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GROUPS IN WHICH PRIME ORDER ELEMENTS COMMUTE

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Abstract. It is known that if G is a finite group in which all the elements of prime power order commute, then G is abelian. However, the same does not hold if *prime power* is replaced by *prime*. In this article, we introduce the study of a class of finite groups G in which the prime order elements commute. In particular, we discuss the relationship between these class of groups with other known classes of finite groups, like simple groups, perfect groups etc. Moreover, we also prove some results on the possible orders of such groups. Finally, we conclude with some open issues and observations supported by computational evidences using GAP.

1. Introduction

It is known that if G is a finite group such that all elements of prime power order commute, then G is abelian. So, what about the finite groups in which all prime order elements commute? Are they necessarily commutative? The answer is negative and Q_{2^n} serves as a family of counterexamples, as it has a unique element of order 2. This motivates the definition of a new class of groups.

Definition 1.1. *A group G is said to have POEC property if all the elements of prime order in G commute.*

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As abelian groups are always $POEC$, we consider only non-abelian $POEC$ groups. The inter-relationship between the classes of simple, perfect and $POEC$ groups, as shown in Figure 1, is the main topic of discussion of the current article. Here T is a perfect POEC group defined in Remark 4.2. Although the definitions of these classes allow the group to be infinite, in what follows, we assume G to be a finite group.

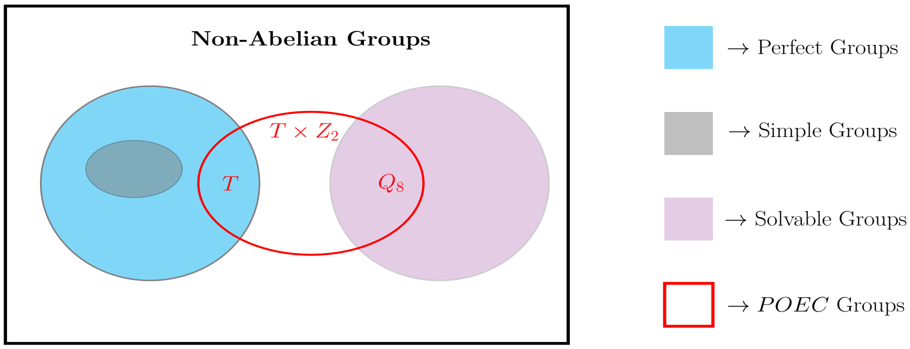

FIGURE 1. Inter-relationship between various types of finite non-abelian groups

1.1. **Preliminaries and Basic Results.** Before going to elaborate results, we define some terminologies and their relations with each other. In this paper, by S_p, we mean Sylow p-subgroup of the group in context, and not the symmetric group on p symbols.

Definition 1.2. *Let G be a finite group, $\pi(G)$ be the set of primes dividing $|G|$ and $p \in \pi(G)$. Define $P[G]$ to be the subgroup generated by all elements of prime order in G, i.e.,*

$$P[G] = \langle \{x \in G : o(x) \text{ is prime}\} \rangle.$$

Also define G_p be the subgroup generated by all elements of order p in G, i.e.,

$$G_p = \langle \{x \in G : o(x) = p\} \rangle.$$

Clearly, $G_p \leq P[G]$ and both are characteristic subgroups of G and hence normal in G. Note that $G = G_p$ for any non-abelian simple group G where $p \mid |G|$. Moreover, $P[G]$ intersects all non-trivial subgroups of G non-trivially, i.e., if X is a non-trivial subgroup of G, then $|X \cap P[G]| > 1$. It is to be noted that for an arbitrary group G, $P[G]$ and G_p may be equal to the entire group G, e.g., simple groups.

It can be shown that if G is a non-abelian $POEC$ group, then $P[G]$ is a proper abelian subgroup of G and hence G is not simple.

1.2. **Organisation of the paper.** In Section 2, we discuss some basic properties and consequences of a $POEC$ group. We study the center of $POEC$ groups and perfect $POEC$ groups in Sections 3 and 4. Our main results include results on order of a perfect $POEC$ (Theorems 4.9 and 4.14). Finally, we conclude with some open issues in Section 5.

