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THE CLIQUE NUMBER OF THE INTERSECTION GRAPH OF SOME CYCLIC GROUPS

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Abstract. For a nontrivial finite group G, the intersection graph $\Gamma(G)$ of G is a simple undirected graph whose vertices are the nontrivial proper subgroups of G and two vertices are joined by an edge if and only if they have a nontrivial intersection. In this paper, we obtain the clique number of the intersection graph of cyclic groups whose orders have four prime divisors. Moreover we find the clique number of the intersection graph of cyclic groups of order n such that all powers of prime divisors of n are equal. As a special case, we find the clique number of this graph for the cyclic groups of the square-free orders.

1. Introduction

Let G be a group. There are several ways to associate a graph to G (see [7] and the references therein). In this paper, we define the intersection graph of G, denoted by $\Gamma(G)$, is a graph having all the non-trivial proper subgroups of G as its vertices and two distinct vertices in $\Gamma(G)$ are adjacent if and only if their intersection is non-trivial. Motivated by [6], Csákány and Pollák [8] defined the intersection graph of a finite group G in 1969. Zelinka in [19] considered the intersection graphs of finite abelian groups. Taeri and Ahmadi [1] characterized all finite groups with planar intersection graphs (see also [12]).

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Kayacan in [11] considered the connectivity of the intersection graphs of finite groups. In 2010, Shen in [17] determined all finite groups with disconnected intersection graphs. In 2021, Freedman [9] proved that $\Gamma(G)$ is connected with diameter at most 5 when G is non-abelian simple. In [14, 15], Shahsavari and Khosravi characterized some families of finite simple groups using their intersection graphs. The reader can see other results about $\Gamma(G)$ in [1, 5, 16, 18]. Until now, there are only a few results on the clique number of the intersection graph of finite groups.

Let Γ be a simple graph. The set of vertices of every complete subgraph of Γ is called a clique of Γ. A maximal clique is a clique that cannot be extended by including one more adjacent vertex. A maximum clique of Γ, is a clique, such that there is no clique with more vertices. Moreover, the clique number of Γ, denoted by $\omega(\Gamma)$, is the number of vertices in a maximum clique in Γ.

In [2], we showed that if G is a finite group such that $\omega(\Gamma(G)) < 13$, then G is solvable and $\omega(\Gamma(A_5)) = 13$ where A_5 is the alternative group on 5 letters. Also, we determined all groups G with $\omega(\Gamma(G)) \leq 4$. In [4], we showed that $\omega(\Gamma(\mathrm{PSL}(2,7))) = 25$ and classify all semisimple groups G with $\omega(\Gamma(G)) \leq 25$. We find $\omega(\Gamma(C_n))$ where C_n is a cyclic group of order n having at most three prime divisors [3]. It seems not easy to find $\omega(\Gamma(C_n))$ in general. So we obtain it for some special cases in this paper.

In the sequel, assume that $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is a positive integer where $r \geq 1$, $p_1, p_2, \ldots, p_r$ are distinct primes and $k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$. We use $\pi(n)$ and $d(n)$ to denote the set of all prime divisors and the number of the divisors of n, respectively. Therefore $d(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$. It is easy to see that $\omega(\Gamma(C_{p_1^{k_1}})) = d(p_1^{k_1}) - 2 = k_1 - 1$ (see [2, Lemma 4.1 (1)]). In [3], we have determined $\omega(\Gamma(C_n))$ when $r \leq 3$ (see Propositions 2.1 and 2.2). In the present paper we first find $\omega(\Gamma(C_n))$ when $r = 4$ (see Theorem 2.3) and then we obtain $\omega(\Gamma(C_n))$ when r is an arbitrary positive integer and $k_1 = k_2 = \cdots = k_r$ (see Theorem 3.2 and 3.3).

In this paper all groups will be finite and we use the usual notation. Also $\mathcal{S}(G)$ is the set of all subgroups of G. The rest of the notation is standard and can be found mainly in [13].

2. Cyclic groups of order with at most four prime divisors

It is known that C_n has a unique subgroup of each order dividing n. It follows that $|\mathcal{S}(C_n)| = d(n)$ and so the number of vertices of $\Gamma(C_n)$ is $d(n) - 2$. For given subgroups H and K of C_n, we have $H \cap K \neq 1$ if and only if $\gcd(|H|, |K|) \neq 1$. This is useful in the sequel.

