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THE INFINITE INTERSECTION PROPERTY OF GROUPS

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Abstract. A group G is said to satisfy the infinite trivial intersection property (ITIP for short), if for every pair of finite subgroups U, V such that $U \cap V = 1$, there exist infinite subgroups X and Y of G such $U \leq X$ and $V \leq Y$ and $X \cap Y = 1$. We shall say that a group G satisfies the infinite non-trivial intersection property (INIP) if every pair of infinite subgroups of G intersect non-trivially. The subject of this paper is to find classes of groups that satisfy ITIP. We prove, among other things, that every periodic locally nilpotent non-Chernikov group satisfies ITIP. The Prüfer-by-finite p-groups are examples of locally nilpotent Chernikov groups that do not satisfy ITIP. We then characterize locally nilpotent groups that satisfy INIP and structure theorems are given in the periodic and the non-periodic case.

1. Introduction

The structure of groups with an intersection property on given systems of subgroups has been investigated for several different choices of the property. See for example [2], [3], [4], [8], [9]. In particular, in [9] G. Walls classified finite p-groups all of whose proper subgroups have trivial intersection. The description of groups with non-trivial intersection property for every pair of non-trivial subgroups was a further step forward in this direction. Abelian groups with non-trivial intersection of every pair of non-trivial subgroups are isomorphic to a subgroup of either the additive group $\mathbb{Q}$, or the Prüfer

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group C_{p^∞} for some prime p. A. I. Sozutov in [8] described the non-abelian residually finite groups with non-trivial intersection of every pair of non-trivial subgroups.

In the present work, we shall find classes of groups that satisfy the *infinite trivial intersection property*. Here a group G is said to have the infinite trivial intersection property (ITIP for short), if for every pair of finite subgroups U, V of G such that $U \cap V = 1$, there exist infinite subgroups X and Y of G such that $U \leq X$ and $V \leq Y$ and $X \cap Y = 1$. It is not difficult to see that ITIP holds for groups that are the direct product of infinitely many non-trivial subgroups. On the other hand there are groups for which this property fails. Tarski groups, that is infinite simple groups whose proper non-trivial subgroups have prime order, are examples of groups that do not satisfy ITIP. Other examples of such groups include the additive group $\mathbb{Q}$ of rational numbers, Prüfer groups and the Prüfer-by-finite groups (as shown in Corollary 2.8 below). The question of whether a Chernikov group does not satisfy ITIP can be reduced to the radicable part -the case where the group is the direct product of finitely many Prüfer groups. In our result (Proposition 2.10) we show that the group does not satisfy ITIP if it is the direct product of finitely many pairwise nonisomorphic Prüfer groups. The direct product of finitely many isomorphic copies of the Prüfer p-group for the prime p also does not satisfy ITIP (Proposition 2.11). Therefore, Chernikov groups fail to satisfy ITIP. At this point, would be convenient for us to exclude from consideration the case of Tarski and Chernikov groups. We prove that a periodic non-Chernikov locally nilpotent group satisfies ITIP (Theorem 2.12). As a consequence, we obtain that every periodic locally nilpotent group with a trivial center satisfies ITIP (Corollary 2.13). For non-periodic locally nilpotent groups that do not satisfy ITIP, we will show that if $G = T \times M$, where T is the torsion subgroup of G and M is a torsion-free subgroup of G, then G is soluble of finite rank (Theorem 2.14).

The last section is devoted to study the *infinite non-trivial intersection property*. A group G is said to have the infinite non-trivial intersection property (abbreviated INIP) provided that the intersection of every pair of infinite subgroups of G is non-trivial. The result that periodic non-Chernikov locally nilpotent groups satisfy ITIP will be used to characterize locally finite groups with INIP. Thus we shall see in Theorem 3.1 that an infinite locally finite group satisfies INIP if and only if it it is a Prüfer-by-finite. We shall also see that a non-periodic locally nilpotent group G satisfies INIP if and only if it is an (torsion-free locally cyclic)-by-finite group (Theorem 3.3).

Most of our notation is standard and can be found in [6].

2. The Infinite Trivial Intersection Property

The first result of this section shows that periodic locally nilpotent non-Chernikov groups satisfy ITIP. This will be done through a series of lemmas. We start with two simple observations.

