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Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. x No. x (2025), pp. xx-xx.

www.ui.ac.ir

A CHARACTERIZATION OF GRAPHS WITH UPPER LOCATING-DOMINATION NUMBER EQUAL TO $n-2$

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Abstract. A set D of vertices in a graph G is called a dominating set of G if every vertex in $V(G)\setminus D$ has at least one neighbor in D. A dominating set D of G is called a locating-dominating set of G if every two vertices in $V(G)\setminus D$ have two distinct neighborhood sets. The upper locating-domination number $\Gamma_L(G)$ is the maximum cardinality of a minimal locating-dominating set of G. In this paper, we characterize the graphs with $\Gamma_L(G) = n-2$.

1. Introduction

In this paper, we consider finite, undirected and simple graphs (no loops or multiple edges). For terminology and notation not defined here, we refer the readers to the book of [4]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The number of vertices $|V|$ is the order of G. A subset S of vertices is called *independent* if no two vertices of S are adjacent. A *clique* is a subset of pairwise adjacent vertices and G is a complete graph if V is a clique. A star is a graph whose vertex set can be partitioned into two subsets, a singleton $\{a\}$ and an independent set L such that every vertex in L is adjacent to a.

MSC(2010): Primary: 05C69; Secondary: 05C70.

Keywords: dominating set, locating-dominating set, upper locating-domination number.

Article Type: Research Paper.

Communicated by Saeid Azam.

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Received: 21 November 2023, Accepted: 20 February 2025, Published Online: 23 February 2025.

Cite this article: M. Mimouni, L. Ouldrabah and N. Ikhlef-Eschouf, A characterization of graphs with upper locating-domination number equal to $n-2$, *Trans. Comb.*, **x** no. x (202x) xx-xx.

http://dx.doi.org/10.22108/toc.2025.139873.2128 .

Given two vertices u and v in V, we say that u *sees* v if u is adjacent to v, and u *misses* v otherwise. Let A and B be two subsets of $V(G)$. We say that v sees (respectively, misses) A if v sees (respectively, misses) every vertex in A. If every vertex in A sees (respectively, misses) B, we say that A sees (respectively, misses) B. Also, we say that A *misses or sees* B if either there is no edge between A and B, or there are all possible edges between A and B. If an edge e may exist or not, then we say that e is *optional*.

A set $D \subseteq V$ is called a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D, and D is a *locating-dominating set* (LDS for short) of G if D is a dominating set such that every two distinct vertices in $V(G) \setminus D$ have distinct neighbors in D. *Locating-domination number* $\gamma_L(G)$ is the minimum cardinality of an LDS of G. The *upper locating-domination number* $\Gamma_L(G)$ is the maximum cardinality of a minimal LDS of G. A minimal LDS of maximum cardinality of G is called a $\Gamma_L(G)$-set.

Locating-domination was first introduced and studied by Slater [14, 15]. He was motivated by his aim to establish a system of surveillance devices that can detect threads, such as fires, intruders (including thieves and saboteurs) and defective processors in multiprocessor systems. These devices, designed to identify such issues, correspond to a dominating set, and to ensure precise and unique localization, this set must also be a locating set. Many results in the literature have discussed locating-domination number, while to our knowledge, few studies have investigated the upper locating-domination number, which is the focus of this paper. Among the various works achieved concerning locating-domination number, we can reference those of Garijo et al. [10], they showed that the locating-domination number of a graph is at most half of its order for certain graphs like 4-cycle free graphs (including trees), graphs with independence number at least half of the order (including bipartite graphs), and graphs having a clique number at least $\lceil \frac{n}{2} \rceil + 1$. More results have been provided byFoucaud et al. [7, 8, 9] for other classes of graphs like split graphs, co-bipartite graphs, cubic graphs and line graphs. For more informations related to this parameter, we can refer to [1, 2, 3, 6].

In addition, a large number of other parameters have been derived from locating-domination by imposing other conditions on the locating-dominating set. Gimbel et al. [11] defined a differentiating-dominating set, also known as an identifying code, as a set S such that for any two vertices u and v of a graph G, $N[u] \cap D \neq N[v] \cap D$. Haynes et al. [12] introduced a total locating-dominating set S in which S has no isolated vertices. Henning and McCoy [13] defined locating-paired-dominating and differentiating-paired-dominating sets, where the dominating set forms a paired structure, inducing a subgraph with a perfect matching.

Our aim here is to investigate the upper locating-domination number. Chellali and Mimouni [5] began the study of this parameter by providing the exact value of $\Gamma_L(P_n)$, where P_n is a path of order $n \geq 2$. They also established an upper bound for the case of a non-trivial tree; in particular, they proved the following result that will be useful later.

http://dx.doi.org/10.22108/toc.2025.139873.2128

In [16], Zahao et al. obtained an exact value of the upper locating-domination number for a cycle of order $n \geq 3$.

From Theorem 1.1, it follows obviously that if G is neither a complete graph nor a star of order n, then $\Gamma_L(G) \leq n-2$.

Theorem 1.1 (Chellali and Mimouni [5]). *Every connected nontrivial graph G of order n satisfies $\Gamma_L(G) \leq n-1$, with equality if and only if G is a complete graph or a star of order n.*

In this paper, we characterize all connected graphs achieving equality for this bound. To this end, we need to define the following three families of graphs.

