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G-DESIGNS FOR THE CONNECTED TRIANGULAR BICYCLIC GRAPHS WITH NINE EDGES

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Abstract. A G-design of order n is a decomposition of the complete graph K_n into isomorphic copies of G. We show that if G is a connected bicyclic graph with nine edges containing two triangles, a G-design of order n exists whenever $n \equiv 0, 1 \pmod{18}$.

1. Introduction

Let n be an integer and G be a subgraph of the complete graph K_n. If the edges of K_n can be partitioned into edge-disjoint copies of G, then we call the partition a *G-design of order n*. We only consider simple connected graphs in this article. A graph is *bicyclic* if it contains exactly two cycles. If the two cycles are both triangles, we say G is a *triangular bicyclic graph*. In this article, we find G-designs for triangular bicyclic graphs with nine edges.

Given a graph G, the G-design spectrum problem asks to find necessary and sufficient conditions for all n such that a G-design exists. This problem has been completely solved for graphs G with six or less edges [13]. For graphs with seven, eight, or nine edges, the spectrum problem has been partially solved (see [1, 2, 3, 4, 7, 10, 11, 12, 14, 15, 16, 17, 18, 19, 21, 23]).

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There are 32 non-isomorphic connected triangular bicyclic graphs with nine edges. Necessarily, such a graph must have exactly eight vertices. Let G be one of these graphs. If a G-design of order n exists, then necessarily $n \equiv 0$ or $1 \pmod 9$, since otherwise $|E(G)| \nmid \binom{n}{2}$. We will use a mix of standard graph labeling techniques and constructions using group divisible designs to construct G-designs of all orders $n \equiv 0$ or $1 \pmod{18}$.

2. Tools

2.1. **Rosa's labelings.** Alex Rosa revolutionized the problem of finding G-designs in 1966 when he reduced the problem to assigning integer values to the vertices of G that satisfied certain properties [22]. He called it a *valuation* of G, but this process has come to be known as a *labeling* of G. For a more detailed exposition of this, we refer the reader to [9].

Definition 2.1. [22] *Let G be a simple graph with n edges. A ρ-labeling is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2n\}$ such that the induced length function $\ell : E(G) \to \{1, 2, \ldots, n\}$ defined as*

$$\ell(uv) = min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\},$$

is a bijection.

A G-design is *cyclic* if the transformation $v \mapsto v + 1$ on the vertices is an automorphism of the design. Rosa showed that a ρ-labeling of a graph G with n edges is equivalent to a cyclic G-design of order $2n + 1$.

Theorem 2.2. [22] *Let G be a graph with n edges. There exists a cyclic G-design of order $2n + 1$ if and only if G admits a ρ-labeling.*

Bunge et al. showed that if some careful restrictions to a ρ-labeling of a tripartite graph are made, then one can produce an infinite class of G-designs of odd order.

Definition 2.3. [6] *Let G be a tripartite graph on n edges and with vertex partition $A \cup B \cup C$. A ρ-tripartite labeling of G is a ρ-labeling f of G such that:*

- *$f(a) < f(v)$ for any edge $av \in E(G)$ where $a \in A$ and $v \in B \cup C$.*
- *For every edge $bc \in E(G)$ where $b \in B$, $c \in C$, there exists a complementary edge $b'c' \in E(G)$ where $b' \in B$, $c' \in C$ such that*

$$|f(b) - f(c)| + |f(b') - f(c')| = 2n.$$

- *For all $b \in B$, $c \in C$,*

$$|f(b) - f(c)| \neq 2n.$$

Theorem 2.4. [6] *Let G be a tripartite graph with n edges which admits a ρ-tripartite labeling. Then there exists a cyclic G-decomposition of K_{2nk+1} for all $k \geq 1$.*

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Thus far, we have considered only G-designs of odd order. The following is a similar set of definitions and theorems that address designs of even order, though G must contain a pendant vertex.

