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APPROXIMATION ALGORITHMS FOR THE FREEZE TAG PROBLEM INSIDE POLYGONS

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Abstract. The *freeze tag problem* (FTP) aims to awaken a swarm of robots with one or more initially awake robots as soon as possible. Each awake robot must touch a sleeping robot to wake it up. Once a robot is awakened, it can assist in awakening other sleeping robots. We study this problem inside a polygonal domain and present approximation algorithms for it.

1. Introduction

Swarm robotics (SR) aims at coordinating multiple robots towards cooperatively performing a particular job. SR has various applications for example in target search and tracking [14, 16], medicine [5], agricultural mechanization [6], road freight [9], and autonomous underwater vehicles [10]. For more discussion, see [13]. Consider the following problem from SR. Given a set of robots denoted by $S = \{s_0, \dots, s_n\}$ with a single initially awake robot s_0 and the remaining asleep robots, in the *freeze tag problem* (FTP), we study how to awaken all robots such that the *makespan* (the total distance traversed from s_0 for awakening the last asleep robot) is minimized. Note that once an asleep robot has been awakened, it can assist in awakening other asleep robots. Each awake robot should move next to an asleep robot to awaken it.

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Observe that S can be modeled as points in some metric space such as vertices of a graph with weighted edges. It is proved that the FTP is NP-hard even on star graphs with an equal number of robots at each leaf, however, a polynomial-time approximation scheme (PTAS) and a greedy approximation algorithm (GA) are given for this case [7]. It is also proved that the approximation factor of the GA, where each robot arriving at the root awakens the unawakened robot at the shortest edge, has a lower bound of 7/3, moreover, a 14-approximation algorithm was proposed for the FTP on star graphs with different numbers of robots at each leaf [7].

Some heuristics are studied in [15]. If there exist more than one initially awake robot, we denote the problem by k-FTP, where k is the number of initially awake robots. A PTAS is presented for the 2-FTP in Euclidean space [12]. It is proved that the FTP in the 2 and 3 dimensional Euclidean spaces is NP-hard in [3] and [11], respectively. An $O(1)$-approximation algorithm and a PTAS are given for the FTP in Euclidean space in [7].

The FTP inside simple polygons was studied in [1]. Depending on the metric used for the distance between robots inside a polygonal domain P, we consider two versions for the FTP: the *geodesic FTP* (GFTP) and *visibility FTP* (VFTP). In the GFTP inside P, the distance between every pair of robots s_i and s_j for $0 \leq i, j \leq n$ is the *geodesic distance* between s_i and s_j, i.e. the length of the shortest path between two robots s_i and s_j inside P denoted by $\pi(s_i, s_j)$ (Figure 1). For the VFTP, the distance between any two robots s_i and s_j is not symmetric; the distance from s_i to s_j, called the *geodesic visibility distance* (GVD) from s_i to s_j, is the shortest path that s_i should travel inside P such that s_j can be *visible* from s_i, i.e., their connecting line segment completely lies inside P (Figure 2). Lubiw and Zeng [1] proved that the GFTP and VFTP are NP-hard. In this paper, we give approximation algorithms for the GFTP and VFTP.

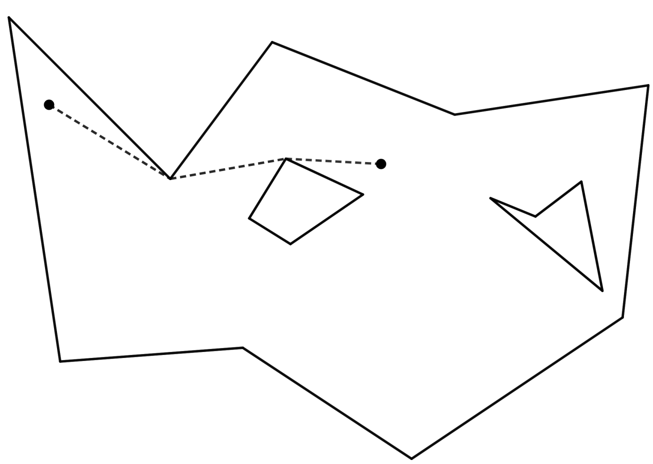

FIGURE 1. A polygonal domain with two holes; the dashed lines represent the geodesic path between the filled circles.