Proposition 2.1. *Let $n = p_1^{k_1} p_2^{k_2}$. Then $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$.*

Proof. See [3, Theorem 3.2]. Note that $d(\frac{n}{p_1}) - 1 = k_1 k_2 + k_1 - 1$ is the number of proper subgroups H of C_n such that p_1 divides $|H|$. □

Proposition 2.2. *Let $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$.*

(1) *If $k_1 \geq k_2 k_3$, then $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$.*

(2) *If* $k_1 < k_2k_3$*, then* $\omega(\Gamma(C_n)) = d(\frac{n}{p_2p_3}) - 1 + k_1(k_2 + k_3)$*.*

Proof. See [3, Theorem 3.3]. Note that if $H, K < C_n$ such that $|\pi(|H|)| \geq 2$ and $|\pi(|K|)| \geq 2$, then $H \cap K \neq 1$. □

In the following, we state and prove the first main result:

Theorem 2.3. *Let* $n = p_1^{k_1}p_2^{k_2}p_3^{k_3}p_4^{k_4}$*.*

(1) *If* $k_1 \geq k_2k_3k_4$*, then* $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$*.*

(2) *If* $k_1 < k_2k_3k_4$*, then*

$$\omega(\Gamma(C_n)) = d_3(n) + k_1k_2 + k_1k_3 + \max\{k_1k_4, k_2k_3\} - 1$$

where $d_3(n)$ *is the number of divisors of* n *with at least three distinct prime divisors.*

Proof. Let $G = C_n$. We introduce some subsets of the vertex set of $\Gamma(G)$ as follows:

- V_{p_i} is the set of all subgroups H of G such that $\pi(|H|) = \{p_i\}$ for $1 \leq i \leq 4$. Clearly $|V_{p_i}| = k_i$.
- $V_{p_ip_j}$ is the set of all subgroups H of G such that $\pi(|H|) = \{p_i, p_j\}$ for $1 \leq i \neq j \leq 4$. Clearly $|V_{p_ip_j}| = k_ik_j$ and $V_{p_ip_j} = V_{p_jp_i}$.
- $V_{p_ip_jp_l}$ is the set of all subgroups H of G such that $\pi(|H|) = \{p_i, p_j, p_l\}$ for $1 \leq i < j < l \leq 4$. Clearly $|V_{p_ip_jp_l}| = k_ik_jk_l$.
- $V_{p_1p_2p_3p_4}$ is the set of all proper subgroups H of G such that $\pi(|H|) = \{p_1, p_2, p_3, p_4\}$. Clearly $|V_{p_1p_2p_3p_4}| = k_1k_2k_3k_4 - 1$.
- $X := (\cup_{1\leq i<j<l\leq 4} V_{p_ip_jp_l}) \bigcup V_{p_1p_2p_3p_4}$.

Clearly,

$$\{V_{p_i}, V_{p_jp_l}, X : \quad 1 \leq i \leq 4, 1 \leq j < l \leq 4\}$$

forms a partition for $\mathcal{S}(G) \setminus \{1, G\}$.

Now set

$$(*) \qquad B_{p_i} := V_{p_ip_j} \cup V_{p_ip_l} \cup V_{p_ip_t}$$

and

$$(**) \qquad A_{p_ip_jp_l} := V_{p_ip_j} \cup V_{p_ip_l} \cup V_{p_jp_l}$$

where i, j, l, t are distinct. Clearly B_{p_i} and $A_{p_ip_jp_l}$ are cliques in $\Gamma(G)$ for every i, j, l. If $T < G$ such that $|\pi(|T|)| = 2$ and $T \notin B_{p_i} \cup A_{p_jp_lp_t}$ for some i, j, l, t, then $B_{p_i} \cup \{T\}$ and $A_{p_jp_lp_t} \cup \{T\}$ are not cliques. So B_{p_i} and $A_{p_jp_lp_t}$ are all maximal cliques whose members are of orders with two distinct prime divisors. Assume first that

$$P_i = \{H < G : \qquad p_i \text{ divides } |H|\},$$

$$D_i = B_{p_i} \bigcup X \qquad\qquad 1 \leq i \leq 4$$

and

$$E_{ijl} := A_{p_ip_jp_l} \bigcup X \qquad 1 \le i < j < l \le 4.$$

It is clear that P_i, D_i and E_{ijl} are cliques in $\Gamma(G)$ for every distinct i,j,l. We claim that all maximal cliques in $\Gamma(G)$ are P_i, D_i and E_{ijl} for all distinct i,j,l.