Definition 2.5. [5] *Let G be a graph on n edges. A 1-rotational ρ-labeling of G is an injection $f : V(G) \to [0, 2n-2] \cup \{\infty\}$ such that:*

- *For some pendant vertex w, $f(w) = \infty$.*
- *f is a ρ-labeling of $G - w$.*

Definition 2.6. [5] *Let G be a tripartite graph on n edges such that $uw \in E(G)$ and $deg(w) = 1$. A 1-rotational ρ-tripartite labeling of a graph G is an injection $h : V(G) \to [0, 2n-2] \cup \{\infty\}$ such that:*

- *h is a 1-rotational ρ-labeling of G with $h(w) = \infty$, where w has degree one.*
- *If the edge $av \in E(G) \backslash uw$, where $a \in A$ and $v \in B \cup C$, then $h(a) < h(v)$.*
- *If $bc \in E(G)$ with $b \in B$, $c \in C$, then there exists an edge $b'c' \in E(G)$ with $b' \in B$, $c' \in C$ such that*

$$|h(b) - h(c)| + |h(b') - h(c')| = 2n.$$

Theorem 2.7. [5] *Let G be a tripartite graph with n edges and a vertex of degree one. If G admits a 1-rotational ρ-tripartite labeling, then there exists a G-design of order $2nk$ for any integer $k \geq 1$.*

2.2. **Group divisible designs.** There is only one triangular bicyclic graph with nine edges that does not contain a pendant edge (we have named it $T(3,3)_1$, see Figure 33). Therefore, we cannot apply Theorem 2.7 to obtain a $T(3,3)_1$-design of order $18k$. To address this case, we use a method related to blow-ups of group divisible designs.

Terminology and results related to group divisible designs can be found in the Handbook of Combinatorial Design [20]. C. J. Colbourn et. al. [8], and L. Zhu [24] showed the existence of a wide range of 3-GDDs. Included in these results is the existence of 3-GDD of type 2^k for $k \equiv 0, 1 \pmod 3$ and 3-GDD of type $2^{k-2}, 4$ for $k \equiv 2 \pmod 3$. We restate this result in terms of graph decomposition. By $K_{a \times b, c}$, we mean the complete multipartite graph with a parts of size b and one part of size c.

Theorem 2.8. *Let $k \geq 3$. Then there exists a K_3 decomposition of $K_{k \times 2}$ if $k \equiv 0, 1 \pmod 3$ and there exists a K_3 decomposition of $K_{k-2 \times 2, 4}$ if $k \equiv 2 \pmod 3$.*

By replacing each vertex in these decompositions with 9 independent vertices and each edge with a $K_{9 \times 9}$, we get the following "blown up" result.

Theorem 2.9. *Let $k \geq 3$. Then there exists a $K_{9 \times 9}$ decomposition of $K_{k \times 18}$ if $k \equiv 0, 1 \pmod 3$ and there exists a $K_{9 \times 9}$ decomposition of $K_{k-2 \times 18, 36}$ if $k \equiv 2 \pmod 3$.*

Using these results, we can build up a decomposition of K_{18k} using a few small designs.

Theorem 2.10. *Let $k \geq 1$. If a graph G decomposes K_{18}, K_{36}, and $K_{9 \times 9}$, then G decomposes K_{18k}.*

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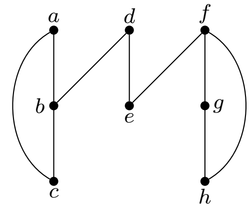

FIGURE 1. $T(3;3)_1$

Proof. Since G decomposes K_{18} and K_{36}, it remains to show that G decomposes K_{18k} for $k \geq 3$.

If $k \equiv 0, 1 \pmod 3$, then by Theorem 2.9, $K_{9\times 9}$ decomposes $K_{k\times 18}$. Then since G decomposes $K_{9\times 9}$, G decomposes $K_{k\times 18}$. Since G decomposes K_{18}, by filling the k sets of independent vertices of $K_{k\times 18}$ with k copies of K_{18}, we get a decomposition of K_{18k}.