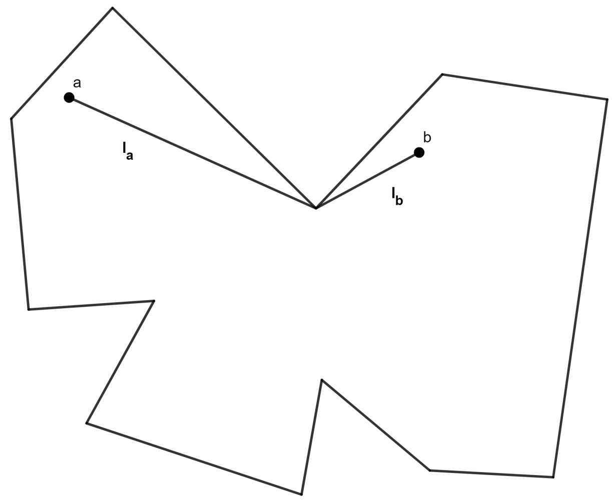

FIGURE 2. An example for illustration of the GVD; l_a (resp. l_b) denotes the GVD from a to b (resp. from b to a).

2. Our Approximation Algorithms

In this section, we first propose a constant factor approximation algorithm which works for two versions of the FTP, the GFTP and VFTP, and then provide a PTAS for them.

2.1. Constant Factor Approximation Algorithm.

In this subsection, we present an $O(1)$-approximation algorithm for the GFTP and VFTP using additional compensatory robots. Let S be a set of n robots (represented as points) inside a polygonal domain P. For any two robots $a, b \in S$, let $\pi(a, b)$ be the geodesic path between a and b inside P (Figure 1), and let $\|\pi(a, b)\|$ denote the corresponding geodesic distance. The *geodesic graph* of S inside P denoted by $GG(S, P)$ is a complete graph with vertex set S such that $e(a, b) = \pi(a, b)$ for $a, b \in S$. Note that $e(a, b)$ denotes the edge between a and b.

The *diameter* of S inside P, denoted by $diam(S, P)$, is the longest geodesic distance between the robots of S inside P, i.e., $diam(S, P) = \max(\|\pi(a, b)\|)_{a,b \in S}$. Let $R = \{r_1, r_2, \ldots, r_m\}$ denote the set of the reflex vertices of P (if exist). And, let $V = \{v_1, v_2, \ldots, v_h\}$ denote the set of the vertices of the polygonal holes inside P (if exist). For each vertex $v \in R \cup V$, if no robot $p \in S$ exists at v, we locate a new robot called the *Steiner* robot. Let $ST = \{st_1, st_2, \ldots, st_{m'}\}$ be the set of the Steiner robots (the robots other than those in S) located at the vertices of $R \cup V$ for $m' \leq h + m$.

A *t-spanner* of a graph $G = (V', E)$ is a subgraph $H = (V', E')$ of G such that the distance between any two vertices $a, b \in V'$ in H is at most t times the distance between a and b in G. Let S' be a set of points in the d dimensional Euclidean space, a *geometric t-spanner* of S' is a graph H' with the vertex set S', such that the length of the shortest path between any two points of S' in H' is at most t times the Euclidean distance between them.

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Theorem 2.1. *Let S be a set of robots inside a polygonal domain P with m reflex vertices and h polygonal hole vertices. There is an $O(1)$-approximation algorithm with the makespan $O(diam(S,P))$ for the GFTP on S using m' Steiner robots, where $m' \leq h+m$.*

Proof. Let $S' = \{s'_0, s'_1, s'_2, \ldots, s'_n\}$ denote a set of robots in the plane. We refer to robots and their corresponding points in Euclidean space interchangeably. We use the idea of the $O(1)$-approximation algorithm proposed in [7] for the FTP on S', called the *constant factor approximation algorithm* (CFA). In the following, we briefly describe the CFA. Note that the distance between any two robots $a, b \in S'$, denoted by $d(a,b)$, is the Euclidean distance. Firstly, they constructed a degree-bounded t-spanner for S' in the plane as follows. They partitioned the plane around each $p \in S'$ into k cones (regions in the plane between two consecutive rays) using rays originating from p at angles $0, 2\pi/k, 2(2\pi/k), 3(2\pi/k), \ldots$.

