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FINITE BCI-GROUPS ARE SOLVABLE

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Abstract. Let S be a subset of a finite group G. The bi-Cayley graph $\mathrm{BCay}(G,S)$ of G with respect to S is an undirected graph with vertex set $G\times\{1,2\}$ and edge set $\{\{(x,1),(sx,2)\}\mid x\in G,\ s\in S\}$. A bi-Cayley graph $\mathrm{BCay}(G,S)$ is called a BCI-graph if for any bi-Cayley graph $\mathrm{BCay}(G,T)$, whenever $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(G,T)$ we have $T=gS^\alpha$ for some $g\in G$ and $\alpha\in\mathrm{Aut}(G)$. A group G is called a BCI-group if every bi-Cayley graph of G is a BCI-graph. In this paper, we prove that every BCI-group is solvable.

1. Introduction

All graphs considered here are assumed to be undirected, finite and simple unless stated otherwise. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, and $\mathrm{Aut}(\Gamma)$ to denote the vertex set, the edge set and the full automorphism group of Γ, respectively. For the most part, our notation and terminology is standard and taken from [4] (for permutation group theory) and [6] (for graph theory).

Let S be a subset of a group G not containing the identity element of G. Recall that the Cayley graph $\Gamma=\mathrm{Cay}(G,S)$ of G with respect to S is the graph with vertex set G, where (x,y) is a directed edge if and only if $yx^{-1}\in S$. Also Γ is undirected if and only if $S=S^{-1}$. A Cayley graph $\mathrm{Cay}(G,S)$ of a group G is called a CI-graph if whenever T is another subset of G such that $\mathrm{Cay}(G,S)\cong\mathrm{Cay}(G,T)$, there exists an automorphism σ of G such that $S^\sigma=T$. For a positive integer m, the group G is said to have the m-CI property if all Cayley graphs of G of valency m are CI-graphs; further, if G has the k-CI property for all $k\le m$, then G is called an m-CI-group, and a $|G|$-CI-group G is called a CI-group. The Cayley isomorphism problem of Cayley graphs, especially determining CI-graphs,

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CI-groups etc., have been an active topic in algebraic graph theory for a long time, see [13] for a survey on this topic.

For a group G, and a subset S (possibly, containing the identity element) of G, the bi-Cayley graph $\mathrm{BCay}(G,S)$ of G with respect to S is the bipartite graph with vertex set $G\times\{1,2\}$ and edge set $\{\{(x,1),(sx,2)\}\mid x\in G, s\in S\}$. A bi-Cayley graph $\mathrm{BCay}(G,S)$ is called a BCI-graph if for any bi-Cayley graph $\mathrm{BCay}(G,T)$, whenever $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(G,T)$ we have $T=gS^{\alpha}$ for some $g\in G$ and $\alpha\in\mathrm{Aut}(G)$. A group G is called a BCI-group, if all bi-Cayley graphs of G are BCI-graphs. Also G is called an m-BCI-group, if all bi-Cayley graphs of G of valency at most m are BCI-graphs (See [7, Definition 1]). Xu, et. al., in [14] found a necessary and sufficient condition for a finite group being a 2-BCI-group.

Recently some authors studied the isomorphisms of bi-Cayley graphs. For example in [7], it is proved that only finite simple non-abelian 3-BCI group is A_5, the alternating group on five symbols. The Sylow subgroups of 3-BCI-groups are classified in [8] and the nilpotent 3-BCI-groups are determined in [10]. Also in [9], the isomorphisms of connected bi-Cayley graphs of cyclic groups, with valency 4 are discussed. In [9, Corollary 2.7], a Babai's type theorem for bi-Cayley graphs of finite cyclic groups is proved and the present authors improved it to arbitrary groups in [1].

Li proved that every finite CI-group is solvable, see [12, Theorem 1.2]. In this paper we prove that every finite BCI-group is also solvable.

Througout the paper we assume that G is a finite group.

2. **Main Results**

In [7, Lemma 2.9], the authors proved that if G is a finite group and S and T are two subsets of G both of which contain the identity, then $\mathrm{Cay}(G,S\setminus\{1\})\cong\mathrm{Cay}(G,T\setminus\{1\})$ implies that $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(G,T)$. By a similar argument to the proof of [7, Lemma 2.9], one can prove that if G and H are two groups, $S\subseteq G$, $T\subseteq H$, $1_G\in S$, $1_H\in T$ and $\mathrm{Cay}(G,S\setminus\{1_G\})\cong\mathrm{Cay}(H,T\setminus\{1_H\})$, then $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(H,T)$.

