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EXPONENTIAL AND WEAKLY EXPONENTIAL SUBGROUPS OF FINITE GROUPS

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Abstract. Sabatini [5] defined a subgroup H of G to be an *exponential subgroup* if $x^{|G:H|} \in H$ for all $x \in G$, in which case we write $H \leqslant_{exp} G$. Exponential subgroups are a generalization of normal (and subnormal) subgroups: all subnormal subgroups are exponential, but not conversely. Sabatini proved that all subgroups of a finite group G are exponential if and only if G is nilpotent. The purpose of this paper is to explore what the analogues of a simple group and a solvable group should be in relation to exponential subgroups. We say that an exponential subgroup $H \leqslant_{exp} G$ is *exp-trivial* if either $H = G$ or the exponent of G, $\exp(G)$, divides $|G : H|$, and we say that a group G is *exp-simple* if all exponential subgroups of G are exp-trivial. We classify finite exp-simple groups by proving G is exp-simple if and only if $\exp(G) = \exp(G/N)$ for all proper normal subgroups N of G, and we illustrate how the class of exp-simple groups differs from the class of simple groups. Furthermore, in an attempt to overcome the obstacle that prevents all subgroups of a generic solvable group from being exponential, we say that a subgroup H of G is *weakly exponential* if, for all $x \in G$, there exists $g \in G$ such that $x^{|G:H|} \in H^g$. If all subgroups of G are weakly exponential, then G is *wexp-solvable*. We prove that all solvable groups are wexp-solvable and almost all symmetric and alternating groups are not wexp-solvable. Finally, we completely classify the groups $\mathrm{PSL}(2,q)$ that are wexp-solvable. We show that if $\pi(n)$ denotes the number of primes less than n and $w(n)$ denotes the number of primes p less than n such that $\mathrm{PSL}(2,p)$ is wexp-solvable, then

$$\lim_{n\to\infty} \frac{w(n)}{\pi(n)} = \frac{1}{4}.$$

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1. Introduction

In this paper, all groups are finite. It is a basic property of a subnormal subgroup $H \lhd\lhd G$ that $x^{|G:H|} \in H$ for all $x \in G$. Subgroups that are not (sub)normal may or may not exhibit this behavior. For instance, the dihedral group D_4 of order 8 contains a nonnormal subgroup H of index 4, but $x^4 \in H$ for all $x \in D_4$. On the other hand, if x and y are distinct transpositions in the symmetric group S_3, then $x^3 \notin \langle y \rangle$.

Definition 1.1. Let G be a finite group. A subgroup $H \leqslant G$ is said to *exponential* with respect to G if, for all $x \in G$, $x^{|G:H|} \in H$. In this case, following [5], we write $H \leqslant_{exp} G$.

It was shown in [5, Proposition 5.5] (see also [6]) that, for a finite group G, $H \leqslant_{exp} G$ for all subgroups $H \leqslant G$ if and only if G is nilpotent. This prompts the question of whether the exponential subgroups, or variations thereof, can be used to identify other classes of finite groups. Our purpose in this paper is to investigate this problem, with an emphasis on simple groups and solvable groups.

As noted above, the exponential property is a weak form of subnormality: if $N \lhd\lhd G$, then N is exponential with respect to G, but not conversely. The question then becomes: how much weaker than subnormality is the exponential property? Just as every group has trivial normal subgroups, there is a class of subgroups that trivially are exponential; for example, $G \leqslant_{exp} G$. Denote the exponent of G – the smallest positive integer k such that $x^k = 1$ for all $x \in G$ – by $\exp(G)$. Then, it is clear that H is exponential with respect to G whenever $\exp(G)$ divides $|G : H|$, and that G always contains such subgroups. As a simple group is one that contains no nontrivial normal subgroups, a group that is "simple" with respect to the exponential property should contain as few exponential subgroups as possible.

Definition 1.2. Let $H \leqslant G$. We say that H is *exponential-trivial* (or *exp-trivial*) if either $H = G$ or $\exp(G)$ divides $|G : H|$. A group G is said to be *exponential-simple* (or *exp-simple*) if the only exponential subgroups of G are exp-trivial.

Which groups are exp-simple? How does the class of simple groups compare to the class of exp-simple groups? We are able to characterize exp-simple groups and answer these questions precisely.