Lemma 2.1. *Let $G = Dr_{i\in\mathbb{N}}G_i$, where G_i is a non-trivial subgroup of G. Then G satisfies ITIP.*

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Proof. Let U and V be finite subgroups with trivial intersection. Then there are $i_1, \ldots, i_n \in \mathbb{N}$ such that $U, V \leq Dr_{j=1}^{n} G_{i_j}$. If we choose $X = U \times G_{i_{n+1}} \times G_{i_{n+3}} \times \cdots \times G_{i_{n+2k+1}} \cdots$ and $Y = V \times G_{i_{n+2}} \times G_{i_{n+4}} \times \cdots \times G_{i_{n+2k}} \cdots$, then $X \cap Y = 1$ and thus G satisfies $ITIP$. □

Lemma 2.2. *Let G be a group and let A be an abelian subgroup of infinite rank of G. Then for every finite subgroup U of G there exists a non-trivial subgroup W of A such that $U \cap W = 1$.*

Proof. Assume for a contradiction that the statement is false, that is, there exists a finite subgroup U of G that intersects non-trivially with all non-trivial subgroups of A. Clearly, A should be periodic. If A has infinitely many components, then it satisfies ITIP by Lemma 2.1. Therefore, A has two infinite subgroups X and W containing $U \cap A$ and $V = 1$ respectively such that $X \cap W = 1$, and hence

$$U \cap W = U \cap (W \cap A) = (U \cap A) \cap W = X \cap W = 1,$$

a contradiction. Now assume that A has a component A_p of infinite rank, so its socle S also has infinite rank and indeed S is isomorphic to a direct product of infinitely many copies of C_p. Let M be a direct factor of S which is isomorphic to C_p. Since $U \cap M \neq 1$ it follows that $S \leq U$ and we get the contradiction that establishes the result. □

The following is a well-known result by W. Möhres [5], which is enormously useful and a key in the proof of our results.

Lemma 2.3. *Let G be a nilpotent p-group and H be a normal subgroup such that G/H is an infinite elementary abelian group. Let U be a finite subgroup of G and $a \in G \setminus U$. Then G has a subgroup L containing U such that $a \notin L$ and LH/H is infinite.*

In connection with Möhres's result one may consider the following property. Let H be a normal subgroup of a group G such that G/H is infinite elementary abelian. We say that G satisfies the property $(\mathfrak{P})$ with respect to H, if for every element $1 \neq a \in G$ and for every finite subgroup U of G with $a \notin U$, there exists a subgroup L of G containing U such that $a \notin L$ and LH/H is infinite. The next result gives a condition under which a locally finite group satisfies $(\mathfrak{P})$.

Lemma 2.4. *Let G be a locally finite group and H be a normal subgroup of G such that G/H is infinite elementary abelian. If*

$$\bigcap_{y \in G \setminus VH} \langle V, y \rangle = V$$

for every finite subgroup V of G, then G satisfies $(\mathfrak{P})$.

Proof. Let U be a finite subgroup of G and let $a \in G \setminus U$. By hypothesis

$$\bigcap_{y \in G \setminus UH} \langle U, y \rangle = U.$$

Then G has an element $y_1 \in G \setminus UH$ such that $a \notin U_1 := \langle U, y_1 \rangle$. Assume that G has elements $y_1, \ldots, y_n$ such that $a \notin U_i := \langle U, y_1, \ldots, y_i \rangle$ and $y_i \notin \langle U, y_1, \ldots, y_{i-1} \rangle H$ where $1 \leq i \leq n$ and $U_0 = U$. By hypothesis

$$\bigcap_{y \in G \setminus U_n H} \langle U_n, y \rangle = U_n.$$

Then there exists $y_{n+1} \in G \setminus U_n H$ such that $a \notin U_{n+1} := \langle U, y_1, \ldots, y_n, y_{n+1} \rangle$. Put $L := \bigcup_{i \in \omega} U_i$, then clearly L is infinite, $U \leq L$ and $a \notin L$ and LH/H is infinite. So G has the property $(\mathfrak{P})$, as claimed. □

Lemma 2.5. *Let G be a nilpotent p-group and H be a normal subgroup such that G/H is an infinite elementary abelian group. Let U and V be finite subgroups of G with $U \cap V = 1$. Then G has a subgroup K such that $U \neq \langle U, K \rangle$ and $V \cap \langle U, K \rangle = 1$.*