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2. POEC Groups

Let G be a non-abelian $POEC$ group. Then all the elements of $P[G]$ are of square free order. Thus, if $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then

$$P[G] \cong G_{p_1} \times G_{p_2} \times \cdots \times G_{p_k} \cong \mathbb{Z}_{p_1}^{\beta_1} \times \mathbb{Z}_{p_2}^{\beta_2} \times \cdots \times \mathbb{Z}_{p_k}^{\beta_k},$$

where $\mathbb{Z}_{p_i}^{\beta_i}$ is the direct product of β_i copies of $\mathbb{Z}_{p_i}$, $1 \leq \beta_i \leq \alpha_i$ for $i = 1, 2 \ldots, k$ and not all $\alpha_i = \beta_i$. Note that for $POEC$ groups,

$$G_p = \langle \{x \in G : o(x) = p\} \rangle = \{x \in G : o(x) = p\} \cup \{e\}.$$

Theorem 2.1. *Let G be a finite group. G is a $POEC$ group if and only if all elements of square-free order forms an abelian subgroup of G.*

Proof. If G is a $POEC$ group, then the theorem holds using the above discussion. Conversely, let H be set of all elements of square-free order which forms an abelian subgroup of G. Clearly, all elements of prime order are in H and H is abelian. Thus G is a $POEC$ group. □

Definition 2.2. *Let G be a group. The subgroup generated by the elements of square-free order in G is defined as $SQF(G)$.*

It is to be noted that in case of a finite $POEC$ group, $SQF(G) = P[G]$.

Proposition 2.3. *Let G be a finite $POEC$ group. Then the following are true:*

(1) *(POEC is subgroup-closed and direct-product closed) If $H \leq G$, then H is a $POEC$ group. If G_1 and G_2 are two $POEC$ groups, then $G_1 \times G_2$ is also a $POEC$ group.*
(2) *If G has a normal Sylow p-subgroup P, then G/P is a $POEC$ group. (the statement is also true if we replace Sylow subgroup by a Hall subgroup)*
(3) *If G is non-abelian, then $|G|$ is not square-free.*
(4) *A positive integer is called almost square-free if it is divisible by p^2 for at most one prime p. If $|G|$ is almost square-free, then G is supersolvable.*
(5) *If $|G|$ is divisible by the square of at most two distinct primes, then G is solvable.*
(6) *If 8 does not divide $|G|$, then G is solvable.*

Proof. (1) The proofs follow from the definition of $POEC$ groups.

(2) Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $|P| = p_1^{\alpha_1}$. As P is normal in G, by Schur – Zassenhaus theorem, P has a complement Q in G with $|Q| = p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. As $G/P \cong Q \leq G$ and Q is $POEC$ by (1), the result follows.

(3) Suppose to contrary that $|G|$ is square-free. Then order of all of its elements are also square-free. Hence $SQF(G) = G$. But this implies that G is abelian, a contradiction. Hence the result holds.

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(4) Suppose to contrary that G is not supersolvable. Thus G is not abelian. Thus by previous result, $|G|$ is not square-free, i.e., there exists a prime p such that p^2 divides $|G|$. If p is the only such prime, i.e., $|G|$ is almost square-free, then $|G| = p^a p_1 p_2 \cdots p_k$ with $a \geq 2$ and $|SQF(G)| = p^b p_1 p_2 \cdots p_k$ with $b \leq a$. Consider K the subgroup of $SQF(G)$ of order $p_1 p_2 \cdots p_k$. Clearly K is cyclic and $K \lhd G$. Now, as G/K is a p-group, it is supersolvable. Now, by [1, Theorem 4.15], we deduce that G is supersolvable, a contradiction. Thus the result holds.

(5) Let $|G| = p^{a_1} q^{b_1} p_1 p_2 \cdots p_k$ with $a_1, b_1 \geq 2$. Then $|SQF(G)| = p^{a_2} q^{b_2} p_1 p_2 \cdots p_k$ with $a_2 \leq a_1, b_2 \leq b_1$ and hence $G/SQF(G)$ is solvable. Moreover, as $SQF(G)$ is solvable, we have the result.

(6) If $|G|$ is m or $2m$, where m is odd, then G is solvable. So we assume that $|G| = 4m$, where m is odd. Let H_2 be the subgroup generated by elements of order 2 in G. Then H_2 is normal in G and $|G/H_2| = m$ or $2m$. In any case, G/H_2 is solvable. Moreover, as H_2 is abelian, and hence solvable. Thus G is solvable.

