{if } n = 1; \\ a_n, & \text{if } n > 1. \end{cases}$$

and let $G = \langle g_k | k \in \mathbb{N}\rangle$. Clearly $G = \mathrm{Dr}_{k\geq 1}\langle g_k\rangle$. If F has characteristic 0, then G is a free abelian group, and if F has characteristic $p > 0$, then G is an elementary abelian p-group. It follows that $Z_{FG}(A)$ is the subspace generated by $\{a_n \mid n > 1\}$, so that the codimension $\mathrm{codim}_F Z_{FG}(A) = 1$. However, $A(\omega FG)$ is also the subspace generated by $\{a_n \mid n > 1\}$, so that $A(\omega FG)$ is infinite dimensional.

To begin to answer the question posed above we shall say that a group G has finite section 0-rank r if every torsion-free abelian section has rank at most r and there is such a section of rank r. Write $r_0(G) = r$. The following result was obtained in [7]

Theorem 4.4. *Let $G \leq GL(F, A)$. Suppose that $codim_F\, Z_{FG}(A) \leq c$. If p is 0 or a prime and if $r_p(G) = r < \infty$ then $dim_F A(\omega FG) \leq \kappa(c, r)$, for some function κ, where p is the characteristic of F.*

Next we give the analogue of Baer's Theorem. Let

$$Z^0_{FG}(A) = 0, Z^1_{FG}(A) = Z_{FG}(A)$$

and for all ordinals α set

$$Z^{\alpha+1}_{FG}(A)/Z^\alpha_{FG}(A) = Z_{FG}(A/Z^\alpha_{RG}(A)),$$

with the usual convention for limit ordinals. The upper FG-central series is

$$0 = Z^0_{FG}(A) \leq Z^1_{FG}(A) \leq Z^2_{FG}(A) \leq \cdots \leq Z^\alpha_{FG}(A) \leq \cdots \leq Z^\gamma_{FG}(A).$$

The last term $Z^\gamma_{RG}(A)$ of this series is called the upper FG-hypercentre of A.

The lower FG-central series can also be defined. We let $A = \gamma^1_{FG}(A)$ and $\gamma^2_{FG}(A) = A(\omega FG)$. Let $\gamma^{\alpha+1}_{FG}(A) = \gamma^\alpha_{FG}(A)(\omega FG)$ for all ordinals α, with the usual convention for limit ordinals. The lower FG-central series is the descending series

$$A = \gamma^1_{FG}(A) \geq \gamma^2_{FG}(A) \geq \cdots \geq \gamma^\alpha_{FG}(A) \leq \gamma^{\alpha+1}_{FG}(A) \geq \cdots .$$

The following result was obtained in [7].

Theorem 4.5. *Let $G \leq GL(F, A)$. Suppose there exists k such that $codim_F\, Z^k_{FG}(A) = c < \infty$. Let p be a prime or 0. If $r_p(G) = r < \infty$ then there exists a function λ such that $dim_F\, \gamma^{k+1}_{FG}(A) \leq \lambda(c, r, k)$, where F is of characteristic p.*

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