Proof. We proceed by induction on $m := |V|$. Let $m = 1$. If we choose $g \in G \setminus U$, then $U \neq \langle U, g \rangle$ and $\langle U, g \rangle \cap V = 1$. So we may assume that there is $1 \neq v \in V$. Now there is $L \leq G$ containing U such that $v \notin L$ and LH/H is infinite by Lemma 2.3. Now $|V \cap L| < |V|$ and $L/(L \cap H)$ is an infinite elementary abelian group. Since L also satisfies the hypothesis of the Lemma it follows, by induction, that L contains a subgroup K such that $U \neq \langle U, K \rangle$ and $\langle U, K \rangle \cap (L \cap V) = 1$ and

$$\langle U, K \rangle \cap V = (\langle U, K \rangle \cap L) \cap V = \langle U, K \rangle \cap (L \cap V) = 1,$$

the result follows. □

We deduce the following lemma which is a tool for our purpose.

Lemma 2.6. *Let G be a nilpotent p-group with a normal subgroup H such that G/H is an infinite elementary abelian group. Then G satisfies ITIP.*

Proof. Let U, V be finite subgroups of G such that $U \cap V = 1$. We note that, by Lemma 2.2 and Lemma 2.5, there are sequences of finite groups

$$U \gneqq U_1 \gneqq U_2 \gneqq \cdots \gneqq U_i \gneqq U_{i+1} \gneqq \cdots$$

and

$$V \gneqq V_1 \gneqq V_2 \gneqq \cdots \gneqq V_i \gneqq V_{i+1} \gneqq \cdots$$

such that $U_i \cap V_j = 1$ for every $i, j \geq 1$.
So if we put $X := \bigcup_{i=1}^{\infty} U_i$ and $Y := \bigcup_{j=1}^{\infty} V_j$, then we obtain $X \cap Y = 1$. This completes the proof. □

The next lemma shows, in particular, that ITIP is inherited by subgroups of finite index.

Lemma 2.7. *Let N be a subgroup of finite index of a group G, and let X be an infinite subgroup of G. Then $X \cap N$ is infinite.*

Proof. Assume that $|X \cap N| < \infty$, then X is finite, since $|X : X \cap N| \leq |XN : N|$ is finite. Thus, we have a contradiction that establishes the result. □

As a special case, we can state the following.

Corollary 2.8. *Let G be a Prüfer-by-finite group. Then all its infinite subgroups intersect non-trivially.*

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many Prüfer groups. In view of Lemma 2.7, the question of whether all Chernikov groups do not satisfy ITIP will be reduced to the direct product of finitely many Prüfer groups. Our first consideration is the case where the group is the direct product of finitely many pairwise nonisomorphic Prüfer groups.

Lemma 2.9. *Let $G = H_1 \times H_2 \times \cdots \times H_n$, where $H_i \simeq C_{p_i^\infty}$ with $1 \leq i \leq n$. Then for every infinite subgroups X and Y of G such that $X \cap Y = 1$, there exists $1 \leq i_1 < \cdots < i_n \leq n$ and $1 \leq j_1 < \cdots < j_n \leq n$ with $i_k \neq j_{k'}$ such that $X \leq H_{i_1} \times H_{i_2} \times \cdots \times H_{i_k}$ and $Y \leq H_{j_1} \times H_{j_2} \times \cdots \times H_{j_{k'}}$ for $k, k' = 1, \ldots, n$.*

Proof. Set $X_i = X \cap H_i$ and $Y_i = Y \cap H_i$, $i = 1, 2, \ldots, n$. Then X_i (resp. Y_i) is the p_i-component subgroup of X (resp. of Y) and $X = X_1 \times X_2 \times \cdots \times X_n$ and $Y = Y_1 \times Y_2 \times \cdots \times Y_n$. Since $X \cap Y = 1$, then $X_i \cap Y_i = 1$ for $i = 1, 2, \ldots, n$. Also, since $X_i, Y_i \leq H_i$, either we have that $X_i = 1$ or $Y_i = 1$. The proof is complete. □

Proposition 2.10. *Let $G = H_1 \times H_2 \times \cdots \times H_n$, where $H_i \simeq C_{p_i^\infty}$ with $1 \leq i \leq n$. Then G does not satisfy ITIP.*

Proof. Suppose that G satisfies ITIP, and take $U = U_1 \times U_2 \times \cdots \times U_n$, where U_i is a non-trivial subgroup of H_i with $1 \leq i \leq n$ and $V = 1$. An application of Lemma 2.9 shows that G has no infinite subgroups containing U and V respectively that intersect trivially, which is a contradiction. □

Now we examine the case of the direct product of finitely many isomorphic copies of the Prüfer p-group.