Suppose that W is a maximal clique in $\Gamma(G)$ and $H \in W$ such that $|\pi(|H|)|$ has the minimum value among all elements in W. It is easy to see that $|\pi(|H|)| \le 2$. Now we consider two cases:

Case 1. If $|\pi(|H|)| = 1$, then $\pi(|H|) = \{p_i\}$ for some $1 \le i \le 4$. Since W is a clique and $H \in W$, p_i divides the order of every element of W and so $W \subseteq P_i$. By the maximality of W, we have $W = P_i$, as claimed.

Case 2. Suppose that $|\pi(|H|)| = 2$. Note that if $K, L < G$ such that $|\pi(|K|)| \ge 2$ and $|\pi(|L|)| \ge 3$, then $\gcd(|K|,|L|) \ne 1$ and so $K \cap L \ne 1$. Since W is a maximal clique in $\Gamma(G)$, we have $X \subset W$. By the properties of B_{p_i}, $A_{p_jp_lp_t}$ and the maximality of W, we get either $W = B_{p_i} \bigcup X$ for some i or $W = A_{p_ip_jp_l} \bigcup X$ for some distinct i,j,l. Therefore either $W = D_i$ for some i or $W = E_{ijl}$ for some distinct i,j,l. This completes the proof of the claim. So

$$\omega(\Gamma(G)) = \max\{|P_i|, |D_i|, |E_{ijl}| : \quad 1 \le i < j < l \le 4\}.$$

Clearly, $|P_i| = d(\frac{n}{p_i}) - 1, |D_i| = |B_{p_i}| + |X|$ for each i and $|E_{ijl}| = |A_{p_ip_jp_l}| + |X|$. By $(*)$ and $(**)$, we see that $|B_{p_i}| = k_ik_j + k_ik_l + k_ik_t$ and $|A_{p_ip_jp_l}| = k_ik_j + k_ik_l + k_jk_l$. Since $k_1 \ge k_2 \ge k_3 \ge k_4$, we conclude that $|P_1| \ge |P_i|$, $|D_1| \ge |D_i|$ for each $i > 1$ and $|E_{123}| \ge |E_{ijl}|$ for each $(i,j,l) \ne (1,2,3)$. Therefore $\omega(\Gamma(G)) = \max\{|P_1|, |D_1|, |E_{123}|\}$.

On the other hand we have $P_1 = V_{p_1} \bigcup B_{p_1} \bigcup (X \setminus \{V_{p_2p_3p_4}\})$ and so $|P_1| = k_1 + |B_{p_1}| + (|X| - k_2k_3k_4)$. Moreover $|D_1| = |B_{p_1}| + |X|$ and $|E_{123}| = |A_{p_1p_2p_3}| + |X|$.

First, assume that $k_1 \ge k_2k_3k_4$. Then $|P_1| \ge |D_1| \ge |E_{123}|$ and so $\omega(\Gamma(G)) = |P_1| = d(\frac{n}{p_1}) - 1$, as desired.

Now assume that $k_1 < k_2k_3k_4$. Then $|D_1| > |P_1|$ and so $\omega(\Gamma(G)) = \max\{|D_1|, |E_{123}|\}$. Note that $|B_{p_1}| = k_1k_2 + k_1k_3 + k_1k_4$ and $|A_{p_1p_2p_3}| = k_1k_2 + k_1k_3 + k_2k_3$. If $k_1k_4 \ge k_2k_3$, then $\omega(\Gamma(G)) = |D_1|$, as wanted. If $k_1k_4 \le k_2k_3$, then $\omega(\Gamma(G)) = |E_{123}|$. This completes the proof. □

3. The powers of all prime divisors of n are equal

In this section we find $\omega(\Gamma(C_n))$ where $n = p_1^k p_2^k \cdots p_r^k$ such that $k \ge 1$ and $r > 2$.