Assume $u_j(p) \in S'$ denotes the closest robot to the robot p in its jth cone. Observe that the robots $u_j(p)$ can be computed in $O(kn \log n)$ total time for all $p \in S'$ and j using standard Voronoi diagrams. Consider the graph $G_K = (S', E_K)$, where there is an edge between each robot $p \in S'$ and the nearest neighbors of p in its cones, i.e., the robots $u_j(p)$. G_K is called a θ-graph, where $\theta = 2\pi/k$, and is a t-spanner for $k \geq 9$ (and for $k \geq 5$ due to [8]).

Assume that the robots $u_1(p), u_2(p), u_3(p), \ldots, u_{k'}(p)$ are in ascending order of their distance from p. Thus, $u_{k'}(p)$ is the farthest neighbor of p. Once the robot $s'_i \in S'$ is awakened, it starts awakening the robots $u_1(s'_i), u_2(s'_i), \ldots, u_{k'}(s'_i)$, respectively, where $u_j(s'_i)$ denotes the jth nearest adjacent vertex of s'_i in G_k for $k' \leq k$. Note that according to the CFA, once s'_i is awakened, it traverses the edge $(s'_i, u_1(s'_i))$ to wake up the robot $u_1(s'_i)$ and travels back to its initial location, then s'_i traverses the edge $(s'_i, u_2(s'_i))$ to wake up the robot $u_2(s'_i)$ and returns to its initial location, and so on.

Observe that if we construct a bounded degree t-spanner for a metric, we can give a constant factor approximation algorithm for the FTP in it using the CFA. A geodesic t-spanner of S inside P is constructed in [2], but the degrees of the vertices are not bounded by a constant integer k. Thus, we first give an algorithm for constructing a degree bounded geodesic t-spanner of S inside P. We construct a 6-spanner of the *visibility graph* of $S \cup R \cup V$ inside P with the degree at most 7 by the algorithm proposed in [17]. The visibility graph of a point set inside P is a graph whose vertices belong to the point set, and there is an edge (connecting line segment) between each two points which are visible from each other [4].

Lemma 2.2. *A visibility t-spanner of $S \cup R \cup V$ inside P is a t-spanner for the geodesic graph of S inside P, i.e., $GG(S,P)$.*

Proof. Let VT be the visibility t-spanner of $S \cup R \cup V$ inside P constructed by the algorithm of [17]. Assume that $\pi'(a,b)$ denotes the shortest path between $a, b \in S$ in VT.

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Observation 1. *Assume that* $\pi(a,b) = a, r'_1, r'_2, \ldots, r'_l, b$ *for* $a, b \in S$*, then we have* $r'_j \in R \cup V$ *for* $1 \le j \le l$*.*

We observe that for every two consecutive vertices p and q in $\pi(a,b)$, p is visible from q (and vice versa). Thus, there exists a path $\pi'(p,q)$ between p and q in VT such that

$$\|\pi'(p,q)\| \le t\|\pi(p,q)\|.$$

Therefore, there exists a path $\pi'(a,b)$ between any two robots $a \in S$ and $b \in S$ in VT such that

$$\begin{aligned}\|\pi'(a,b)\| \le & t\|\pi(a,r'_1)\| + t\|\pi(r'_1,r'_2)\| \\ & + \cdots + t\|\pi(r'_{l-1},r'_l)\| \\ & + t\|\pi(r'_l,b)\|.\end{aligned}$$

And, thus

$$\begin{aligned}\|\pi'(a,b)\| \le & t(\|\pi(a,r'_1)\| + \|\pi(r'_1,r'_2)\| \\ & + \cdots + \|\pi(r'_{l-1},r'_l)\| \\ & + \|\pi(r'_l,b)\|) = t\|\pi(a,b)\|.\end{aligned}$$