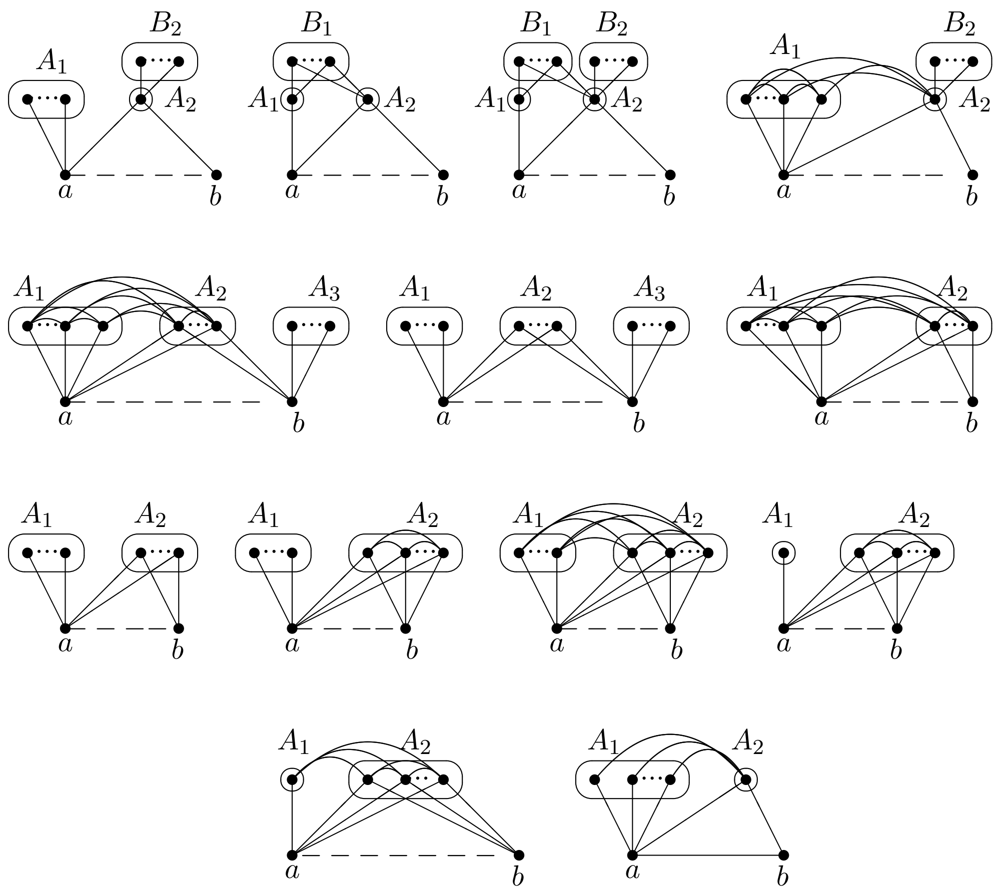

FIGURE 1. Family $\mathcal{F}_1$

Family $\mathcal{F}_1$. A graph G is in $\mathcal{F}_1$ if its vertex-set can be partitionned into six sets $\{a, b\}$, A_1, A_2, A_3, B_1 and B_2 such that

- At least one of A_3 and $B_1 \cup B_2$ is empty, while A_1 and A_2 both are nonempty.
- A_3 and $B_1 \cup B_2$ both are independent.

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- a sees $A_1 \cup A_2$, b sees $A_2 \cup A_3$ and ab is optional.
- If $B_1 \cup B_2 \neq \emptyset$, then $|A_2| = 1$. In particular, if $B_1 \neq \emptyset$, then $|A_1| = 1$ and B_1 sees $A_1 \cup A_2$, while if $B_2 \neq \emptyset$, then B_2 sees A_2.
- For each $i \in \{1, 2\}$, A_i is either a clique or an independent set such that one of the following conditions holds.
  - $A_1 \cup A_2$ is either a clique or an independent set.
  - A_1 is independent set and A_2 is a clique each of order at least 2 in which case A_1 sees or misses A_2 and $A_3 \cup B_1 \cup B_2 = \emptyset$.
  - A_1 is independent with $|A_1| \geq 2$ and $|A_2| = 1$ in which case A_1 sees A_2 and further a sees b and $A_3 \cup B_1 \cup B_2 = \emptyset$.
  - $|A_1| = 1$ and A_2 is a clique with $|A_2| \geq 2$ in which case $B_1 \cup B_2 = \emptyset$ and further either A_1 misses A_2 and $A_3 = \emptyset$ or A_1 sees A_2.