Theorem 1.3. *Let G be a finite group. Then, G is exp-simple if and only if* $\exp(G) = \exp(G/N)$ *for all proper normal subgroups N of G.*

Recall that a proper subgroup N of a group G is a *maximal normal subgroup* if N is normal in G and there does not exist a normal subgroup K of G such that $N < K < G$, and a group G is *quasisimple* if $G = [G, G]$ and $G/Z(G)$ is a simple group. Using Theorem 1.3, we can prove the following:

Corollary 1.4. *Let G be a finite group.*

(1) If G is simple, then G is exp-simple.

(2) If G is solvable, then G is exp-simple if and only if G is a p-group with exponent p.

(3) If G is the direct product of simple groups, then G is exp-simple if and only if each simple direct factor of G has the same exponent.

(4) If G has a unique maximal normal subgroup N, then G is exp-simple if and only if $\exp(G) = \exp(G/N)$. In particular, if G is quasisimple, then G is exp-simple if and only if $\exp(G) = \exp(G/Z(G))$.

See Section 2 for the proofs of Theorem 1.3 and its corollary.

Recall that all subgroups H of G are exponential if and only if G is nilpotent. We would like to modify the definition of exponential subgroups in such a way as to capture all solvable groups. Consider the following example.

Example 1.5. Let $G = S_3$. This group will have subgroups that are not exponential since G has more than one Sylow 2-subgroup. Indeed, take $x = (1\ 2)$ and $H = \langle (2\ 3) \rangle$. Then, $|G : H| = 3$, and hence $x^{|G:H|} = x \notin H$, showing that H is not exponential. We will run into such a problem whenever a group G has more than one Sylow p-subgroup. Suppose $P_1 \neq P_2$ are such subgroups and $x \in P_1 \backslash P_2$. Then, $|G : P_2|$ is coprime to p, so $\langle x^{|G:P_2|} \rangle = \langle x \rangle$. Thus, $x^{|G:P_2|} \notin P_2$.

To avoid the obstruction discussed in Example 1.5, we can use the fact that all Sylow p-subgroups of a finite group are conjugate. This leads to the following natural generalization of exponential subgroups.

Definition 1.6. Let G be a finite group. A subgroup $H \leqslant G$ is said to be *weakly exponential* with respect to G if, for all $x \in G$, there exists $g \in G$ such that $x^{|G:H|} \in H^g$. In this case, we write $H \leqslant_{wexp} G$.

It should be immediately observed that all subgroups of S_3 are weakly exponential, and, indeed, if P is a Sylow p-subgroup of G, we will have $P \leqslant_{wexp} G$, fixing the issues raised in Example 1.5. With this in mind, we make the following definition.

Definition 1.7. If every subgroup of G is weakly exponential, then G is called a *weakly exponential-solvable group* (or a *wexp-solvable group*). If G is not wexp-solvable, then G is said to be a *weakly exponential-nonsolvable group* (or *wexp-nonsolvable group*).

As the next theorem shows, the use of the term "solvable" in "wexp-solvable" is justified.

Theorem 1.8. *All finite solvable groups are wexp-solvable groups.*

It turns out that the class of wexp-solvable groups is strictly larger than the class of solvable groups (and hence a group being wexp-solvable is a weaker condition than being solvable). In particular, the

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alternating groups A_5 and A_6 are wexp-solvable; see Lemmas 5.2 and 5.3. This also means that a group being wexp-nonsolvable is a strictly stronger condition than being nonsolvable. It is not difficult to show (see Lemma 3.3) that if any proper quotient of a finite group G is wexp-nonsolvable, then G is itself wexp-nonsolvable. This motivates the next definition, which is analogous to that of *σ-elementary* when studying covering numbers of groups (see [3]).

Definition 1.9. A finite group G is said to be *minimal wexp-nonsolvable* if G is wexp-nonsolvable (that is, G is not wexp-solvable) but all proper quotients of G are wexp-solvable.

Thus, determining which finite groups are wexp-solvable (or, equivalently, which finite groups are wexp-nonsolvable) comes down to determining which groups are minimal wexp-nonsolvable. It would be quite interesting to classify the minimal wexp-nonsolvable groups. Any almost simple group is potentially minimal wexp-nonsolvable, and natural examples to consider are symmetric and alternating groups. Apart from some small examples, these groups are all minimal wexp-nonsolvable.