□

Remark 2.4. *POEC is not quotient-closed, i.e., if $H \lhd G$, then G/H may not be a POEC group. Consider the group $G = (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$ with GAP ID (32,2) which is defined by the following presentation: $\langle a, b, x \mid a^4 = b^2 = x^4, ab = ba, bx = xb, xax^{-1} = ab \rangle$. It is a POEC group. Now G has a normal subgroup N isomorphic to Klein's 4-group, such that the quotient G/N is isomorphic to D_4, the dihedral group of order 8, which itself is not a POEC group.*

Proposition 2.5. *Suppose that G is a nilpotent group. Then G is POEC if and only if all of its Sylow subgroups are POEC.*

Proof. Let G be a nilpotent $POEC$ group. As $POEC$ is a subgroup-closed property, any subgroup of G and in particular Sylow subgroups are $POEC$. Conversely, let G be a nilpotent group such that its Sylow subgroups P_i's are $POEC$. As $POEC$ is direct product closed, G is a $POEC$ group. □

The above proposition suggests that we should try to explore $POEC$ p-groups. Our focus is on p-groups of order p^n, where $n \geq 3$, as p-groups are commutative for $n \leq 2$. It is an interesting fact that, for an odd prime p, there is a non-abelian $POEC$ group of order p^3, namely $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$. Using this, we can always construct a non-abelian $POEC$ group of order p^n for all $n \geq 3$, as $(\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_{p^{n-3}}$ serves our purpose. Moreover, this result also holds for 2-groups due to Q_{2^n} for $n \geq 3$. Note that for a non-abelian $POEC$ p-group (p is odd) of order p^n, by [3, Theorem 5.4.10.ii, p.199], we have $p < |G_p| < p^n$. In fact for all r with $2 \leq r \leq n-1$, we can construct a non-abelian $POEC$ p-group of order p^n such that $|G_p| = p^r$. For $p = 2$, along with the above values of r, we can also get $|G_p| = 2$. On the other hand, we would like to mention that there always exists a non-$POEC$ group of order p^n for all $n \geq 3$ due to existence of Heisenberg group, $Heis(\mathbb{Z}_p)$ and Dihedral groups.

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3. Center of $POEC$ groups

It is observed via numerical examples that $POEC$ groups have non-trivial center. In this section, we prove some partial results in this direction.

Proposition 3.1. *If G is a $POEC$ group such that $[G : P[G]]$ is a prime power, then $Z(G)$ is non-trivial.*

Proof. Let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i's are distinct primes and without loss of generality, let $[G : P[G]] = p_1^{\beta_1}$ where $\beta_1 \leq \alpha_1$. Thus $G_{p_i} = S_{p_i}$ for $i = 2, \ldots, k$ where S_{p_i} denotes Sylow p_i-subgroup of G. Observe that $H = S_{p_2} S_{p_3} \cdots S_{p_k}$ is a normal Hall subgroup of G and $G = HS_{p_1}$. Choose $a \in Z(S_{p_1})$ such that $o(a) = p_1$. Note that any element $g \in G$ is of the form $g = g_{p_2} g_{p_3} \cdots g_{p_k} \cdot b$, where $g_{p_i} \in G_{p_i}$ for all $i \geq 2$ and $b \in S_{p_1}$. Then

$$\begin{aligned} a \cdot g &= a \cdot (g_{p_2} g_{p_3} \cdots g_{p_k}) \cdot b \\ &= ((g_{p_2} g_{p_3} \cdots g_{p_k}) \cdot a) \cdot b, \quad \text{as } G \text{ is POEC} \\ &= (g_{p_2} g_{p_3} \cdots g_{p_k}) \cdot (b \cdot a), \quad \text{as } a \in Z(S_{p_1}) \\ &= g \cdot a \end{aligned}$$

Thus $a \in Z(G)$. □

Corollary 3.2. *If G is a $POEC$ group such that $|G|$ is almost square-free, then $Z(G)$ is non-trivial.*

Remark 3.3. *The above corollary is not true in general, as A_4 has trivial center.*

Proposition 3.4. *If G is a $POEC$ group such that $|G| = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $p_i \nmid (p_1 - 1)$ for all i, then $Z(G)$ is non-trivial.*

Proof. Using N/C-theorem on G_{p_1}, we get

$$G/C_G(G_{p_1}) \leq Aut(\mathbb{Z}_{p_1}).$$

Note that $|Aut(\mathbb{Z}_{p_1})| = p_1 - 1$ and as $p_i \nmid (p_1 - 1)$ for all i, we have $G = C_G(G_{p_1})$, i.e., $G_{p_1} \subseteq Z(G)$. □

Corollary 3.5. *If G is a $POEC$ group such that $|G| = p_1 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_1 is the smallest prime factor of $|G|$, then $G_{p_1} \subseteq Z(G)$.*