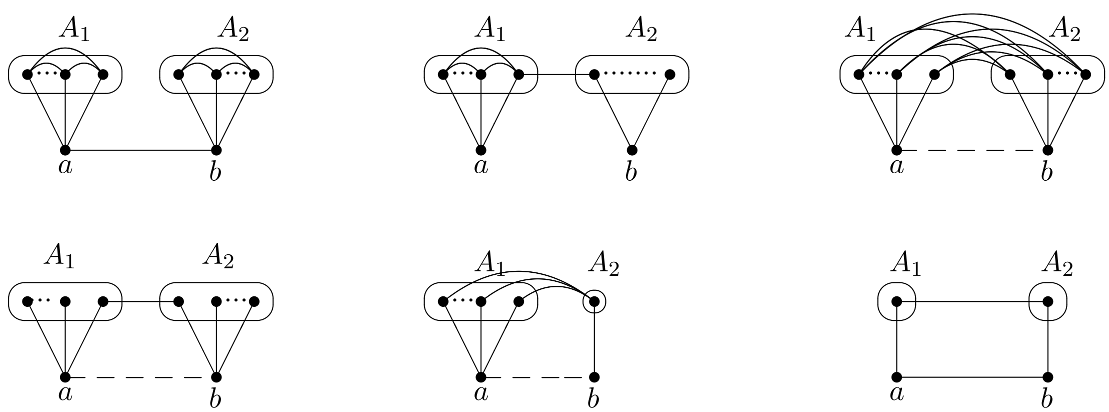

FIGURE 2. Family $\mathcal{F}_2$

Family $\mathcal{F}_2$. A graph G is in $\mathcal{F}_2$ if its vertex-set can be partitioned into three nonempty sets $\{a, b\}$, A_1 and A_2 such that

- a sees A_1 and b sees A_2.
- For each $i \in \{1, 2, \}$, A_i is either a clique or an independent set such that one of the following conditions holds.
  - For each $i \in \{1, 2\}$, A_i is a clique of order at least 2 such that a sees b.
  - For each $i \in \{1, 2\}$, A_i is an independent set of order at least 2 such that ab is optional and further either A_1 sees A_2 or there is exactly one edge between A_1 and A_2.
  - A_1 is a clique with $|A_1| \geq 2$ and A_2 is independent in which case there is exactly one edge between A_1 and A_2.
  - A_1 is independent with $|A_1| \geq 2$ and $|A_2| = 1$ such that A_2 sees A_1 and ab is optional.
  - $|A_1| = 1$ and $|A_2| = 1$ in which case A_1 sees A_2 and a sees b.

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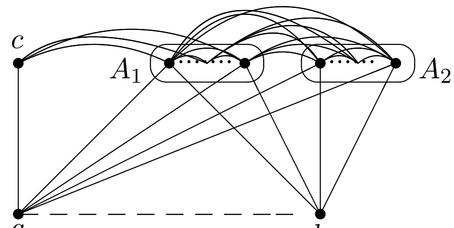

FIGURE 3. Family $\mathcal{F}_3$.

Family $\mathcal{F}_3$. A graph G is in $\mathcal{F}_3$ if its vertex-set can be partitioned into three nonempty sets $\{a, b, c\}$, A_1 and A_2 such that

- $A_1 \cup A_2$ is a clique.
- a sees $A_1 \cup A_2 \cup \{c\}$, b sees $A_1 \cup A_2$ and c sees $A_1 \cup \{a\}$ and ab is optional.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Notice that there is no other edge than those mentioned in the definition of the three families.

Our main result is stated as follows.

Theorem 1.2. *Let G be a connected graph of order $n \geq 4$. Then $\Gamma_L(G) = n - 2$ if and only if $G \in \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are described above.*

We start by proving the following proposition.

Proposition 1.3. *If G is a member of $\mathcal{F}$, then $\Gamma_L(G) = n - 2$.*

Proof 1.4. *By the definition of $\mathcal{F}$, it is clear that G is neither a star nor a complete graph. So, by Theorem 1.1, we have $\Gamma_L(G) \leq n - 2$. To show equality it suffices to construct a minimal locating-dominating set D of G of size $n-2$. To do this, we consider three situations. In the first one, assume that $G \in \mathcal{F}_1$. If $B_1 \cup B_2 \neq \emptyset$, then $D = A_1 \cup A_2 \cup B_1 \cup B_2$, where at most one of B_1 and B_2 is possibly empty. Whereas, if $B_1 \cup B_2 = \emptyset$, then $D = A_1 \cup A_2 \cup A_3$, where A_3 is possibly empty. In the second situation, assume that $G \in \mathcal{F}_2$. If either A_1 is a clique of order at least 2 and A_2 is an independent set or A_1 and A_2 are independent sets of order at least 2 then denote by xy be the unique edge joining A_1 to A_2 such that $x \in A_1$ and $y \in A_2$. In this case, $D = ((A_1 - \{x\}) \cup \{a\}) \cup ((A_2 - \{y\}) \cup \{b\})$. For the remaining cases, we take $D = A_1 \cup A_2$. In the last situation, assume that $G \in \mathcal{F}_3$. In this case, we take $D = A_1 \cup A_2 \cup \{c\}$. In each situation, it is a routine matter to check that D is a minimal LDS of G of order $n-2$. This implies that $\Gamma_L(G) \geq n - 2$.*

2. Definitions and remarks

Next we introduce the following definitions and notations. Straightforward, let S be a $\Gamma_L(G)$-set of size $n - 2$.

- $V - S = \{x, y\}$.
- $A_x = \{v \in S : v$ sees x and misses $y\}$.

- $A_y = \{v \in S : v$ sees y and misses $x\}$.
- $A_{xy} = \{v \in S : v$ sees $\{x, y\}\}$ and $B = \{v \in S : v$ misses $\{x, y\}\}$.
- $B_x = \{v \in B : v$ sees $A_x \cup A_{xy}$ and misses $A_y\}$,
- $B_y = \{v \in B : v$ sees $A_y \cup A_{xy}$ and misses $A_x\}$.

From these definitions, we deduce the following three remarks.