Proposition 2.11. *Let $G = H_1 \times H_2 \times \cdots \times H_n$, where $H_i \simeq C_{p^\infty}$ with $1 \leq i \leq n$. Then G does not satisfy ITIP.*

Proof. Let S be the socle of G. Then S is a non-trivial finite subgroup of G, and any non-trivial subgroup of G has non-trivial intersection with S. Hence, G does not satisfy ITIP. □

We turn next to periodic non-Chernikov locally nilpotent groups and we prove our first main Theorem.

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Theorem 2.12. *Let G be a periodic non-Chernikov locally nilpotent group. Then G satisfies ITIP.*

Proof. Let U, V be finite subgroups of G such that $U \cap V = 1$. Since G is periodic locally nilpotent, it is the direct product of its primary components. If there are infinitely many of such components, then G satisfies the ITIP by Lemma 2.1. Therefore we may assume that finitely many primary components may exist, that is, $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_n}$ where, for each $i \in \{1, 2, \ldots, n\}$, p_i is a prime and G_{p_i} is a p_i-component of G. Since G is not Chernikov, then at least one of the primary components of G is not Chernikov. Assume that G_{p_1} is not Chernikov. Since U, V are finite subgroups of G such that $U \cap V = 1$, there exist finite subgroups $U_1, V_1 \leq G_{p_1}$, $U_2, V_2 \leq G_{p_2}, \ldots$ and $U_n, V_n \leq G_{p_n}$ such that

$$U \leq U_1 \times U_2 \times \cdots \times U_n \text{ and } V \leq V_1 \times V_2 \times \cdots \times V_n.$$

Since G_{p_1} is not Chernikov and since $U_1, V_1 \leq G_{p_1}$, $C_{G_{p_1}}(\langle U_1, V_1 \rangle)$ contains an infinite elementary abelian subgroup E (applying [1, Lemma 2.4.]). Put $W := E \langle U_1, V_1 \rangle$, then W is a nilpotent p_1-group and $W / \langle U_1, V_1 \rangle$ is infinite and elementary abelian. Therefore W and hence G_{p_1} satisfies $ITIP$ by Lemma 2.6. It follows that there exist two infinite subgroups X_1 and Y_1 of G_{p_1} containing U_1 and V_1, respectively, such that $X_1 \cap Y_1 = 1$. Now let

$$X := X_1 \times U_2 \times \cdots \times U_n \text{ and } Y := Y_1 \times V_2 \times \cdots \times V_n$$

Then $X \cap Y = 1$, and so G satisfies ITIP. This completes the proof. □

Theorem 2.12 has the following immediate consequence.

Corollary 2.13. *Every periodic locally nilpotent group with trivial center satisfies ITIP.*

Proof. Let U, V be finite subgroups of G such that $U \cap V = 1$. Since a locally nilpotent Chernikov group is hypercentral, we see that G is certainly non-Chernikov. So by Theorem 2.12, G satisfy ITIP and so the result follows. □

We consider now the case of a locally nilpotent group G that does not satisfy ITIP such that $G = T \times M$, where T is the torsion subgroup of G and M is a torsion-free subgroup. Our next theorem shows in particular that such groups are soluble of finite rank.

Theorem 2.14. *Let G be a locally nilpotent group such that $G = T \times M$, where T is the torsion subgroup of G and M is a torsion-free subgroup of G. If G does not satisfies ITIP, then T is a Chernikov group and $M \simeq B \leq \mathbb{Q}$.*

Proof. By Theorem 2.12, the torsion subgroup of G is Chernikov. We claim that $\langle a \rangle \cap \langle b \rangle \neq 1$ for all non-trivial elements a and b in M. For otherwise, assume that M contains two non-trivial elements a and b such that $\langle a \rangle \cap \langle b \rangle = 1$. If U and V are two finite subgroups of G such that $U \cap V = 1$, then by putting $X = U \times \langle a \rangle$ and $Y = V \times \langle b \rangle$, we shall have that G satisfies ITIP. Hence M has no any pair of distinct cyclic subgroups that intersect trivially. Since finitely generated torsion-free abelian groups can be expressed as the direct sum of infinite cyclic groups, we deduce that M must be locally cyclic. Thus $M \simeq B \leq \mathbb{Q}$ as required. □

3. The Infinite Non-Trivial Intersection Property

In this short section we prove that locally nilpotent groups satisfying the infinite non-trivial intersection property (INIP) are (locally cyclic)-by-finite. To this purpose, we examine the classes of periodic groups and non-periodic locally nilpotent groups.