For positive integers $i \le r$, we denote $C(r,i) = \frac{r!}{i!(r-i)!}$. Also we define $C(r,0) = C(r,r) = 1$. Clearly, $C(r,i) = C(r, r-i)$ for each $i \le r$. It is well-known that if Ω is a set with r elements, then the number of subsets of Ω with i elements is $C(r,i)$. So we have

$$C(r,0) + C(r,1) + \cdots + C(r,r-1) + C(r,r) = 2^r. \tag{1}$$

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If $r>1$ is odd, then

$$\sum_{i=1}^{\frac{r-1}{2}} C(r,i)=\sum_{i=\frac{r+1}{2}}^{r-1} C(r,i)=2^{r-1}-1. \tag{2}$$

If r is even, then

$$\sum_{i=1}^{\frac{r}{2}-1} C(r,i)+\frac{1}{2}C(r,\frac{r}{2})=\sum_{i=\frac{r}{2}+1}^{r-1} C(r,i)+\frac{1}{2}C(r,\frac{r}{2})=2^{r-1}-1. \tag{3}$$

The following result is useful in the sequel.

Lemma 3.1. *(see [13, 1.3.11]) Let G be a group and H and K be subgroups of G. Then $|G:H\cap K|\leq |G:H||G:K|$, with equality if the indices $|G:H|$ and $|G:K|$ are coprime.*

First, we state and prove the main result for the cyclic groups of orders having odd prime divisors with equal powers.

Theorem 3.2. *Let $n=p_1^kp_2^k\cdots p_r^k$ where $r>1$ is an odd integer. Then*

$$\omega(\Gamma(C_n))=\sum_{i=\frac{r+1}{2}}^{r-1} C(r,i)k^i+k^r-1.$$

Proof. Assume that $G=C_n$ and $\omega=\sum_{i=\frac{r+1}{2}}^{r-1} C(r,i)k^i+k^r-1$. Also $\mathfrak{S}_i$ denotes the set of all subgroups of G of orders having i prime divisors for $1\leq i\leq r$. It is easy to see that $|\mathfrak{S}_i|=C(r,i)k^i$.

If $H\in\mathfrak{S}_m$ and $T\in\mathfrak{S}_l$ such that $m\geq l\geq\frac{r+1}{2}$, then $|G:H|$ and $|G:T|$ have $r-m$ and $r-l$ prime divisors respectively. Therefore $|G:H\cap T|$ has at most $2r-(m+l)$ prime divisors by Lemma 3.1. Since $m+l\geq r+1$, we have $|G:H\cap T|$ has at most $r-1$ prime divisors. But $|G|$ has r prime divisors which implies that $H\cap T\neq 1$. Hence $\mathfrak{S}_{\frac{r+1}{2}}\cup\mathfrak{S}_{\frac{r+1}{2}+1}\cup\cdots\cup\mathfrak{S}_{r-1}$ forms a clique in $\Gamma(G)$. Moreover, if $H\in\mathfrak{S}_r\setminus\{G\}$ and $K\in\mathcal{S}(G)\setminus\{1,G\}$, then $\gcd(|H|,|K|)\neq 1$ and so $H\cap K\neq 1$. Consequently $(\bigcup_{i=\frac{r+1}{2}}^{r-1}\mathfrak{S}_i)\cup(\mathfrak{S}_r\setminus\{G\})$ forms a clique in $\Gamma(G)$. By the definition of clique number of a graph, we have $\omega(\Gamma(G))\geq|(\bigcup_{i=\frac{r+1}{2}}^{r-1}\mathfrak{S}_i)\cup(\mathfrak{S}_r\setminus\{G\})|$. Since $|\mathfrak{S}_i|=C(r,i)k^i$ and $\mathfrak{S}_i\cap\mathfrak{S}_j=\emptyset$ for each $i\neq j$, we deduced that

$$\omega(\Gamma(G))\geq\sum_{i=\frac{r+1}{2}}^{r-1} C(r,i)k^i+k^r-1=\omega. \tag{$*$}$$

Now suppose that W is an arbitrary clique in $\Gamma(G)$. It is enough to show that $|W|\leq\omega$.