Remark 2.1. *For each $s, t \in \{x, y\}$ with $s \neq t$, we have*

$$B_s \cup \{s\} \text{ sees } A_s \cup A_{xy} \text{ and } t \text{ misses } A_s \cup B_s.$$

Remark 2.2. *The sets A_x, A_y, A_{xy} and B are pairwise disjoint sets. So, we can write*

$$S = A_x \cup A_y \cup A_{xy} \cup B.$$

Remark 2.3. *By defintion of S, one of A_x and A_y must be nonempty. So, we may assume, up to symmetry, that*

$$|A_x| \geq |A_y| \text{ with } |A_x| \geq 1 \text{ and } A_y \text{ posssibly empty.}$$

3. Preliminary Lemmas

In the remainder, to prove Theorem 1.2 we need the following serial lemmas. So, first let G be a connected graph and S a $\Gamma_L(G)$-set of size $n-2$. Moreover, according to remark 2.3, A_x is not empty.

Lemma 3.1. *If $B \neq \emptyset$, then*

(i) *B is an independent set, $B_x \cap B_y = \emptyset$ and $B = B_x \cup B_y$.*

(ii) *$|A_{xy}| \leq 1$ and for each $s \in \{x, y\}$, if $B_s \neq \emptyset$, then $|A_s| \leq 1$.*

(iii) *$A_y \neq \emptyset$ if and only if $A_{xy} = \emptyset$.*

Proof 3.2. *(i) If v is a vertex in B that sees some vertices in B, then it is easy to check that $S - \{v\}$ is an LDS of G, but this contradicts the minimality of S. So B is an independent set. The second part follows from the definition of B_x and B_y. Now, let us now prove the last part. Notice that $B_x \cup B_y \subseteq B$ by definition of B_x and B_y. Thus, it suffices to show that $B \subseteq B_x \cup B_y$. Suppose this is not true. Then $B - (B_x \cup B_y)$ has at least one vertex, say v. Since B is independent and G is connected, v must see at least one vertex in $S - B$. But this leads to the same contradiction as before. Thus $B - (B_x \cup B_y) = \emptyset$, so $B = B_x \cup B_y$.*

(ii) If $|A_{xy}| \geq 2$ (respectively, $|A_s| \geq 2$), then for every vertex v in A_{xy} (respectively, in A_s), $S - \{v\}$ is an LDS of G since v sees B (respectively, B_s), a contradiction. Thus $\max\{|A_{xy}|, |A_s|\} \leq 1$.

(iii) The 'only-if part' is obvious. So it suffices to show the 'if part'. If $A_{xy} \neq \emptyset$, then by item (ii), we can let $A_{xy} = \{v\}$. In this case, since v sees B and $\{x, y\}$ misses B, we see that $S - \{v\}$ should be an LDS of G, a contradiction.

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Lemma 3.3. *Let $D \in \{A_x, A_y, A_{xy}\}$ be a nonempty subset of S. Then D is either a clique or an independent set in G.*

Proof 3.4. *The case $|D| = 1$ or 2 is obvious. Thus, assume that $|D| \geq 3$. If D is neither a clique nor an independent set, then it has at least two nonadjacent vertices such that one of them, say v is not isolated in D. In this case, $S - \{v\}$ should be an LDS of G, a contradiction.*

Lemma 3.5. *Let $B = \emptyset$, $z \in A_{xy}$ and suppose that $A_y \neq \emptyset$, then we have the following.*

(i) *If z sees A_x (respectively, A_y), then z misses A_y (respectively, A_x).*

(ii) *A_x misses A_y.*

Proof 3.6. *(i) For otherwise, $S - \{z\}$ should be an LDS of G, a contradiction.*

(ii) To the contrary, suppose without loss of generality that u and v are two vertices with $u \in A_x$ and $v \in A_y$ such that u sees v. By item (i), we may suppose by symmetry that z misses u. In this case, since A_{xy} is nonempty, we see that $S - \{u\}$ is an LDS of G, which is impossible.

Lemma 3.7. *Let $B = \emptyset$, if A_{xy} is nonempty and A_x is a clique of order at least two, then*

(i) *$A_x \cup A_{xy}$ is a clique,*

(ii) *if $A_y \neq \emptyset$, then A_y is an independent set that misses $A_x \cup A_{xy}$.*

Proof 3.8. *(i) Let $u \in A_x$ and $z \in A_{xy}$. Since A_x is a clique, it is enough to show that z sees $A_x \cup A_{xy}$. Indeed, if z misses u, then $S - \{u\}$ is an LDS of G, a contradiction. Thus A_{xy} sees A_x. Furthermore, z sees all vertices in A_{xy}, otherwise A_{xy} must be an independent set according to Lemma 3.3. Using this together with the fact that z sees A_x, we see that $S - \{z\}$ is an LDS of G, a contradiction.*

(ii) From Lemma 3.7-(i) together with Lemma 3.5-(i) and (ii), we obtain that A_y misses $A_x \cup A_{xy}$. Now, let us show that A_y is an independent set. Indeed the case $|A_y| = 1$ is obvious. So, assume that $|A_y| \geq 2$ and suppose that A_y is not an independent set. Thus, A_y is a clique according to Lemma 3.3. In this case, Lemma 3.5-(i) shows that $A_y \cup A_{xy}$ is a clique, which contradicts the fact that A_y misses $A_x \cup A_{xy}$.