Remark 3.6. *If G is a $POEC$ group such that $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where $\alpha_i \geq 1$, then $Z(G)$ is non-trivial if and only if $G_{p_j} \cap Z(G)$ is non-trivial for some j. The above p_j is not necessarily the smallest prime divisor of $|G|$, e.g., $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$ is a $POEC$ group with a center of order 3.*

4. Perfect $POEC$ groups

As mentioned in Section 1.1, $POEC$ groups are not simple and so we now focus on perfect $POEC$ groups.

http://dx.doi.org/10.22108/ijgt.2025.143874.1939

Theorem 4.1. *The smallest non-solvable POEC group must be perfect.*

Proof. Let G be the smallest non-solvable $POEC$ group. If $G' \subsetneq G$, then G' is a proper subgroup of G and hence G' is a $POEC$ group and hence solvable. Also as G/G' is abelian, it is solvable. Thus we must have G to be solvable. Thus $G' = G$, i.e., G is perfect. □

Remark 4.2. *There is a perfect POEC group T (GAP:* PerfectGroup$(1215000, 9)$*) of order* $1215000 = 2^3 \cdot 3^5 \cdot 5^4$. *$T$ has the structure $(\mathbb{Z}_{30} \times \mathbb{Z}_{15} \times \mathbb{Z}_{15} \times \mathbb{Z}_3) \cdot A_5$ and its center is $\mathbb{Z}_2$. As $P[T] \cong (\mathbb{Z}_{30} \times \mathbb{Z}_{15} \times \mathbb{Z}_{15} \times \mathbb{Z}_3)$, it is a POEC group. Using GAP [2], one can check that it is the smallest non-solvable POEC group. In fact, this the unique perfect POEC group of order* 1215000. *We will denote this group by T throughout the paper.*[1] *Since direct product of perfect groups are perfect, there exist infinitely many perfect POEC groups.*

Remark 4.3. *There exist POEC groups which are neither solvable nor perfect. Let P be a perfect POEC group and A be any abelian group. Set $G = P \times A$. Then G is a POEC group. As $P \leq G$, G is non-solvable and as $G' \cong P' \times A' \cong P < G$, G is not perfect.*

Remark 4.4. *The smallest order of a POEC group which is neither solvable nor perfect is* $2 \times 1215000 = 2430000$*: Let G be any such group. As G is non-solvable and non-perfect, it must have a proper perfect POEC subgroup. As T is the smallest perfect POEC group, G must have a proper subgroup at least as large as $|T|$. Thus $|G| \geq 2|T|$. Now $T \times \mathbb{Z}_2$ is a valid candidate of order* $2 \times 1215000 = 2430000$.

We now recall a result on finite perfect groups which will be used in the next theorem.

Proposition 4.5. [4, Theorem 10.1.4] *If G is a finite perfect group and $[G : Z(G)] = t$, then $g^t = e$ for all $g \in G$.*

Theorem 4.6. *If G is a POEC group with a cyclic Sylow p-subgroup, then G is not perfect.*

Proof. Assume that G is a perfect group. Let $S_p = \langle a \rangle$ be the cyclic Sylow p-subgroup of G of order p^k and G_p be the subgroup generated by elements of order p in G. Then $\mathbb{Z}_p \cong G_p \leq S_p$ and G_p is normal in G. Then by N/C - theorem, $G/C_G(G_p)$ is isomorphic to a subgroup of $\mathbb{Z}_p^*$. As $\mathbb{Z}_p^*$ is cyclic, $G/C_G(G_p)$ is abelian. Again as G is perfect, $G/C_G(G_p)$ is also perfect. Thus $G/C_G(G_p)$ is trivial, i.e., $G = C_G(G_p)$, i.e., $G_p \leq Z(G)$.

Let $[G : Z(G)] = t$. Then p^k does not divide t and hence $a^t \neq e$. But by Proposition 4.5, $g^t = e$ for all $g \in G$, a contradiction. Thus G is not perfect. □

Corollary 4.7. *If G is a perfect POEC group with $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $\alpha_i > 1$ for all i.*

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[1] The authors are grateful to Professor Alexander Hulpke for pointing out this example.