Set $W_i=W\cap\mathfrak{S}_i$ for each $1\leq i\leq r$. If $1\leq i\leq\frac{r-1}{2}$, then define $f_i:W_i\cup W_{r-i}\longrightarrow\mathfrak{S}_{r-i}$ by

$$f_i(H)=\begin{cases}H & H\in W_{r-i}\\ H' & H\in W_i\end{cases}$$

where H' is defined as follows.

Note that if $H \in W_i$, then $|H| = p_{j_1}^{\alpha_{j_1}} \cdots p_{j_i}^{\alpha_{j_i}}$ such that $1 \leq j_1 < j_2 < \cdots < j_i \leq r$ and $1 \leq \alpha_{j_t} \leq k$ for every t. Suppose that

$$\{p_1, p_2, \ldots, p_r\} \setminus \{p_{j_1}, \ldots, p_{j_i}\} = \{p_{l_1}, \ldots, p_{l_{r-i}}\}$$

where $l_1 < l_2 < \cdots < l_{r-i}$. Now we consider $H' < G$ such that $|H'| = p_{l_1}^{\alpha_{j_1}} \cdots p_{l_i}^{\alpha_{j_i}} p_{l_{i+1}} \cdots p_{l_{r-i}}$. Therefore $H \cap H' = 1$ and $H' \in \mathfrak{S}_{r-i}$. Since $H \in W_i$, we have $H' \in \mathfrak{S}_{r-i} \setminus W_{r-i}$. We claim that f_i is a one to one map for each $1 \leq i \leq \frac{r-1}{2}$.

Assume that $H_1, H_2 \in W_s \cup W_{r-s}$ for some $1 \leq s \leq \frac{r-1}{2}$ such that $f_s(H_1) = f_s(H_2)$. If $H_1 \in W_{r-s}$ and $H_2 \in W_s$, then $H_1 = H_2'$. By the definition of H_2', we have $H_2 \cap H_2' = 1$ and so $H_2 \cap H_1 = 1$, a contradiction since $H_1, H_2 \in W$. Therefore either $H_1, H_2 \in W_s$ or $H_1, H_2 \in W_{r-s}$. The first case gives $H_1' = H_2'$ since $f_s(H_1) = f_s(H_2)$. It follows that

$$|H_1'| = |H_2'| = p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \cdots p_{i_s}^{\alpha_{i_s}} p_{i_{s+1}} \cdots p_{i_{r-s}}.$$

By the definition of f_s, we get $|H_1| = |H_2| = p_{j_1}^{\alpha_{i_1}} \cdots p_{j_s}^{\alpha_{i_s}}$ where

$$\{p_{j_1}, p_{j_2}, \ldots, p_{j_s}\} = \{p_1, \ldots, p_r\} \setminus \{p_{i_1}, p_{i_2}, \ldots, p_{i_{r-s}}\}.$$

Since G is cyclic, we have $H_1 = H_2$, as wanted. The second case gives $H_1 = H_2$ by the definition of f_s. This completes the proof of the claim. Hence $|W_i \cup W_{r-i}| \leq |\mathfrak{S}_{r-i}|$ for each $1 \leq i \leq \frac{r-1}{2}$. Since $W_r \subseteq \mathfrak{S}_r \setminus \{G\}$, we deduce that

$$\begin{aligned} |W| &= |\bigcup_{i=1}^{r} W_i| = |\bigcup_{i=1}^{\frac{r-1}{2}} (W_i \cup W_{r-i})| + |W_r| \\ &\leq (\sum_{i=1}^{\frac{r-1}{2}} |\mathfrak{S}_{r-i}|) + k^r - 1 = \omega \end{aligned} \tag{3.1}$$

Hence $(*)$ and (3.1) give the result. □

In the following, we prove the third main result. More precisely, we find the clique number of the intersection graph of cyclic groups of orders having even prime divisors with equal powers.