Lemma 3.9. *Let $B = \emptyset$. Suppose that A_{xy} is nonempty and A_x is an independent set of order at least two, then*

(i) *if $A_y \neq \emptyset$, then A_x misses A_{xy},*

(ii) *if $A_y \neq \emptyset$ is an independent set and S is not independent, then $|A_y| = 1$ and $A_y \cup A_{xy}$ is a clique.*

(iii) *if $A_y = \emptyset$ and $|A_{xy}| \geq 2$, then A_{xy} is a clique and A_x sees or misses A_{xy}.*

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Proof 3.10. *(i) Let $u \in A_x$. By Lemma 3.5-(ii), u misses A_y. This implies that u also misses A_{xy}, for otherwise $S - \{u\}$ should be an LDS of G, a contradiction. So, A_x misses A_{xy}.*

(ii) Let $z \in A_{xy}$. First, note that by Lemma 3.1-(iii), since A_{xy} and A_y are not empty, $B = \emptyset$. Now, let us show the first part. To the contrary suppose that $|A_y| \geq 2$. Then by item (i) and by symmetry A_y misses A_{xy}. Taking this together with (i) into consideration, we conclude that A_{xy} misses $A_x \cup A_y$. Using this together with Lemma 3.5-(ii) and the fact that S is not independent, we see that A_{xy} is a clique of order at least two. But in this case, $S - \{z\}$ is an LDS of G, a contradiction. Thus, $|A_y| = 1$. Now, it remains to prove the second part, that is $A_y \cup A_{xy}$ is a clique. In view of Lemmas 3.9-(i) and 3.5-(ii), A_x misses $A_y \cup A_{xy}$. It follows that if $|A_{xy}| = 1$, then $A_y \cup A_{xy}$ is a clique since $|A_y| = 1$ and S is not independent. Now, let us consider the case $|A_{xy}| \geq 2$ and let $A_y = \{v\}$. If A_{xy} is an independent set, then since S is not independent and A_x misses $A_y \cup A_{xy}$, v sees at least a vertex in A_{xy}, say z. In this case, $S - \{z\}$ should be an LDS of G, a contradiction. So A_{xy} is a clique. On the other part, if v misses some vertex, say z in A_{xy}, then $S - \{z\}$ is an LDS of G, a contradiction. Thus v sees A_{xy} implying that $A_y \cup A_{xy}$ is a clique.

(iii) Since $|A_{xy}| \geq 2$, from the first part of Lemma 3.1-(ii), $B = \emptyset$. First, suppose that A_{xy} is not a clique, then by Lemma 3.3, A_{xy} is an independent set. Knowing that A_x and A_{xy} are independent, A_y is empty and S is not independent, there is at least a vertex in A_{xy}, say z that sees at least a vertex in A_x. But in this case, $S - \{z\}$ is an LDS of G, a contradiction. So A_{xy} is a clique. Now let us show the second part. To the contrary suppose there is a vertex, say w belonging to A_x (respectively, A_{xy}) that sees a vertex and misses another one in A_{xy} (respectively, A_x). Since A_y is empty, $S - \{w\}$ should be an LDS of G, a contradiction.

Lemma 3.11. *If $|A_x| = |A_y| = 1$, $A_{xy} \neq \emptyset$ and S is not an independent set, then $A_x \cup A_{xy}$ or $A_y \cup A_{xy}$ is a clique.*

Proof 3.12. *Let $A_x = \{u\}$ and $A_y = \{v\}$. First, let us check the case $|A_{xy}| = 1$. By Lemma 3.5, u misses v. Since S is not independent, one of u and v must see the unique vertex in A_{xy}, so the assertion holds for $|A_{xy}| = 1$. Now, suppose that $|A_{xy}| \geq 2$. Then according to Lemma 3.1-(ii), $B = \emptyset$. Furthermore, since A_{xy} and A_y are nonempty, A_x misses A_y (by Lemma 3.5-(ii)). We begin by proving that A_{xy} is a clique. Suppose not. Suppose to the contrary that A_{xy} is an independent set. Then since $B = \emptyset$ and S is not independent, one of u and v, say u (by symmetry) must see at least a vertex in A_{xy}, say z. In this case, $S - \{z\}$ should be an LDS of G, a contradiction. Thus, A_{xy} is a clique. We shall show now that A_{xy} sees A_x or A_y. Indeed, if A_{xy} misses $A_x \cup A_y$, then for every vertex z in A_{xy}, $S - \{z\}$ is an LDS of G, a contradiction. So, there is at least an edge, say wv between A_{xy} and $A_x \cup A_y$ such that $w \in A_x \cup A_y$ and $z \in A_{xy}$. Up to symmetry, we may assume that $w = u$. If u misses a vertex in $A_{xy} - \{z\}$, then $S - \{u\}$ should be an LDS of G, a contradiction. So, u sees A_{xy}, implying that $\{u\} \cup A_{xy}$ is a clique. Thus, Lemma 3.11 holds.*

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4. Proof of Theorem 1.2

Proof 4.1. *From Proposition 1.3, the sufficiency is satisfied. Let us show now the necessity condition. Let S be a $\Gamma_L(G)$-set of size $n-2$ and $V-S=\{x,y\}$. In view of Lemmas 3.1 and 3.3, we may assume throughout this proof that $B=B_x\cup B_y$ and each set of A_x, A_y and A_{xy} is either a clique or an independent set. Next, we separate the proof in two cases, depending on whether A_{xy} is empty or not.*

Case 1. $A_{xy}\neq\emptyset$.