In the following lemma we construct some non-BCI graphs. Let us denote by C_n, K_n, $\overline{\Gamma}$, and $\Gamma_1[\Gamma_2]$, a cylcel of length n, a complete graph on n vertices, the complement of a graph Γ, and the lexicographic product of graphs Γ_1 and Γ_2, respectively. Recall that the lexicographic product graph of $\Gamma_1=(V_1,E_1)$ and $\Gamma_2=(V_2,E_2)$ has vertex set $V_1\times V_2$, and $u=(u_1,u_2)$ is adjacent with $v=(v_1,v_2)$ whenever $(u_1,v_1)\in E_1$ or $u_1=v_1$ and $(u_2,v_2)\in E_2$.

Lemma 2.1. *Assume that G has two subgroups $H=\langle a\rangle\cong\mathbb{Z}_{n^2}$ and $K=\langle b\rangle\times\langle c\rangle\cong\mathbb{Z}_n\times\mathbb{Z}_n$, where $n\geq 2$. Then G has a bi-Cayley graph of valency $n+1$ which is not a BCI-graph.*

Proof. Firstly we claim that if G is an arbitrary finite group with a subset S such that $\mathrm{BCay}(G,S)$ is a BCI-graph, then for each $T\subseteq G$ with $\mathrm{BCay}(\langle SS^{-1}\rangle,S)\cong\mathrm{BCay}(\langle TT^{-1}\rangle,T)$, we have $\langle SS^{-1}\rangle\cong\langle TT^{-1}\rangle$. To prove the claim we note that since $\mathrm{BCay}(\langle SS^{-1}\rangle,S)\cong\mathrm{BCay}(\langle TT^{-1}\rangle,T)$, by Lemma 2.8 of [7], $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(G,T)$. Thus there exist $\sigma\in\mathrm{Aut}(G)$ and $g\in G$ such that $T=gS^{\sigma}$.

Therefore we have

$$\langle SS^{-1}\rangle^\sigma = \langle (SS^{-1})^\sigma\rangle = \langle S^\sigma (S^{-1})^\sigma\rangle = \langle g^{-1}TT^{-1}g\rangle = g^{-1}\langle TT^{-1}\rangle g$$

and thus $\langle SS^{-1}\rangle \cong \langle SS^{-1}\rangle^\sigma = g^{-1}\langle TT^{-1}g\rangle \cong \langle TT^{-1}\rangle$. This proves the claim.

Now to prove the lemma, we put $S = a\langle a^n\rangle$ and $T = b\langle c\rangle$. Then $\mathrm{Cay}(H,S) \cong C_n[\overline{K_n}] \cong \mathrm{Cay}(K,T)$ and since $H = \langle S\rangle$ and $K = \langle T\rangle$, we have $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ (see [13, Example 2.8 and p. 310]). Let $S' = S \cup \{1\}$ and $T' = T \cup \{1\}$. By the discussion preceding the lemma, we have $\mathrm{BCay}(G,S') \cong \mathrm{BCay}(G,T')$ and $\mathrm{BCay}(H,S') \cong \mathrm{BCay}(K,T')$. Suppose, for a contradiction, that $\mathrm{BCay}(G,S')$ is a BCI-graph. Since $1 \in S' \cap T'$, $\langle S'S'^{-1}\rangle = \langle S'\rangle = \langle S\rangle = H$ and $\langle T'T'^{-1}\rangle = \langle T'\rangle = \langle T\rangle = K$ (see [7, p. 1259]). Hence by the above claim, $H \cong K$, which is a contradiction. □

In 1976, Babai and Frankl proved that if G is a CI-group and H a characteristic subgroup of G, then G/H is a CI-group (see [13, Lemma 8.2]). Later, in 2013, Dobson and Morris proved that the quotient group of every CI-group is also a CI-group [5]. The following lemma is the BCI-version of the result of Babai and Frankl.