Theorem 3.3. *Let $n = p_1^k p_2^k \cdots p_r^k$ be a positive integer such that $r \geq 4$ is even. Then*

$$\omega(\Gamma(C_n)) = \sum_{i=\frac{r}{2}+1}^{r-1} C(r,i)k^i + \frac{1}{2}C(r, \frac{r}{2})k^{\frac{r}{2}} + k^r - 1.$$

Proof. Assume that $G = C_n$ and $\omega = \sum_{i=\frac{r}{2}+1}^{r-1} C(r,i)k^i + \frac{1}{2}C(r, \frac{r}{2})k^{\frac{r}{2}} + k^r - 1$. Let $\mathfrak{S}_i$ denote the set of all subgroups of G of orders having i prime divisors where $1 \leq i \leq r$. It is easy to see that $|\mathfrak{S}_i| = C(r,i)k^i$ for every i.

By a similar argument to that of the proof of Theorem 3.2, we see that $\mathfrak{S}_{\frac{r}{2}+1} \cup \mathfrak{S}_{\frac{r}{2}+2} \cup \cdots \cup \mathfrak{S}_{r-1} \cup (\mathfrak{S}_r \setminus \{G\})$ is a clique in $\Gamma(G)$. Set

$$\mathfrak{S}^1_{\frac{r}{2}} := \{H \in \mathfrak{S}_{\frac{r}{2}} : \ p_1 \text{ divides } |H|\}.$$

Therefore $\mathfrak{S}^1_{\frac{r}{2}}$ is a clique in $\Gamma(G)$ and $|\mathfrak{S}^1_{\frac{r}{2}}| = C(r-1, \frac{r}{2}-1)k^{\frac{r}{2}} = \frac{1}{2}C(r, \frac{r}{2})k^{\frac{r}{2}}$. Moreover,

$$\mathfrak{S}_{\frac{r}{2}+1} \cup \mathfrak{S}_{\frac{r}{2}+2} \cup \cdots \cup \mathfrak{S}_{r-1} \cup (\mathfrak{S}_r \setminus \{G\}) \cup \mathfrak{S}^1_{\frac{r}{2}}$$

forms a clique in $\Gamma(G)$ (similar to the proof of Theorem 3.2, applying Lemma 3.1). By the definition of the clique number, we have

$$(*) \qquad \omega(\Gamma(G)) \geq \sum_{i=\frac{r}{2}+1}^{r-1} C(r,i)k^i + \frac{1}{2}C(r, \frac{r}{2})k^{\frac{r}{2}} + k^r - 1 = \omega.$$

Now suppose that W is an arbitrary clique in $\Gamma(G)$. We prove that $|W| \leq \omega$.

Set $W_i = W \cap \mathfrak{S}_i$ for each $1 \leq i \leq r$. As in the proof of Theorem 3.2, we have $|W_i \cup W_{r-i}| \leq |\mathfrak{S}_{r-i}|$ for each $1 \leq i \leq \frac{r}{2} - 1$. It remains to prove $|W_{\frac{r}{2}}| \leq \frac{1}{2}|\mathfrak{S}_{\frac{r}{2}}|$.

Define $g : W_{\frac{r}{2}} \longrightarrow \mathfrak{S}_{\frac{r}{2}} \setminus W_{\frac{r}{2}}$ by $g(H) = H'$ where $|H| = p_{j_1}^{\alpha_{j_1}} \cdots p_{j_{\frac{r}{2}}}^{\alpha_{j_{\frac{r}{2}}}}$, $|H'| = p_{l_1}^{\alpha_{j_1}} p_{l_2}^{\alpha_{j_2}} \cdots p_{l_{\frac{r}{2}}}^{\alpha_{j_{\frac{r}{2}}}}$ and $\{p_1, \ldots, p_r\} = \{p_{j_1}, \ldots, p_{j_{\frac{r}{2}}}\} \cup \{p_{l_1}, \ldots, p_{l_{\frac{r}{2}}}\}$ such that $j_1 < \cdots < j_{\frac{r}{2}}$ and $l_1 < \cdots < l_{\frac{r}{2}}$. It is easy to see that g is one to one.