Theorem 1.10. *The symmetric group S_n is minimal wexp-nonsolvable if and only if $n = 5$ or $n \geqslant 7$.*

Theorem 1.11. *The alternating group A_n is minimal wexp-nonsolvable if and only if $n \geqslant 7$.*

These theorems might lead one to conjecture that almost all (almost) simple groups are minimal wexp-nonsolvable. However, the reality is far more complicated, as evidenced by the following result on projective special linear groups.

Theorem 1.12. *Let $q = p^d$ be a prime power. Then, $\mathrm{PSL}(2,q)$ is a wexp-solvable group if and only if $q = 4, 9$ or*

$$p \equiv 2, 3, 5, 7, 17, 43, 53, 67, 77, 103, 113 \pmod{120}$$

and d is odd. Consequently, $\mathrm{PSL}(2,q)$ is wexp-solvable for infinitely many values of q and is wexp-nonsolvable (and hence minimal wexp-nonsolvable) for infinitely many values of q

Other than the three exceptional cases $p = 2, 3, 5$, by Theorem 1.12, $\mathrm{PSL}(2,p)$ is wexp-solvable if and only if p lies in one of eight congruence classes modulo 120. If φ denotes the Euler totient function, then the prime numbers are asymptotically equally distributed among the $\varphi(120) = 32$ congruence classes modulo 120 (this is the “stronger version” of Dirichlet’s Theorem on Arithmetic Progressions). This gives us the last of our major results.

Corollary 1.13. *Let $\pi(n)$ denote the number of primes less than n and $w(n)$ denote the number of primes p less than n such that $\mathrm{PSL}(2,p)$ is wexp-solvable. Then,*

$$\lim_{n\to\infty} \frac{w(n)}{\pi(n)} = \frac{1}{4}.$$

Remark 1.14. Computation in GAP [2] shows that the only group with order at most 1000 that is wexp-nonsolvable and does not have S_5 as a proper quotient is $\mathrm{PSL}(2, 11)$. In particular, the only minimal wexp-nonsolvable groups with order at most 1000 are S_5 and $\mathrm{PSL}(2, 11)$.

This paper is organized as follows. Section 2 is dedicated to the study of exp-simple groups and the proofs of Theorem 1.3 and Corollary 1.4. In Section 3, we present some basic results that are useful when studying weakly exponential subgroups. Section 4 is dedicated to the proof of Theorem 1.8, which shows that all solvable groups are wexp-solvable. In Section 5, we prove Theorems 1.10 and 1.11, which determines precisely which symmetric and alternating groups are minimal wexp-nonsolvable. Finally, in Section 6, we prove Theorem 1.12, which classifies the prime powers q such that $\mathrm{PSL}(2, q)$ is wexp-solvable.

2. Exponential-simple groups

Recall that a group G is exp-simple if the only exponential subgroups of G are G itself, and those proper subgroups $H \leqslant G$ such that $\exp(G)$ divides $|G : H|$. As pointed out in [5, Remark 4.2], G will contain exponential subgroups $H \neq \{1\}$ as long as G is not a cyclic group of prime order. So, one should expect that there exist nontrivial examples of exp-simple groups, and this is indeed the case. In this section, we will prove Theorem 1.3—which completely describes exp-simple groups—as well as its corollaries, which give several interesting examples of such groups.

One part of our classification of exp-simple groups will follow from the next lemma; see also [5, Lemma 4.3].

Lemma 2.1. *Let G be a finite group, $H \leqslant G$, and*

$$K := \bigcap_{g \in G} H^g.$$

If H is exponential with respect to G, then, for all $x \in G$, $x^{|G:H|} \in K$. In particular, $\exp(G/K)$ divides $|G : H|$.

Proof. Suppose H is exponential and let $x \in G$. For all $g \in G$, we have

$$(x^{|G:H|})^{g^{-1}} = (x^{g^{-1}})^{|G:H|} \in H.$$

Thus, $x^{|G:H|} \in H^g$ for all $g \in G$, and so $x^{|G:H|} \in K$. Finally, $K \lhd G$, so $\overline{x}^{|G:H|} = \overline{1}$ for all $\overline{x} \in G/K$, proving the result. □

We can now prove Theorem 1.3, which characterizes exp-simple groups.