Similarly, if $k \equiv 2 \pmod 3$, then $K_{9\times 9}$ decomposes $K_{k-2\times 18,36}$, and so G decomposes $K_{k-2\times 18,36}$. Since G decomposes K_{18} and K_{36}, by filling the $k-2$ independent vertices of $K_{k-2\times 18,36}$ with $k-2$ copies of K_{18} and one copy of K_{36}, we get a decomposition of K_{18k}. □

It then remains to find decompositions of these smaller designs, which is done in the proof of Theorem 2.11. Throughout this section, let $T[a,b,c,d,e,f,g,h]$ be a copy of $T(3;3)_1$ on the vertices $\{a,b,c,d,e,f,g,h\}$, as shown in Figure 1. Let $\mathbb{Z}_m^*$ be the set $\{0^*, 1^*, \ldots, m-1^*\}$. By labeling the vertices with the elements from $\mathbb{Z}_a \times \mathbb{Z}_b$ or $\mathbb{Z}_a \times \mathbb{Z}_b \cup \mathbb{Z}_m^*$ and considering the cyclic action on $\mathbb{Z}_b$, you can get a decomposition of K_{ab}, K_{ab+m}, $K_{a\times b}$, or $K_{ab+m}\backslash K_m$ (that is , K_{ab+m} with a hole of size m) depending on what edges are present.

Theorem 2.11. *For $x \geq 1$, there exists a $T(3;3)_1$-design of order $18x$.*

Proof. Let $V(K_{18}) = \mathbb{Z}_2 \times \mathbb{Z}_7 \cup \mathbb{Z}_4^*$ and let P be the set of blocks

$$T[(0,1+i),(0,i),(0,3+i),(1,2+i),(0,2+i),(1,i),(1,1+i),(1,3+i)],$$
$$T[(0,i),0^*,(1,1+i),1^*,2^*,3^*,(0,1+i),(1,i)],$$
$$T[(0,i),1^*,(1,3+i),3^*,0^*,2^*,(0,3+i),(1,i)]$$

for $i \in \mathbb{Z}_7$. Then P is a $T(3;3)_1$-decomposition of K_{18}.

Let $V(K_{10}) = \mathbb{Z}_2 \times \mathbb{Z}_5$ and let P be the set of blocks $T[(1,i),(0,i),(1,1+i),$ $(0,3+i),(1,2+i),(1,4+i),(0,1+i),(0,2+i)]$ for $i \in \mathbb{Z}_5$. Then P is a $T(3;3)_1$-decomposition of K_{10}.

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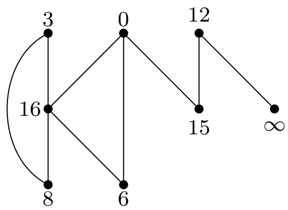

FIGURE 2. $T(0;4)_1$

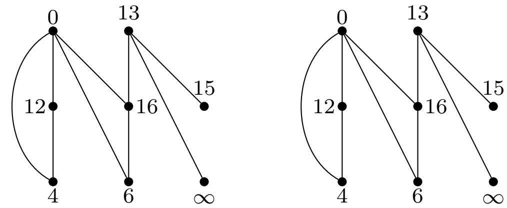

FIGURE 3. $T(0;4)_2$

Let $V(K_{36}\backslash K_{10}) = \mathbb{Z}_2 \times \mathbb{Z}_{13} \cup \mathbb{Z}_{10}^*$ and let P be the set of blocks

$$T[0^*, (0, i), (1, 1 + i), (0, 1 + i), (0, 3 + i), (0, 6 + i), (1, 8 + i), 1^*],$$
$$T[2^*, (0, i), (1, 3 + i), (0, 4 + i), (0, 9 + i), (0, 3 + i), (1, 7 + i), 3^*],$$
$$T[4^*, (1, i), (0, 1 + i), (1, 1 + i), (1, 3 + i), (1, 6 + i), (0, 8 + i), 5^*],$$
$$T[6^*, (1, i), (0, 3 + i), (1, 4 + i), (1, 9 + i), (1, 3 + i), (0, 7 + i), 7^*],$$
$$T[8^*, (0, 6 + i), (1, 11 + i), (1, i), (0, i), (1, 6 + i), (0, 11 + i), 9^*],$$

for $i \in \mathbb{Z}_{13}$. Then P is a $T(3; 3)_1$-decomposition of $K_{36}\backslash K_{10}$. Since $T(3; 3)_1$ decomposes K_{10}, by replacing the independent vertices in $K_{36}\backslash K_{10}$ with a copy of K_{10}, we see $T(3; 3)_1$ decomposes K_{36}.