Since $W_r \subseteq \mathfrak{S}_r \setminus \{G\}$, we conclude that

$$\begin{aligned} |W| &= |\bigcup_{i=1}^{r} W_i| = |(\bigcup_{i=1}^{\frac{r}{2}-1}(W_i \cup W_{r-i})| + |W_{\frac{r}{2}}| + |W_r| \\ &\leq \sum_{i=1}^{\frac{r}{2}-1} |\mathfrak{S}_{r-i}| + \frac{1}{2}|\mathfrak{S}_{\frac{r}{2}}| + k^r - 1 = \omega \end{aligned} \tag{3.2}$$

By $(*)$ and (3.2), we have the result. □

As an application, we find $\omega(\Gamma(C_n))$ when n is square-free.

Corollary 3.4. *Let $n = p_1p_2 \cdots p_r$. Then $\omega(\Gamma(C_n)) = 2^{r-1} - 1$.*

Proof. Apply Theorems 3.2 and 3.3 for $k = 1$. If $r > 1$ is odd, then by Theorem 3.2, we have $\omega(\Gamma(C_n)) = \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i)$ and so the result follows from (2).

If r is even, then by Theorem 3.3, we have

$$\omega(\Gamma(C_n)) = \sum_{i=\frac{r}{2}+1}^{r-1} C(r,i) + \frac{1}{2}C(r, \frac{r}{2}),$$

and hence the result follows from (3). This completes the proof. □

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References

[1] H. Ahmadi and B. Taeri, Planarity of the intersection graph of subgroups of a finite groups, *J. Algebra Appl.*, **15** no. 3 (2016) 19 pp.

[2] A. Beheshtipour and S. M. Jafarian Amiri, The clique number of the intersection graph of a finite group, *Bull. Iran. Math. Soc.*, **49** no. 5 (2023) 16 pp.

[3] A. Beheshtipour and S. M. Jafarian Amiri, The clique number of the intersection graph of a cyclic group of order with at most three prime factors, *Journal of Mathematics and Society*, **8** no. 4 (2023) 21–25.

[4] A. Beheshtipour and S. M. Jafarian Amiri, The clique number of the intersection graph of $\mathrm{PSL}(2,7)$, submitted.

[5] M. H. Bien and D. H. Viet, Intersection graphs of general linear groups, *J. Algebra Appl.*, **20**no. 3 (2021) 14 pp.

[6] J. Bosak, The graphs of semigroups, *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, Publ. House Czech. Acad. Sci., Prague, 1964 119–125.

[7] P. J. Cameron, Graphs defined on groups, *Int. J. Group Theory*, **11** (2022) 53–107.

[8] B. Csákány and G. Pollák, The graph of subgroups of a finite group, *Czech. Math. J.*, **19** (1969) 241–247.

[9] Saul D. Freedman, The intersection graph of a finite simple group has diameter at most 5, *Arch. Math. (Basel)*, **117** no. 1 (2021) 1–7.

[10] M. Herzog, P. Longobardi and M. Maj, On a graph related to maximal subgroups of a group, *Bull. Austral. Math. Soc.*, **81** (2010) 317–328.

[11] S. Kayacan, Connectivity of intersection graphs of finite groups, *Comm. Algebra*, **46** no. 4 (2018) 1492–1505.

[12] S. Kayacan and S. E. Yaraneri, Finite groups whose intersection graphs are planar, *J. Kor. Math. Soc.* **52** no. 1 (2015) 81–96.

[13] D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag New York, 1996.

[14] H. Shahsavari and B. Khosravi, Characterization by intersection graph of some families of finite simple groups, *Publ. Math. Debrecen*, **102** no. 1-2 (2023) 45–60.

[15] H. Shahsavari and B. Khosravi, Characterization of some families of simple groups by their intersection graphs, *Comm. Algebra*, **48** (2020) 1266–1280.

[16] H. Shahsavari and B. Khosravi, On the intersection graph of a finite group, *Czech. Math. J.*, **67**no. 4 (2017) 1145–1153.

[17] R. Shen, Intersection graphs of subgroups of finite groups, *Czech. Math. J.*, **60** (2010) 945–950.

[18] H. Su and L. Zhu, Thickness of the subgroup intersection graph of a finite group, *AIMS Math.*, **6** no. 3 (2021) 2590–2606.

[19] B. Zelinka, Intersection graphs of finite abelian groups, *Czech. Math. J.*, **25** (1975) 171–174.

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