The first result provides necessary and sufficient conditions for a locally finite group to satisfy INIP, the proof requires the result that periodic non-Chernikov abelian groups satisfy ITIP (Theorem 2.12).

Theorem 3.1. *An infinite locally finite group G satisfies INIP if and only if it is a Prüfer-by-finite group.*

Proof. Let G be an infinite locally finite group satisfying INIP. If every abelian subgroup of G is Chernikov, then G is Chernikov by [7] and therefore it is a Prüfer-by-finite group. Hence we may assume that G has an abelian non-Chernikov subgroup A. But a non-Chernikov periodic abelian group satisfies ITIP by Theorem 2.12, so A must be a Chernikov group, and we have a contradiction.

The converse follows by Corollary 2.8. □

Next, we identify the structure of torsion-free locally nilpotent groups with INIP.

Lemma 3.2. *Let G be a torsion-free locally nilpotent group. Then, G satisfies INIP if and only if G is locally cyclic and hence it is isomorphic to a subgroup of the additive group $\mathbb{Q}$ of rational numbers.*

Proof. Let E be a finitely generated subgroup of G. Then E is a residually finite p-group for every prime p. Since G is torsion-free, every proper non-trivial subgroup of G is infinite, and hence by the hypothesis, all proper subgroups of G intersect non-trivially. If G is non-abelian, then by Theorem of [8], G is a finite (generalized) quaternion group, which is a contradiction. Therefore G is abelian, and so E is the direct sums of infinite cyclic subgroups, and we deduce that E must be cyclic. The lemma is proved. □

We finish by describing non-periodic locally nilpotent groups that satisfy INIP.

Theorem 3.3. *A non-periodic locally nilpotent group G satisfies INIP if and only if it is a (torsion-free locally cyclic)-by-finite group..*

Proof. Let T be the torsion subgroup of G. Since G is a non-periodic locally nilpotent group in which all infinite subgroups intersect non-trivially, T must be finite. Since the automorphism group of a finite group is finite, $G/C_G(T)$ is finite. On the other hand, we claim that the factor group G/T is a torsion-free locally cyclic group. By Lemma 3.2, it is enough to show that G/T satisfies INIP. Otherwise there would exist two infinite cyclic subgroups with trivial intersection in G/T, hence also in G which is a contradiction. So, by requirement, G/T is torsion-free and locally cyclic. It follows that $C_G(T)/T \cap C_G(T)$ is also torsion-free and locally cyclic. But since $T \cap C_G(T)$ is contained in the centre of $C_G(T)$, we deduce that $C_G(T)$ is abelian. Now if $C_G(T)$ is periodic, then it is obviously finite. If $C_G(T)$ is non-periodic, let F be its torsion subgroup, then it is possible to write $C_G(T) = F \times L$, where L is torsion-free locally cyclic. It follows that $C_G(T)$ is (torsion-free locally cyclic)-by-finite. Therefore, G is (torsion-free locally cyclic)-by-finite and the theorem is proved. □

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References

[1] A. O. Asar, Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov, *J. London Math. Soc. (2)* **61** no. 2 (2000) 412–422.

[2] C. J. B. Brookes and H. Heineken, Locally nilpotent groups with an intersection property, *Arch. Math. (Basel),* **44** no. 6 (1985) 488–492.

[3] J. Hempel, The finitely generated intersection property for Kleinian groups, *Knot theory and manifolds,* Springer, Berlin, Heidelberg, (1985) 18–24.

[4] S. R. Li and X. Y. Guo, Finite p-groups whose abelian subgroups have a trivial intersection, *Acta Math. Sin. (Engl. Ser.)* **23** no. 4 (2007) 731–734.

[5] W. Möhres, Torsionsgruppen, deren Untergruppen alle subnormal sind, *Geom. Dedicata*, **31** no. 2 (1989) 237–244.

[6] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part I, II, Springer-Verlag, Berlin, 1972.

[7] V. P. Shunkov, On locally finite groups with a minimality condition for Abelian subgroups, *Algebr Logic*, **9** (1970) 350–370.

[8] A. I. Sozutov, Residually finite groups with nontrivial intersection of every subgroup pair, *Siberian. Math. J.*, **41** no. 2 (2000) 362–365.

[9] G. Walls, Trivial intersection groups, *Arch. Math. (Basel)*, **32** no. 1 (1979) 1-4.

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