Let $V(K_{3\times 9}) = \mathbb{Z}_3 \times \mathbb{Z}_9$ and let P be the set of blocks

$$T[(1, i), (0, i)(2, 1 + i), (1, 8 + i), (2, 8 + i), (0, 2 + i), (1, 7 + i), (2, i)],$$
$$T[(1, 1 + i), (0, i), (2, 5 + i), (1, 7 + i), (2, 1 + i), (0, 8 + i), (1, 2 + i), (2, 7 + i)],$$
$$T[(1, 2 + i), (0, i), (2, i), (1, 6 + i), (2, 3 + i), (0, 8 + i), (1, 3 + i), (2, 2 + i)]$$

for $i \in \mathbb{Z}_9$. Then P is a $T(3; 3)_1$-decomposition of $K_{3\times 9}$. Then since $K_{3\times 9}$ decomposes $K_{9\times 9}$, $T(3; 3)_1$ decomposes $K_{9\times 9}$.

Since $T(3; 3)_1$ decomposes K_{18}, K_{36}, and $K_{9\times 9}$, $T(3; 3)_1$ decomposes K_{18x}. □

3. Main result

To catalog the 32 connected triangular bicyclic graphs with nine edges, we let $T(d; \Delta)_i$ denote the i^{th} graph with the property that the two triangles are distance d apart and Δ is the maximum degree taken over all vertices. The left side of each of the following figures shows a 1-rotational ρ-tripartite labeling (with the exception of Figure 33), while the right side shows a ρ-tripartite labeling of each graph G. The vertices of each graph are aligned in three rows to distinguish the partite sets: A (top), B (middle), and C (bottom).

The following theorem summarizes our results.