□

Observe that in the CFA, there exists one robot at every vertex of G_k, which starts awakening its adjacent vertices after being awakened by another robot. Thus, we use the set of the Steiner robots $ST = \{st_1, st_2, \ldots, st_{m'}\}$ with $m' = |R \cup V|$ as follows. We apply the CFA to the set of robots $S \cup ST$ with $VT = (S \cup ST, E'')$ as a t-spanner. By Lemma 2.2, there exists a path $\pi'(s_0, s_i)$ in VT such that

$$\|\pi'(s_0,s_i)\| \le t\|\pi(s_0,s_i)\|,$$

for all $s_i \in S$. Consider two consecutive robots $s''_y, s''_z \in S \cup ST$ in the path $\pi'(s_0, s_i)$. By the CFA, when the robot s''_y is awakened, it traverses the distance $awakedist(s''_y, s''_z)$ in VT to awaken the robot s''_z. Let $s''_z = u_j(s''_y)$, i.e., s''_z is the jth nearest adjacent vertex of s'_y in VT. Note that $\|\pi'(s''_y, u_{j'}(s''_y))\| \le \|\pi'(s''_y, u_j(s''_y))\|$ for all $j' \le j$. Thus,

$$awakedist(s''_y, u_j(s''_y)) \le (2j-1)\|\pi'(s''_y, u_j(s''_y))\|.$$

In other words, we have

$$awakedist(s''_y, s''_z) \le (2j-1)\|\pi'(s''_y, s''_z)\|.$$

Note that we have $j \le k$, therefore

$$awakedist(s''_y, s''_z) \le (2k-1)\|\pi'(s''_y, s''_z)\|.$$

Recall that k is the maximum degree of the vertices in VT. Thus, for the distance traversed from s_0 for awakening each robot s_i, i.e. $awakedist(s_0, s_i)$, we have:

$$\begin{aligned} awakedist(s_0, s_i) &\leq (2k-1)\|\pi'(s_0, s_i)\| \\ &\leq t(2k-1)\|\pi(s_0, s_i)\|. \end{aligned}$$

Note that we have $k = 7$ and $t = 6$, thus

$$awakedist(s_0, s_i) \leq 78\|\pi(s_0, s_i)\|,$$

for all $s_i \in S$. Therefore, we conclude Theorem 2.1. □

As above, we give an $O(1)$-approximation algorithm for the VFTP inside P.

Theorem 2.3. *For the VFTP on $S \cup ST$ inside P, there exists an $O(1)$-approximation algorithm with the makespan $O(diam(S, P))$.*

Proof. The *geodesic visibility path* (GVP) from $a \in S \cup ST$ to $b \in S \cup ST$ (resp. from b to a) is the shortest path that a (resp. b) travels inside P until b (resp. a) can be visible from a (resp. b).

Let $GVP(a, b)$ denote the GVP from a to b in P for $a, b \in S \cup ST$. The *geodesic visibility graph* of S inside P, denoted by $GVG(S, P)$, is a directed complete graph with the vertex set S such that $e(a, b) = GVP(a, b)$ and $e(b, a) = GVP(b, a)$ for $a, b \in S$.

Lemma 2.4. *From the visibility t-spanner of $S \cup R \cup V$ inside P, we get a t-spanner for the geodesic visibility graph of S inside P, i.e. $GVG(S, P)$.*

Proof. Assume $\pi(a, b) = a, r'_1, r'_2, \ldots, r'_l, b$. Observe that we have:

$$GVP(a, b) = \pi(a, r'_l),$$

and

$$GVP(b, a) = \pi(b, r'_1).$$

We observe that VT is a t-spanner for $GVG(S, P)$. Since, there exist the paths $\pi'(a, r'_l)$ and $\pi'(b, r'_1)$ in VT such that

$$\|\pi'(a, r'_l)\| \leq t\|\pi(a, r'_l)\| = t\|GVP(a, b)\|,$$

and

$$\|\pi'(b, r'_1)\| \leq t\|\pi(b, r'_1)\| = t\|GVP(b, a)\|.$$

□

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We use the CFA with VT as a t-spanner for $GVG(S,P)$ to awaken the robots in $S \cup ST$. Let $\pi(s''_i, u_j(s''_i)) = s''_i, r''_1, r''_2, \ldots, r''_{l'}, u_j(s''_i)$. Observe that after the robot $s''_i \in S \cup ST$ travels the path $\pi'(s''_i, r''_{l'})$ in VT to awaken the robot $u_j(s''_i)$, it returns to its initial location through the same path, i.e. the path $\pi'(s''_i, r''_{l'})$, although $GVP(s''_i, u_j(s''_i)) \neq GVP(u_j(s''_i), s''_i)$ for $s''_i \in S \cup ST$. Thus, as Theorem 2.1, we can prove Theorem 2.3. □

2.2. **A PTAS for the GFTP.** Here, we give an $O(1+\epsilon)$-approximation algorithm, PTAS, for the GFTP on a set of robots (modeled as points) S inside a polygonal domain P, using the idea of the PTAS presented in [7]. Let S' be a set of robots (points) in Euclidean space of any fixed dimension such that at most one robot is located at any position. An *awakening tree* for S' is a spanning tree rooted at the initially awake robot, where the root has exactly one child and all other vertices have at most two children.