Proof of Theorem 1.3. First, let G be a finite exp-simple group and let N be a proper normal subgroup of G. Certainly, $\exp(G/N)$ divides $\exp(G)$. Since N is a normal subgroup, $N \leqslant_{exp} G$; but, G is exp-simple, so N is exp-trivial, and hence $\exp(G)$ divides $|G : N|$.

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Let $\overline{G} = G/N$, and suppose $\exp(G) > \exp(G/N)$. Then, there exists a prime p and positive integer r such that p^r divides $\exp(G)$ but p^r does not divide $\exp(\overline{G})$. Now, p^r divides $\exp(G)$, so p^r divides $|G:N| = |\overline{G}|$. Suppose a Sylow p-subgroup of $\overline{G}$ has order p^d and let $\overline{H}$ be a subgroup of $\overline{G}$ of order $p^{d-(r-1)}$. Let $\phi : G \to \overline{G}$ be the natural homomorphism and $H := \phi^{-1}(\overline{H}) \leqslant G$. Then,

$$|G:H| = |\overline{G}:\overline{H}| = p^{r-1} \cdot |\overline{G}|_{p'},$$

where $|\overline{G}|_{p'} := |\overline{G}|/p^d$ denotes the p'-part of $|\overline{G}|$. Since the highest power of p dividing $\exp(\overline{G})$ is less than p^r, $\exp(\overline{G})$ divides $|G:H|$, but p^r does not divide $|G:H|$, so $\exp(G)$ does not divide $|G:H|$.

Let $x \in G$. In $\overline{G}$, $\overline{x}^{|G:H|} = \overline{1}$, so $x^{|G:H|} \in N \leqslant H$. Thus, $H \leqslant_{exp} G$, but $\exp(G) \nmid |G:H|$, so H is not exp-trivial. This contradicts G being exp-simple. Thus, if G is exp-simple, then $\exp(G/N) = \exp(G)$ for all proper normal subgroups N.

Conversely, assume that $\exp(G) = \exp(G/N)$ for all proper normal subgroups N of G. Let $H < G$, and assume $H \leqslant_{exp} G$. Let

$$K := \bigcap_{g \in G} H^g.$$

By Lemma 2.1, $\exp(G/K)$ divides $|G:H|$. By hypothesis, $\exp(G/K) = \exp(G)$, meaning H is exp-trivial. Therefore, G is exp-simple, as desired. □

Corollary 2.2. *If G is a finite simple group, then G is exp-simple.*

Proof. This is clear, since in a finite simple group the only proper normal subgroup is the identity subgroup. (This result also follows from [5, Lemma 4.3].) □

Although all simple groups are exp-simple, the class of exp-simple groups is strictly larger than the class of simple groups, as the following results show.

Corollary 2.3. *Let G be a solvable group. Then, G is exp-simple if and only if G is a p-group with exponent p for some prime p.*

Proof. Let G be a solvable group. Assume first that G is a p-group with exponent p. Then, the exponent of every proper quotient also has exponent p, and so G is exp-simple by Theorem 1.3.

Conversely, assume that G is a solvable group that is exp-simple. Since G is solvable, there exists a normal subgroup N of G such that $|G:N| = p$ for some prime p. By Theorem 1.3, this implies $\exp(G) = \exp(G/N) = p$, and hence G is a p-group with exponent p. □

Corollary 2.4. *Let $T_1, \dots, T_n$ be simple groups. The direct product*

$$G := T_1 \times \cdots \times T_n$$

is exp-simple if and only if all T_i have the same exponent.

Proof. Assume first that G is exp-simple. Since G naturally projects onto each T_i, by Theorem 1.3, this implies $\exp(T_i) = \exp(G)$ for all i. Conversely, assume each T_i has the same exponent m. If $N \lhd G$, then G/N is isomorphic to a direct product of some collection of T_i's, and so $\exp(G/N) = m = \exp(G)$. By Theorem 1.3, G is exp-simple. $\square$

Example 2.5. The groups $\mathrm{PSL}(4,2) \cong A_8$ and $\mathrm{PSL}(3,4)$ are not isomorphic, but both groups have exponent 420. Thus, $\mathrm{PSL}(4,2) \times \mathrm{PSL}(3,4)$ is exp-simple by Corollary 2.4.