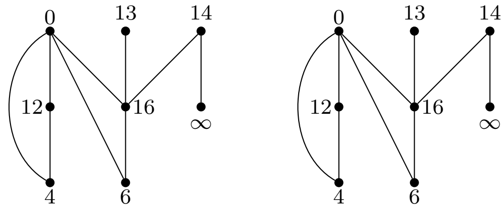

Figure 4. $T(0;4)_3$

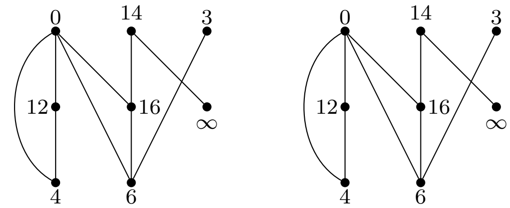

Figure 5. $T(0;4)_4$

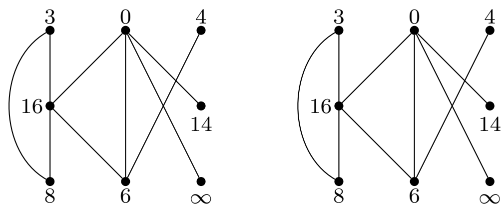

Figure 6. $T(0;4)_5$

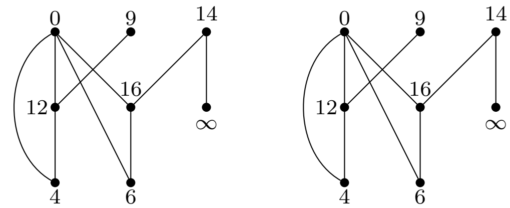

Figure 7. $T(0;4)_6$

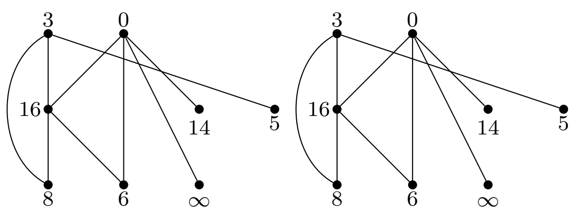

Figure 8. $T(0;4)_7$

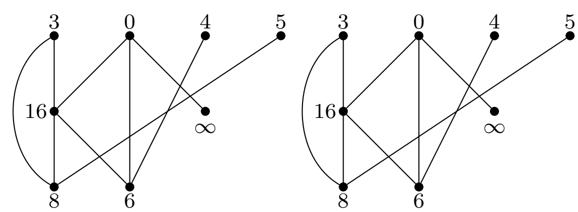

Figure 9. $T(0;4)_8$

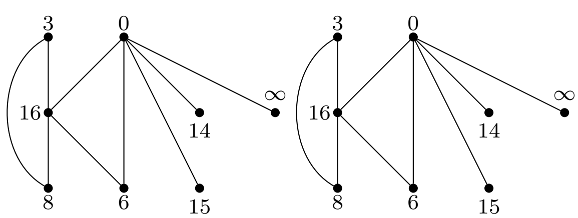

Figure 10. $T(0;5)_1$

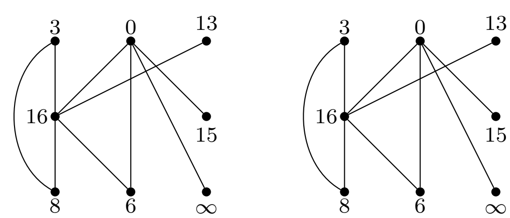

Figure 11. $T(0;5)_2$

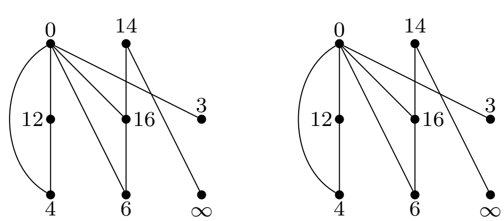

Figure 12. $T(0;5)_3$

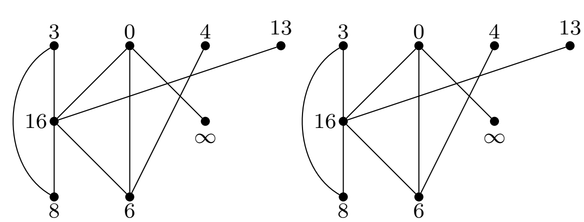

Figure 13. $T(0;5)_4$

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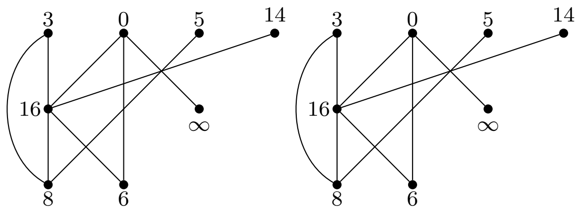