If S is an independent set, then in view of Remarks 2.1 and 2.2, we have $B=\emptyset$. In this case, we obtain that $G\in\mathcal{F}_1$ such that

$$a=x, b=y, A_1=A_x,\ A_2=A_{xy},\ A_3=A_y, B_1=B_x, B_2=B_y, \tag{4.1}$$

where $B_1\cup B_2=\emptyset$ and A_3 is possibly empty.

Now, suppose that S is not an independent set. Since A_{xy} and A_x are nonempty, Lemma 3.1-(ii) and (iii) show that

$$\text{if } B_s\neq\emptyset \text{ for } s\in\{x,y\},\ \text{then } |A_{xy}|=1 \text{ and } A_y=\emptyset. \tag{4.2}$$

Next, We distinguish the following two subcases according to the size of A_x.

Subcase 1.1. $|A_x|\geq 2$.

By the second part of Lemma 3.1-(ii), $B_x=\emptyset$. We discuss the structure of G depending on whether A_x is a clique or an independent set. Assume first that A_x is a clique. Since $B_x=\emptyset$, Lemma 3.7 shows that $G\in\mathcal{F}_1$ such that (4.1) is fulfilled, where $B_1=\emptyset$ and $A_3=A_y$ is possibly empty. Furthermore, using (4.2) we notice that if $|A_y|\geq 2$ or $|A_{xy}|\geq 2$, then $B_2=B_y=\emptyset$. However, if $|A_y|\leq 1$ and $|A_{xy}|=1$, then B_2 is possibly nonempty.

Now, assume that A_x is an independent set. It follows by symmetry that the case A_y is a clique with at least two vertices is already treated above. Therefore, for the rest of Case 1.1 we will assume that A_y (if exists) is an independent set. According to Lemma 3.9-(ii) and (iii), $|A_y|\leq 1$.

First assume that $|A_y|=1$. Note that by (4.2), we have $B_y=\emptyset$. Then since $B_x=\emptyset$, $B=\emptyset$. In this case, by taking Lemmas 3.5-(ii) and 3.9-(i) and (ii) into consideration, we obtain that $G\in\mathcal{F}_1$, with $a=y$, $b=x$, $A_1=A_y$, $A_2=A_{xy}$, $A_3=A_x$ and $B_1\cup B_2=\emptyset$.

Now, suppose that $A_y=\emptyset$. We have the following two possibilities according to the size of A_{xy}.

Possibility 1. $|A_{xy}|\geq 2$.

By Lemma 3.1-(ii), $B=\emptyset$. Thus, Lemma 3.9-(iii), implies that $G\in\mathcal{F}_1$, where (4.1) is satisfied and $A_3=B_1\cup B_2=\emptyset$.

Possibility 2. $|A_{xy}|=1$. *Let $A_{xy}=\{z\}$.*

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If z misses A_x, then $B_y \neq \emptyset$ since S is not independent. Therefore, we obtain again that $G \in \mathcal{F}_1$, where (4.1) is satisfied, $B_2 \neq \emptyset$ and $B_1 \cup A_3 = \emptyset$. Now, assume that z sees a vertex in A_x, say u and define the following set.

$$R_1 = \{w \in A_x \cup \{y\} : w \text{ sees } \{x, z\}\}.$$

Observe that $R_1 \neq \emptyset$ since u sees $\{x, z\}$. If $y \notin R_1$, then x misses y and so $G \in \mathcal{F}_1$ such that

$$\begin{aligned} &a = z,\ b = x,\ A_1 = B_y \cup \{y\},\ A_2 = R_1,\ A_3 = A_x - R_1, \\ &\text{where } B_1 \cup B_2 = \emptyset \text{ and } A_x - R_1 \text{ is possibly empty.} \end{aligned} \tag{4.3}$$

Assume now that $y \in R_1$; so x sees y. In this case, $G \in \mathcal{F}_1$ such that either $B_y \neq \emptyset$, so (4.3) is fulfilled with $A_1 = B_y$; or $B_y = \emptyset$, and in this case $a = x$, $A_3 \cup B_1 \cup B_2 = \emptyset$ and one of the following options holds.

- *$A_x - R_1 \neq \emptyset$, so $b = z$, $A_1 = A_x - R_1$ and $A_2 = R_1$.*
- *$A_x - R_1 = \emptyset$, so $b = y$, $A_1 = A_x$ and $A_2 = \{z\}$.*

Subcase 1.2. $|A_x| = 1$.

In view of Remark 2.3, we have $|A_y| \leq 1$. If $|A_y| = 1$, then since A_{xy} and A_y are nonempty, Lemma 3.1-(iii) shows that $B = \emptyset$. Thus, in view of Lemma 3.11 and up to symmetry, we may suppose that $A_x \cup A_{xy}$ is a clique. By Lemma 3.5 and the fact that $B = \emptyset$, we get that $G \in \mathcal{F}_1$, where (4.1) is fulfilled and $B_1 \cup B_2 = \emptyset$.