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Proof. If $\alpha_i = 1$ for some i, then the corresponding Sylow p_i-subgroup is cyclic and hence by the above theorem, G is not perfect, a contradiction. □

Proposition 4.8. *If G is a perfect POEC group such that 2^4 does not divide $|G|$, then the Sylow 2-subgroup of G is isomorphic to the quaternion group, Q_8 and $2 \mid |Z(G)|$.*

Proof. By Proposition 2.3(6) and as 16 does not divide $|G|$, we have $|G| = 8p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Again as G/G_2 is perfect, i.e., 4 divides $|G/G_2|$, we have $|G_2| = 2$. As G_2 is the subgroup generated by elements of order 2 in G, we conclude that the Sylow 2-subgroup S_2 of G has exactly one element of order 2, thereby enforcing $G_2 \leq Z(G)$. Again, since S_2 is a group of order 8, the only possibilities are $S_2 \cong \mathbb{Z}_8$ or Q_8. As $\mathbb{Z}_8$ is cyclic, by Theorem 4.6, $S_2 \cong Q_8$. □

From Proposition 2.3(5 & 6), it is observed that if G is a perfect *POEC* group, then there exist two odd primes p and q such that $8p^2q^2$ divides $|G|$. Now, we are in a position to say something more.

Theorem 4.9. *If G is a perfect POEC group, then there exist two distinct odd primes p, q such that $8p^3q^3$ divides $|G|$.*

Proof. We prove the result by contradiction. If the theorem does not hold, then there exists a perfect *POEC* group G such that $|G| = 2^\alpha p_1^\beta p_2^2 \cdots p_t^2$, where $\alpha \geq 3, \beta \geq 2, t \geq 2$ and p_i's are distinct odd primes. This follows from Proposition 2.3(5 & 6) and Corollary 4.7.

Observe that $S_{p_i} = G_{p_i} \cong \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ for all $i \geq 2$, using Theorem 4.6. Consider the normal subgroup $H = G_{p_2}G_{p_3}\cdots G_{p_t}$ of G. Then, as $|G/H| = 2^\alpha p_1^\beta$, G/H is solvable and being a quotient of a perfect group G/H is perfect, a contradiction. □

Corollary 4.10. *If G is a non-solvable POEC group, then there exist two odd primes p, q such that $8p^3q^3$ divides $|G|$.*

Proof. This follows from the above theorem and the fact that every non-solvable group is either perfect or has a perfect subgroup. □

Remark 4.11. *It follows from Corollary 4.7, that if G is a perfect POEC group and p is a prime dividing $|G|$, then p^2 divides $|G|$. In light of Theorem 4.9, it is natural to ask the following question: If G is a perfect POEC group and p is a prime dividing $|G|$, is it necessary that p^3 divides $|G|$? We provide a partial answer to this.*

We again recall a result on perfect subgroups of $SL(2, p)$ which will be used in the next theorem.

Proposition 4.12. [5, Theorem 6.17] *Let p be an odd prime. The only non-trivial perfect subgroups of $SL(2, p)$ are:*

- *$SL(2, p)$ itself, and*
- *$SL(2, 5) \cong 2.A_5$.*

The second case occurs only if $p \equiv \pm 1 \ (mod\ 10)$.

Theorem 4.13. *Let* G *be a perfect* $POEC$ *group and* p *be a prime dividing* $|G|$. *If one of the following conditions hold:*

- 2^4 *or* 3 *or* 5 *does not divide* $|G|$;
- $p \not\equiv \pm 1 \ (mod\ 10)$,

then $p^3 \mid |G|$.

Proof. As $p \mid |G|$, it follows from Corollary 4.7 that p^2 divides $|G|$. Suppose $p^3 \nmid |G|$. Then we have $S_p = G_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

By N/C theorem, $G/C_G(G_p)$ is isomorphic to a subgroup of $Aut(G_p) \cong GL(2,p)$. Note that any perfect subgroup of $GL(2,p)$ is also a subgroup of $SL(2,p)$ as $det([A,B]) = det(ABA^{-1}B^{-1}) = 1$ for all $A, B \in GL(2,p)$. Now, as $G/C_G(G_p)$ is perfect, $G/C_G(G_p)$ is isomorphic to a subgroup of $SL(2,p)$. As $G/C_G(G_p)$ is perfect, $p \nmid |G/C_G(G_p)|$ and $p \mid |SL(2,p)|$, from Proposition 4.12, it follows that either $G/C_G(G_p)$ is trivial or $G/C_G(G_p) \cong SL(2,5)$.