Example 2.6. Let $\exp_p(G)$ denote the highest power of p dividing $\exp(G)$. If p is an odd prime and $n = p^k$, then A_n contains an element of order p^k, while A_{n-1} does not. So, if n is a power of p, then $\exp_p(A_n) \neq \exp_p(A_{n-1})$. Otherwise, $\exp_p(A_n) = \exp_p(A_{n-1})$. When $p = 2$, elements of order 2^k are present in A_n if and only if $n \geqslant 2^k + 2$. Hence, if $n \neq 2^k + 2$, then $\exp_2(A_n) = \exp_2(A_{n-1})$. Thus, if $n \geqslant 6$ is not a power of an odd prime nor equal to $2^k + 2$ for some k, then $\exp(A_n) = \exp(A_{n-1})$, and so $A_{n-1} \times A_n$ is exp-simple for all such n by Theorem 1.3.

Corollaries 2.2, 2.3, and 2.4 might lead one to believe that all composition factors of an exp-simple group need to have the same exponent. This is not the case. Recall that a *maximal normal subgroup* of a group G is a proper subgroup N that is normal in G such that there does not exist a normal subgroup K with $N < K < G$; equivalently, N is normal in G and G/N is simple. Furthermore, recall that a group G is *quasisimple* if $G = [G,G]$ and $G/Z(G)$ is a simple group.

Corollary 2.7. *Let G be a finite group with a unique maximal normal subgroup N. Then, G is exp-simple if and only if $\exp(G) = \exp(G/N)$. In particular, if G is quasisimple, then G is exp-simple if and only if $\exp(G) = \exp(G/Z(G))$.*

Proof. Let G be a finite group with a unique maximal normal subgroup N. If G is exp-simple, then $\exp(G) = \exp(G/N)$ by Theorem 1.3. Conversely, assume $\exp(G) = \exp(G/N)$, and let K be a proper normal subgroup of G. Since N is the unique maximal normal subgroup of G, $1 \leqslant K \leqslant N$, and hence

$$\exp(G/N) \leqslant \exp(G/K) \leqslant \exp(G).$$

Thus, $\exp(G/K) = \exp(G)$, and hence G is exp-simple by Theorem 1.3.

Finally, if G is quasisimple, then $Z(G)$ is the unique maximal normal subgroup of G, and the result follows. $\square$

Example 2.8. Consider the quasisimple group $G = 3.A_6$ (this is SmallGroup(1080,260) in GAP [2]). Note that $Z(G) \cong C_3$ and $G/Z(G) \cong A_6$, so G has distinct composition factors C_3 and A_6. Since $\exp(G) = \exp(A_6) = 60$, G is exp-simple by Corollary 2.7. Thus, an exp-simple group can have composition factors with distinct exponents.

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3. Preliminary results related to weakly exponential subgroups

Recall that a subgroup H of G is *weakly exponential* (denoted $H \leqslant_{wexp} G$) if, for all $x \in G$, there exists $g \in G$ such that $x^{|G:H|} \in H^g$. We say that G is *wexp-solvable* if $H \leqslant_{wexp} G$ for all subgroups H of G. Here, we collect some lemmas that are useful for determining whether or not a group is wexp-solvable.

Lemma 3.1. *Let G be a finite group. Let Y be a subgroup of G such that Y is wexp-solvable and Y is weakly exponential with respect to G. If $H \leqslant Y$, then H is weakly exponential with respect to G.*

Proof. Let G, Y, H be as in the statement of the lemma. Since $Y \leqslant_{wexp} G$, for each $x \in G$ there exists $g \in G$ such that $x^{|G:Y|} \in Y^g$. Since $Y^g \cong Y$, Y^g is wexp-solvable, i.e., there exists $y \in Y^g$ such that

$$x^{|G:H|} = \left(x^{|G:Y|}\right)^{|Y^g:H^g|} \in H^{gy},$$

which shows that $H \leqslant_{wexp} G$. □

Lemma 3.2. *Let G be a group such that all maximal subgroups of G are themselves wexp-solvable groups. Then, G is wexp-solvable if and only if each maximal subgroup is weakly exponential with respect to G.*

Proof. First, if M is maximal and M is not weakly exponential with respect to G, then G is not wexp-solvable by definition.