FIGURE 14. $T(0;5)_5$

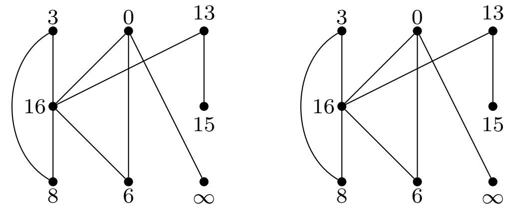

FIGURE 15. $T(0;5)_6$

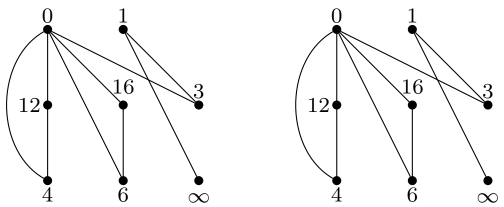

FIGURE 16. $T(0;5)_7$

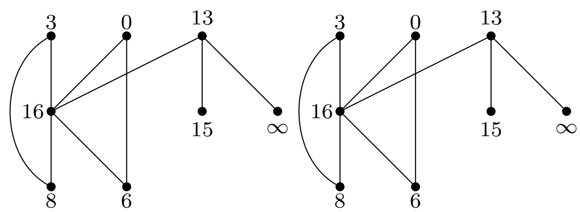

FIGURE 17. $T(0;5)_8$

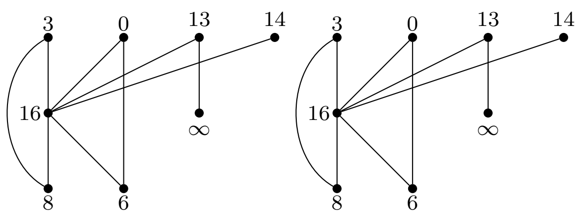

FIGURE 18. $T(0;6)_1$

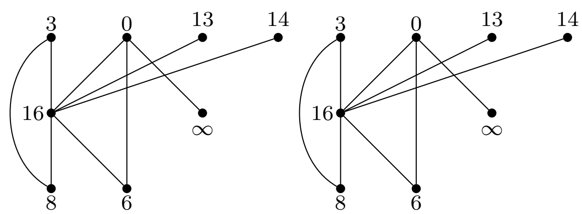

FIGURE 19. $T(0;6)_2$

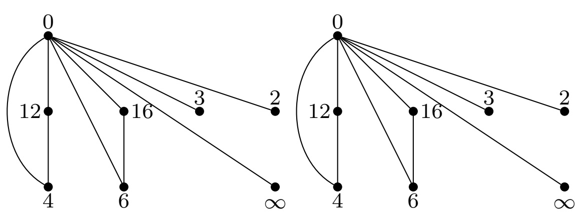

FIGURE 20. $T(0;7)_1$

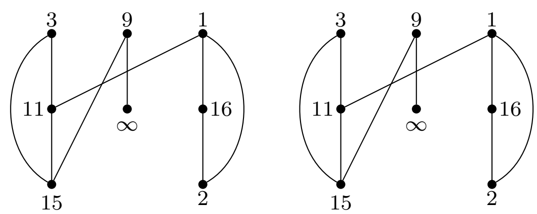

FIGURE 21. $T(1;3)_1$

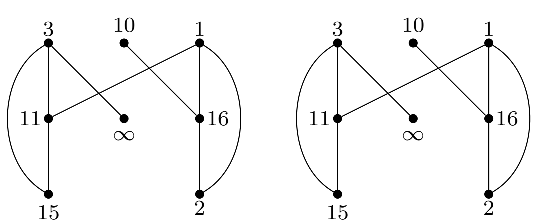

FIGURE 22. $T(1;3)_2$

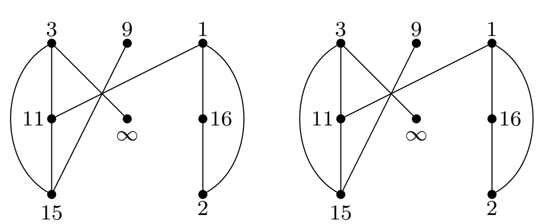

FIGURE 23. $T(1;3)_3$

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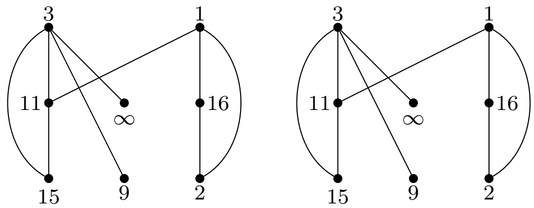