In the later case, we have $|G/C_G(G_p)| = 120 = 2^3 \cdot 3 \cdot 5$. Since G is a $POEC$ group, $G_2 \subseteq C_G(G_p)$ and hence $2^4 \cdot 3 \cdot 5$ divides $|G|$. Also as $G/C_G(G_p) \cong SL(2,5)$, by Proposition 4.12, $p \equiv \pm 1 \ (mod\ 10)$. Both of these violate the given conditions. Thus $G/C_G(G_p)$ is trivial, i.e., $G_p \cong \mathbb{Z}_p \times \mathbb{Z}_p \leq Z(G)$, i.e., $p \nmid [G : Z(G)]$. Since G contains elements of order p, by Proposition 4.5, we get a contradiction. Thus $p^3 \mid |G|$. □

Theorem 4.14. *If* G *is a perfect* $POEC$ *group such that* $3 \mid |G|$, *then* $3^4 \mid |G|$.

Proof. By Theorem 4.13, it follows that $3^3 \mid |G|$. Now, we assume that $3^4 \nmid |G|$. Consider the Sylow 3-subgroup S_3 of G of order 27. Then S_3 must be isomorphic to either $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, because the other possibilities of S_3, namely $\mathbb{Z}_{27}$ and $Heis(\mathbb{Z}_3)$, can be ruled out.

If $S_3 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 = G_3$, then by N/C theorem, $G/C_G(G_3)$ is isomorphic to a subgroup of $SL(3,3)$. Now, as $G/C_G(G_3)$ is perfect and $SL(3,3)$ has no non-trivial perfect subgroup (checked in GAP), we must have either $G/C_G(G_3) \cong SL(3,3)$ or $|G/C_G(G_3)| = 1$. If $G/C_G(G_3) \cong SL(3,3)$, then clearly $3^6 \mid |G|$, because $3^3 \mid |SL(3,3)|$ and $G_3 \subseteq C_G(G_3)$. If $|G/C_G(G_3)| = 1$, then $G = C_G(G_3)$, i.e., $G_3 \leq Z(G)$, i.e., 27 divides $|Z(G)|$. Thus $3 \nmid [G : Z(G)] = t$ (say). Now, by Proposition 4.5, $g^t = e$ for all $g \in G$. However, this can not hold as G contains elements of order 3.

Thus $S_3 \neq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, i.e., S_3 must be isomorphic to either $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ and in both the cases, $G_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Again, by N/C theorem, $G/C_G(G_3)$ is isomorphic to a subgroup of $SL(2,3)$. If $C_G(G_3)$ is a proper subgroup of G, then as G is perfect, $G/C_G(G_3)$ is also perfect, but $SL(2,3)$, being a group of order 24, has no perfect subgroup. Thus $G = C_G(G_3)$, i.e., $G_3 \leq Z(G)$, i.e., 9 divides $|Z(G)|$.

Let $[G : Z(G)] = t$. Clearly 3 divides t and 9 does not divide t. By Proposition 4.5, $g^t = e$ for all $g \in G$. In particular if g belongs to a Sylow 3-subgroup S_3 of G, this implies that $g^3 = e$, i.e., every element of S_3 is of order 3. However, as S_3 is isomorphic to either $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, we get a contradiction. Thus the theorem holds. □

It is known that if G is a finite non-trivial perfect group, then 4 divides $|G|$ and if 8 does not divide $|G|$, then 3 does. A stronger result holds for perfect $POEC$ groups, as a corollary of Theorem 4.14.

Corollary 4.15. *If G is a perfect $POEC$ group such that $2^4 \nmid |G|$, then $3^4 \mid |G|$.*

Proof. If $2^3 \nmid |G|$, then as G is a perfect group, $|G|$ is divisible by 3. Now, by Theorem 4.14, the corollary follows. If $2^3 \mid |G|$, then consider the group G/G_2. It is a perfect group with $2^3 \nmid |G/G_2|$ and hence $3 \mid |G/G_2|$, and thereby $3 \mid |G|$. Now the corollary follows from Theorem 4.14. □

5. Conclusion and Open Issues

We conclude with some possible directions and open issues.

(1) In Propositions 3.1, 3.4 and 4.8, we have shown that $POEC$ groups, under certain conditions, admit a non-trivial center. We strongly believe that this holds for all finite $POEC$ groups without imposing further constraints,

Open Issue 1: If G is a finite $POEC$ group, then $|Z(G)| > 1$.

(2) In Theorem 4.13, it was shown that if a prime p divides the order of a perfect $POEC$ group G, then under certain conditions on p and the prime factorization of $|G|$, p^3 divides $|G|$. We ask whether the same holds for any prime divisor of any $POEC$ group.

Open Issue 2: If G is a finite perfect $POEC$ group and $p \mid |G|$, then $p^3 \mid |G|$.

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