So, assume that $A_y = \emptyset$. We argue by considering the following three possibilities for $A_x \cup A_{xy}$. First, we may assume that $A_x \cup A_{xy}$ is a clique. If $|A_{xy}| = 1$, then $G \in \mathcal{F}_1$, where (4.1) is satisfied, $A_3 = \emptyset$ and each of B_1 and B_2 is possibly empty. If $|A_{xy}| \geq 2$, then by Lemma 3.1-(ii), $B = \emptyset$. In this case, again $G \in \mathcal{F}_1$, where (4.1) holds and $A_3 \cup B_1 \cup B_2 = \emptyset$.

Next, assume that $A_x \cup A_{xy}$ is an independent set. Then since S is not independent, one of B_x and B_y must be nonempty. On the other hand, Lemma 3.1-(ii) shows that since $A_{xy} \neq \emptyset$, A_{xy} must be a singleton. But, again $G \in \mathcal{F}_1$, where (4.1) is fulfilled, $A_3 = \emptyset$ and at least one of B_1 and B_2 is nonempty.

Finally, we may assume that $A_x \cup A_{xy}$ is neither a clique nor an independent set. Then $|A_{xy}| \geq 2$ and so by Lemma 3.1-(ii), $B = \emptyset$. If A_x misses A_{xy}, then A_{xy} is a clique since S is not independent. In this case, $G \in \mathcal{F}_1$, with $a = x$, $b = y$, $A_1 = A_x$, $A_2 = A_{xy}$ and $A_3 \cup B_1 \cup B_2 = \emptyset$. So, assume that A_x sees at least a vertex in A_{xy}, say z. If A_{xy} is independent, then since $|A_{xy}| \geq 2$ and $B = \emptyset$, $S - \{z\}$ should be an LDS of G, a contradiction. So, A_{xy} is a clique. Consequently, let $A_x = \{u\}$ and R_2 be a set defined as follows.

$$R_2 = \{w \in A_{xy} : w \text{ sees } u\}.$$

Clearly $R_2 \neq \emptyset$ (since u sees z), and also $A_{xy} - R_2 \neq \emptyset$ since $\{u\} \cup A_{xy}$ is not a clique. In this case, $G \in \mathcal{F}_3$ with $a = x$, $b = y$, $c = u$, $A_1 = R_2$ and $A_2 = A_{xy} - R_2$.

Case 2. $A_{xy} = \emptyset$.

Then $A_y \neq \emptyset$. *By Remark 2.3, we have* $|A_x| \geq |A_y| \geq 1$. *Let* $u \in A_x$ *and* $v \in A_y$, *we assume first that* S *is an independent set of* G. *It follows that* $B = \emptyset$, *and* x *sees* y *since* G *is connected. We consider the case* $|A_x| = 1$. *This yields* $A_x = \{u\}$ *implying by Remark 2.3 that* $A_y = \{v\}$. *In this case* $G \in \mathcal{F}_2$ *with* $a = x$, $b = y$, $A_1 = \{u\}$ *and* $A_2 = \{v\}$. *If* $|A_x| \geq 2$, *then* $G \in \mathcal{F}_1$ *with* $a = x$, $b = v$, $A_1 = A_x$, $A_2 = \{y\}$, $A_3 \cup B_1 = \emptyset$ *and* $B_2 = A_y - \{v\}$ *(possibly empty).*

From now on, we may assume that S *is not an independent set. We distinguish two subcases*

Subcase 2.1.

$|A_x| \geq 2$. *Then by Lemma 3.1-(ii), we have* $B_x = \emptyset$. *We discuss by considering whether* A_x *misses* A_y *or not.*

First, we assume that A_x *misses* A_y. *Then* x *sees* y *since* G *is connected. The following three possibilities can be considered.*

Possibility 1. A_x *and* A_y *both are independent.*

Since S *is not independent,* $B_y \neq \emptyset$ *and consequently by Lemma 3.1-(ii), we have* $A_y = \{v\}$. *In this case,* $G \in \mathcal{F}_1$ *such that* $a = x$, $b = v$, $A_1 = A_x$, $A_2 = \{y\}$, $A_3 = B_y$ *and* $B_1 \cup B_2 = \emptyset$.

Possibility 2. A_x *is a clique.*

Let $u \in A_x$. *Assume first that* A_y *is independent. If* $|A_y| \geq 2$, *then* $B_y = \emptyset$ *(by Lemma 3.1-(ii)) and so* $G \in \mathcal{F}_1$ *such that* $a = u$, $b = y$, $A_1 = A_x - \{u\}$, $A_2 = \{x\}$, $A_3 = A_y$ *and* $B_1 \cup B_2 = \emptyset$. *Now consider the case* $|A_y| = 1$. *This means that* $A_y = \{v\}$, *so we distinguish the following two options.*

- *If* $B_y = \emptyset$, *then* $G \in \mathcal{F}_1$ *such that* $a = u$, $b = y$, $A_1 = A_x - \{u\}$, $A_2 = \{x\}$, $A_3 = \{v\}$ *and* $B_1 \cup B_2 = \emptyset$.
- *If* $B_y \neq \emptyset$, *then* $G \in \mathcal{F}_2$ *such that* $a = u$, $b = v$, $A_1 = (A_x - \{u\}) \cup \{x\}$, $A_2 = B_y \cup \{y\}$.

Possibility 3. A_y *is a clique of order at least 2.*

This means that $B_y = \emptyset$ *and therefore* $G \in \mathcal{F}_2$ *with* $a = x$, $b = y$, $A_1 = A_x$, $A_2 = A_y$.