Lemma 2.2. *Let G be a BCI-group and H a characteristic subgroup of G. Then G/H is a BCI-group.*

Proof. Let $\mathrm{BCay}(G/H,S) \cong \mathrm{BCay}(G/H,T)$, where $S,T \subseteq G/H$. Let $\pi : G \to G/H$ be the natural projection, and set $S_1 := \{g \in G \mid g^\pi \in S\}$, $T_1 := \{g \in G \mid g^\pi \in T\}$. We claim that $\mathrm{BCay}(G,S_1) \cong \mathrm{BCay}(G/H,S)[\overline{K_{|H|}}]$. Let R be a right transversal of H in G. Then for any $g \in G$ there exists a unique $r \in R$ such that $Hg = Hr$. Define

$$\begin{aligned} \varphi : G \times \{1,2\} &\to (G/H \times \{1,2\}) \times H \\ (g,i) &\mapsto ((g^\pi,i), gr^{-1}), \end{aligned}$$

where $Hg = Hr$ $(r \in R)$. Clearly φ is well-defined. Let $(g_1,i_1)^\varphi = (g_2,i_2)^\varphi$. Then $((g_1^\pi,i_1), g_1r_1^{-1}) = ((g_2^\pi,i_2), g_2r_2^{-1})$, where $Hg_1 = Hr_1$ and $Hg_2 = Hr_2$, for some $r_1,r_2 \in R$. So $i_1 = i_2$, $g_1^\pi = g_2^\pi$ and $g_1r_1^{-1} = g_2r_2^{-1}$, which imply that $r_1 = r_2$. Hence $g_1 = g_2$ and $i_1 = i_2$, i.e., φ is 1-1. Now let $((Hx,i),h) \in (G/H \times \{0,1\}) \times H$. Then there exists $r \in R$ such that $Hx = Hr$. So $(hr,i)^\varphi = ((Hhr,i), hrr^{-1}) = ((Hx,i),h)$. Hence φ is onto. Now we show that φ preserves the adjacency.

Let $\Gamma = \mathrm{BCay}(G,S_1)$ and $\Sigma = \mathrm{BCay}(G/H,S)$. Let $(x,1),(y,2) \in V(\Gamma)$. Then $Hx = Hr_1$ and $Hy = Hr_2$, for some $r_1,r_2 \in R$. We have

$$\begin{aligned} \{(x,1),(y,2)\} \in E(\Gamma) &\Rightarrow yx^{-1} \in S_1 \\ &\Rightarrow (yx^{-1})^\pi \in S \\ &\Rightarrow y^\pi (x^\pi)^{-1} \in S \\ &\Rightarrow \{(x^\pi,1),(y^\pi,2)\} \in E(\Sigma) \\ &\Rightarrow \{(x,1)^\varphi,(y,2)^\varphi\} \in E(\Sigma[\overline{K_{|H|}}]), \end{aligned}$$

and

$$\{(x,1)^\varphi,(y,2)^\varphi\} \in E(\Sigma[\overline{K_{|H|}}]) \Rightarrow \{((x^\pi,1),xr_1^{-1}),((y^\pi,2),yr_2^{-1})\} \in E(\Sigma[\overline{K_{|H|}}])$$

$$\begin{aligned}
&\Rightarrow \ \{(x^\pi,1),(y^\pi,2)\}\in E(\Sigma)\\
&\Rightarrow \ y^\pi(x^\pi)^{-1}\in S\\
&\Rightarrow \ (yx^{-1})^\pi\in S\\
&\Rightarrow \ yx^{-1}\in S_1\\
&\Rightarrow \ \{(x,1),(y,2)\}\in E(\Gamma).
\end{aligned}$$

This shows that φ is an isomorphism. Similarly we can show that $\mathrm{BCay}(G/H,T)[\overline{K_{|H|}}]\cong \mathrm{BCay}(G,T_1)$. Now we have

$$\mathrm{BCay}(G,S_1)\cong \mathrm{BCay}(G/H,S)[\overline{K_{|H|}}]\cong \mathrm{BCay}(G/H,T)[\overline{K_{|H|}}]\cong \mathrm{BCay}(G,T_1).$$

Since G is a BCI-group, there exist $\alpha\in \mathrm{Aut}(G)$ and $g\in G$ such that $T_1=gS_1^\alpha$. Also H is a characteristic subgroup of G and so $\overline{\alpha}: G/H\to G/H$, defined by $(Hx)^{\overline{\alpha}}:=Hx^\alpha$ is an automorphism. Now $T=T_1^\pi=g^\pi(S_1^\alpha)^\pi=g^\pi S^{\overline{\alpha}}$. This completes the proof. □