In [7], a PTAS was presented for the FTP on S' as follows. They first partitioned Euclidean space into m^2 subspaces called *pixels*, where $m = 1/\epsilon$. Then, they selected an arbitrary robot of each pixel as its representative, and found a *pseudo balanced awakening tree* (SBAT) for the representative robots by examining all possible awakening trees on them in $O(2^{m^2 \log m})$ time. They converted the SBAT to an awakening tree for all robots $p \in S'$, using the awakening tree found by the CFA for each pixel (they replaced the representative of each pixel with the awakening tree of the robots inside that pixel). Note that a tree with n vertices is *pseudo balanced* if the length of each path from the root to a leaf of the tree is $O(\log^2 n)$. The steps of their PTAS, PTAS1, are as follows:

- Divide the unit square containing the robots into m^2 pixels, and select a representative robot for each pixel.
- Find an SBAT for representative robots.
- Convert the above SBAT to an awakening tree for all robots $p \in S'$ by running the CFA on the robots in each pixel.

Now, we propose a PTAS for the GFTP on S inside P using the idea of the above PTAS. We assume that there exists at most one robot in each position inside P. We first decompose the polygonal domain P into convex partitions as follows. Let $SQ(P)$ denote the unit square containing P. We divide $SQ(P)$ into $m * m$ squares with side $1/m$ for $m = O(1/\epsilon)$ (the coordinates of the robots have been rescaled such that they lie in $SQ(P)$). We consider the intersection of P and the squares of $SQ(P)$ as the decomposition (pixels) of P (Figure 3). Note that the number of pixels of P may be less than $m * m$. We observe that the diameter of each pixel of P is $O(1/m)$, therefore all theorems and lemmas corresponding to PTAS1 (which do not depend on the type of the distance between robots, although the geodesic distance is a metric, the same as the Euclidean distance) also hold for our PTAS. Thus, a PTAS for the GFTP can be given as follows:

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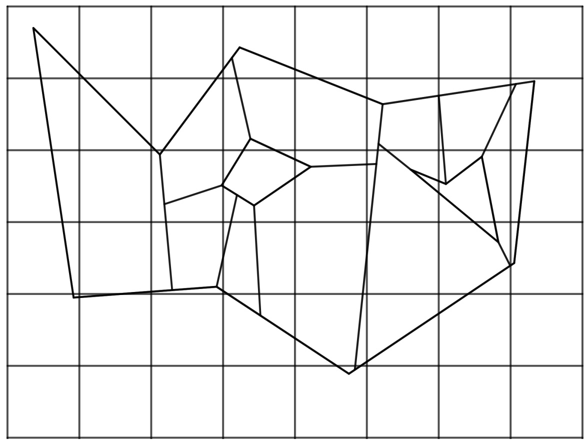

FIGURE 3. An example for decomposing the polygonal domain P into convex partitions (pixels).

- We decompose P into m^2 or fewer pixels (convex sub polygons), and select an arbitrary robot in each pixel (if exists) as its representative.
- We find an SBAT for the representative robots.
- Convert the SBAT to an awakening tree of all robots. Observe that each pixel is convex, thus we can apply the CFA to the robots in each pixel to find an awakening tree for the robots in that pixel.

Theorem 2.5. *There is a PTAS for the GFTP in a polygonal domain P.*

Observe that in the above PTAS, for any two adjacent robots $s''_y, s''_z \in S \cup ST$ in VT, either s''_y awakens s''_z, or s''_z awakens s''_y. Thus, we can assume w.l.o.g. that the geodesic visibility graph of S inside P, $GVG(S, P)$, is an undirected graph. Therefore, as above, we can prove the following theorem.

Theorem 2.6. *There is a PTAS for the VFTP in a polygonal domain P.*

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