So, we may assume that A_x *does not miss* A_y *and there are two vertices* $u \in A_x$ *and* $v \in A_y$ *such that* u *sees* v. *In consequence,* A_x *must be an independent set, for otherwise* $S - \{u\}$ *is an LDS of* G, *a contradition. We consider two situations.*

Situation 1. $|A_y| \geq 2$.

Then by Lemma 3.1-(ii), $B_y = \emptyset$, *implying that* $B = \emptyset$ *(since* $B_x = \emptyset$*) and by symmetry,* A_y *is also an independent set. Then we assert that* A_x *sees* A_y. *Indeed, if* u *misses some vertex in* A_y, *then* $S - \{u\}$ *is an LDS of* G *(since* u *sees* v*), a contradiction. Thus* u *sees* A_y *and by symmetry,* v *also sees* A_x. *Similar arguments applied to every vertex in* $A_x \cup A_y$ *show that* A_x *sees* A_y. *So, the assertion is true.*

Since A_x *and* A_y *both are independent sets of order at least two and* $B = \emptyset$, *we deduce that* $G \in \mathcal{F}_2$ *such that* $a = x$, $b = y$, $A_1 = A_x$ *and* $A_2 = A_y$.

Situation 2. $|A_y| = 1$.

http://dx.doi.org/10.22108/toc.2025.139873.2128

Then $A_y = \{v\}$. Let us define the following set.

$$R_3 = \{w \in A_x : w \text{ sees } v\}.$$

Clearly $R_3 \neq \emptyset$, since v sees u. If $B_y \neq \emptyset$, then $G \in \mathcal{F}_1$ with $a = v$, $b = x$, $B_1 \cup B_2 = \emptyset$, $A_3 = A_x - R_3$ (possibly empty). Furthermore, we have two options.

- *If x sees y, then $A_1 = B_y$ and $A_2 = R_3 \cup \{y\}$.*
- *If x misses y, then $A_1 = B_y \cup \{y\}$ and $A_2 = R_3$.*

Assume now that $B_y = \emptyset$. If x misses y, then $G \in \mathcal{F}_1$ with $a = v$, $b = x$, $A_1 = \{y\}$, $A_2 = R_3$, $B_1 \cup B_2 = \emptyset$ and $A_3 = A_x - R_3$ (possibly empty). If x sees y, then we have two following options.

- *$A_x - R_3 = \emptyset$, implying that $G \in \mathcal{F}_2$ with $a = x$, $b = y$, $A_1 = A_x$, $A_2 = A_y$.*
- *$A_x - R_3 \neq \emptyset$, implying that $G \in \mathcal{F}_1$ with $a = x$, $b = v$, $A_1 = A_x - R_3$, $A_2 = R_3 \cup \{y\}$ and $A_3 \cup B_1 \cup B_2 = \emptyset$.*

Subcase 2.2.

$|A_x| = 1$. Then by Remark 2.3, we have $|A_y| = 1$. Let $A_x = \{u\}$ and $A_y = \{v\}$. First, assume that $B = \emptyset$. Since S is not independent, u sees v. If x sees y, then $G \in \mathcal{F}_2$ with $a = x$, $b = y$, $A_1 = \{u\}$ and $A_2 = \{v\}$. Otherwise, if x misses y, then $G \in \mathcal{F}_1$ with $a = u$, $b = y$, $A_1 = \{x\}$, $A_2 = \{v\}$ and $A_3 \cup B_1 \cup B_2 = \emptyset$.

Next, assume that the independent set B is nonempty; so at least one of B_x and B_y (say B_x) is nonempty. As G is connected, either x sees y or u sees v. Assume first that x sees y and u misses v, this leads to the following two options.

- *If $B_y = \emptyset$, then $G \in \mathcal{F}_1$ with $a = u$, $b = y$, $A_1 = B_x$, $A_2 = \{x\}$, $A_3 = \{v\}$ and $B_1 \cup B_2 = \emptyset$.*
- *If $B_y \neq \emptyset$, then $G \in \mathcal{F}_2$ with $a = u$, $b = v$, $A_1 = B_x \cup \{x\}$ and $A_2 = B_y \cup \{y\}$.*

Now, assume that x misses y and u sees v. In this case, $G \in \mathcal{F}_1$ with $a = u$, $b = y$, $A_1 = B_x \cup \{x\}$, $A_2 = \{v\}$, $B_2 = B_y$ (possibly empty) and $A_3 \cup B_1 = \emptyset$.

Finally, assume x sees y and u sees v. Again we consider two options.

- *If $B_y = \emptyset$, then $G \in \mathcal{F}_1$ with $a = u$, $b = y$, $A_1 = B_x$, $A_2 = \{x, v\}$ and $A_3 \cup B_1 \cup B_2 = \emptyset$.*
- *If $B_y \neq \emptyset$, then $G \in \mathcal{F}_2$ with $a = u$, $b = v$, $A_1 = B_x \cup \{x\}$ and $A_2 = B_y \cup \{y\}$.*

This achieves the proof of Theorem 1.2.

Acknowledgments

The authors wish to thank the referee for their comments and valuable suggestions, which have helped improve the manuscript $\cdots$

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http://dx.doi.org/10.22108/toc.2025.139873.2128