Lemma 2.3. *A_5 is not 6-BCI group.*

Proof. It is shown in [2] that $\mathrm{Cay}(A_5,S\setminus\{\mathrm{id}\})\cong \mathrm{Cay}(A_5,T\setminus\{\mathrm{id}\})$, where

$$S:=\{(12435),(15342),(14)(25),(14)(35),(25)(34),\mathrm{id}\},$$
$$T:=\{(12435),(15342),(12)(45),(13)(24),(15)(34),\mathrm{id}\},$$

and S,T are not conjugate in $\mathrm{Aut}(A_5)$. Also by [7, Lemma 2.9], $\mathrm{BCay}(G,S)\cong\mathrm{BCay}(G,T)$. Suppose, for a contradiction, that there exist $g\in A_5$ and $\sigma\in\mathrm{Aut}(A_5)$ such that $S^\sigma=g^{-1}T$. Since S and T are not conjugate in $\mathrm{Aut}(A_5)$, $g\neq \mathrm{id}$. Also $\mathrm{id}\in S$ implies that $g\in T$. We distinguish three cases:
Case I. $g=(12435)$ or $g=(15342)$ or $g=(15)(34)$. In the first case $(254)\in S^\sigma$, in the second case $(123)\in S^\sigma$ and in the third case $(245)\in S^\sigma$. So in all cases S must have an element of order 3, which is a contradiction.

Case II. $g=(12)(45)$. Then $S^\sigma=\{(14)(35),(25)(34),\mathrm{id},(14523),(12534),(12)(45)\}$. Therefore $\{(12435)^\sigma,(15342)^\sigma\}=\{(14523),(12534)\}$. This implies that $\mathrm{id}=(12435)^\sigma(15342)^\sigma=(243)$ or (135), which is a contradiction.

Case III. $g=(13)(24)$. Then $S^\sigma=\{(15)(23),(14)(35),(13254),\mathrm{id},(14235),(13)(24)\}$. This implies that $\mathrm{id}=(12435)^\sigma(15342)^\sigma=(152)$ or (345), which is a contradiction.

Hence in all cases we obtain a contradiction. This completes the proof. □

Recall that two elements a and b of a group G are said to be fused or inverse-fused if there exists $\sigma\in\mathrm{Aut}(G)$ such that $a=b^\sigma$ or $a=(b^{-1})^\sigma$, respectively. G is an FIF-group if every pair of elements of the same order is fused or inversed-fused, see [11]. Now we are ready to prove that finite BCI-groups are solvable.

Theorem 2.4. *Every finite BCI-group is solvable.*

Proof. Let G be an insolvable BCI-group. Then G is a 2-BCI-group and so by [14] (see also [7, Lemma 2.4(2)]) is an FIF-group. Now by [11, Corollary 1.3], $G = A \times B$, where A and B have coprime orders, A is solvable and B is one of the groups $PSL_2(q)$, $q = 5, 7, 8, 9$, $PSL_3(4)$, $Sz(8)$, M_{11}, M_{23}, $SL_2(q)$, $q = 5, 7, 9$ or $(C \times Sz(8)) \rtimes \mathbb{Z}_{3^s m}$, where $s, m \geq 1$ and C is an abelian group. Since $(|A|, |B|) = 1$, A and B are characteristic subgroups of G. Hence by Lemma 2.2, $B \cong G/A$ is a BCI-group. Since $|B| > 3$, B is a 3-BCI-group. If B is one of the simple non-abelian groups $PSL_2(q)$, $q = 5, 7, 8, 9$, $PSL_3(4)$, $Sz(8)$, M_{11}, M_{23}, then by [7, Theorem 1.1], $B = PSL_2(5) \cong A_5$, which is not a BCI-group, by Lemma 2.3. Now suppose that $B = SL_2(q)$, where $q = 5, 7, 9$. Since $PSL_2(q) = SL_2(q)/Z(SL_2(q))$ and $Z(SL_2(q))$ is a characteristic subgroup of $SL_2(q)$, so by Lemma 2.2, $PSL_2(q)$ must be a 3-BCI-group. Thus again by [7, Theorem 1.1], $q = 5$ which implies that $B = SL_2(5)$ is a BCI-group. So $A_5 \cong PSL_2(5)$ is a BCI-group, which is a contradiction, by Lemma 2.3.

The remaining case is $B = (C \times Sz(8)) \rtimes \mathbb{Z}_{3^s m}$. By [3], $Sz(8)$ (and so B) has two subgroups isomorphic to $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence Lemma 2.1 implies that B is not a BCI-group. This completes the proof. □

By [11, Corollary 1.3(3)], [14] (or [7, Lemma 2.4(2)]) and Theorem 2.4, every finite BCI-group G is a semidirect product of a group A by a group B such that $(|A|, |B|) = 1$, A is a nilpotent FIF-group and every Sylow subgroup of B is cyclic or Q_8. Note that by [7, Lemma 2.4(2)], A is a 2-BCI-group and the structure of all Sylow subgroups of A are given in [11, Corollary 1.3 (2)].

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