FIGURE 24. $T(1;4)_1$

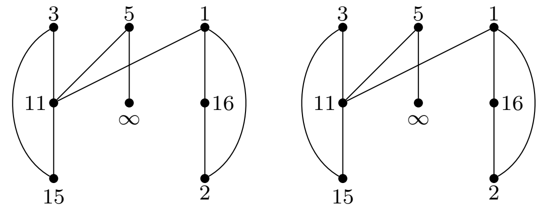

FIGURE 25. $T(1;4)_2$

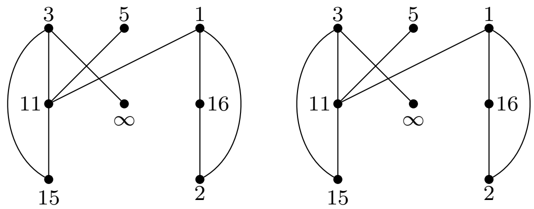

FIGURE 26. $T(1;4)_3$

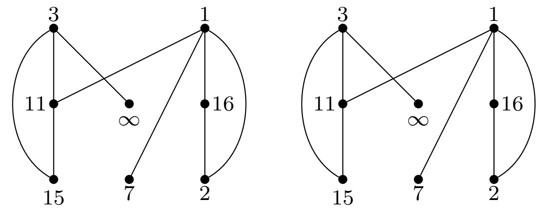

FIGURE 27. $T(1;4)_4$

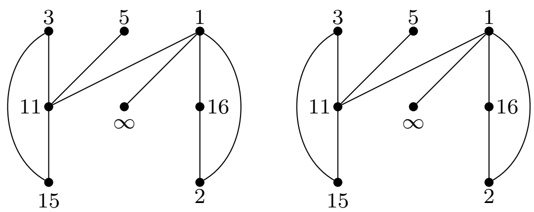

FIGURE 28. $T(1;4)_5$

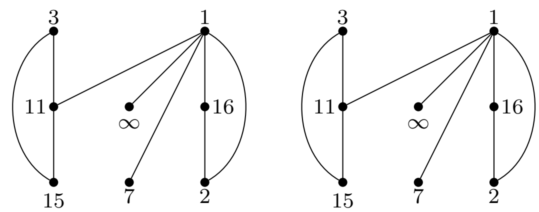

FIGURE 29. $T(1;5)_1$

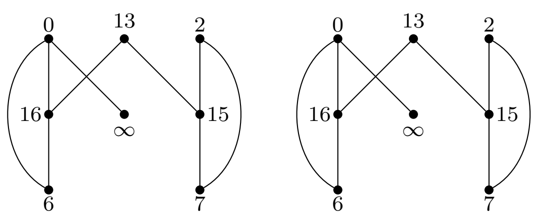

FIGURE 30. $T(2;3)_1$

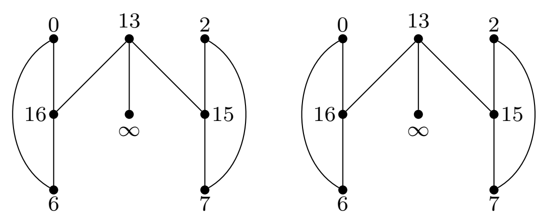

FIGURE 31. $T(2;3)_2$

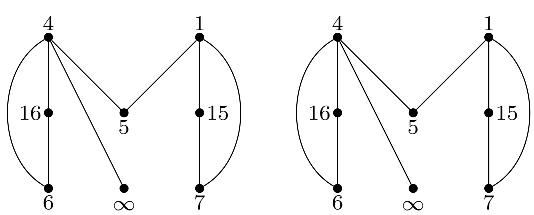

FIGURE 32. $T(2;4)_1$

1-rotational
ρ-tripartite labeling
DNE

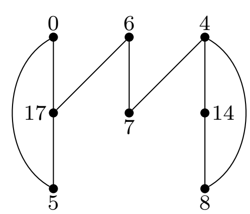

FIGURE 33. $T(3;3)_1$

Theorem 3.1. *Let G be a connected triangular bicyclic graph with exactly nine edges. There exists a G-design of order n whenever $n \equiv 0, 1 \pmod{18}$.*

Proof. If $n \equiv 1 \pmod{18}$, the G-design exists by Theorem 2.4 and the ρ-tripartite labelings in Figures 2 through 33. If $n \equiv 0 \pmod{18}$ and $G \cong T(3;3)_1$, the design is by Theorem 2.11. Otherwise, the result follows from Theorem 2.7 and the 1-rotational ρ-tripartite labelings in Figures 2 through 33. □

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