Conversely, assume all maximal subgroups of G are weakly exponential with respect to G and are themselves wexp-solvable groups. Let $x \in G$ and $H < G$. Since $H < G$, there exists a maximal subgroup M of G such that $H \leqslant M < G$. Since $M \leqslant_{wexp} G$ and M is wexp-solvable, $H \leqslant_{wexp} G$ by Lemma 3.1. Since H was arbitrary, G is wexp-solvable. □

Lemma 3.3. *If G is a wexp-solvable group, then all quotients of G are wexp-solvable.*

Proof. Let G be a wexp-solvable group, and let $\phi : G \to \overline{G}$ be a surjective homomorphism. Let $\overline{x} \in \overline{G}$ and $\overline{H} \leqslant \overline{G}$. Define $H := \phi^{-1}(\overline{H}) \leqslant G$, and let x be any element of G in $\phi^{-1}(\overline{x})$. Since G is wexp-solvable, there exists $g \in G$ such that

$$x^{|G:H|} \in H^g,$$

and hence

$$\overline{x}^{|\overline{G}:\overline{H}|} = \overline{x}^{|G:H|} \in \overline{H}^{\overline{g}}.$$

Thus, $\overline{H} \leqslant_{wexp} \overline{G}$. Since $\overline{H}$ was arbitrary, $\overline{G}$ is wexp-solvable. □

4. **Solvable groups are wexp-solvable**

The goal of this section is to prove that the issues raised in Example 1.5 are essentially all that prevent subgroups of solvable groups from being exponential; that is, all subgroups of solvable groups are weakly exponential (and hence all solvable groups are wexp-solvable, justifying the terminology). To prove this, we need the following lemma on the exponent of a Sylow p-subgroup of the affine general linear group $\mathrm{AGL}(n,p)$.

Lemma 4.1. *Let P be a Sylow p-subgroup of $\mathrm{AGL}(n,p)$. Then, $\exp(P) \leqslant p^n$.*

Proof. Since $\mathrm{AGL}(n,p) = C_p^n \rtimes \mathrm{GL}(n,p)$, $P \cong C_p^n \rtimes Q$, where Q is a Sylow p-subgroup of $\mathrm{GL}(n,p)$. Thus, $\exp(P) \leqslant \exp(Q) \cdot p$.

Let $g \in \mathrm{GL}(n,p)$. By the Cayley-Hamilton Theorem, g satisfies $\chi(g) = 0$, where $\chi(x) := \det(xI - g)$ is the characteristic polynomial of g. Thus, g generates a subalgebra of $\mathrm{GF}(p)[g]$ containing at most $p^n - 1$ nonzero elements, which implies $|g| \leqslant p^n - 1$. If $g \in Q$, this means $|g| \leqslant p^{n-1}$. The result follows. $\square$

We are now ready to prove that all solvable groups are wexp-solvable.

Proof of Theorem 1.8. Assume G is a finite solvable group. Note that, since all subgroups of nilpotent groups are exponential, all nilpotent groups are wexp-solvable. We proceed by induction on $|G|$; that is, assume all solvable groups with order less than $|G|$ are wexp-solvable. By Lemma 3.2, it suffices to prove that all maximal subgroups of G are weakly exponential. Let M be a maximal subgroup of G.

Let N be a nontrivial normal subgroup of G, and let $x \in G$. Assume first that $N \leqslant M$. Let $\overline{G} = G/N$ and $\overline{M} = M/N \leqslant \overline{G}$. By the inductive hypothesis, $\overline{G}$ is wexp-solvable, so

$$\overline{x}^{|\overline{G}:\overline{M}|} \in \overline{M}^{\overline{g}} \tag{4.1}$$

for some $Ng = \overline{g} \in \overline{G}$, where $\overline{x} = Nx$ and $g \in G$. Since $|\overline{G} : \overline{M}| = |G : M|$, (4.1) implies that $x^{|G:M|} \in M^g$, showing that M is weakly exponential with respect to G.

Now, assume that $N \not\leqslant M$ for all nontrivial normal subgroups N of G. Since G is solvable, it contains a minimal normal subgroup $N \cong C_p^n$ (see, for example, [4, Lemma 3.11]), where n is a positive integer and p is a prime. Since M is maximal and $N \not\leqslant M$, we have $G = NM$. Moreover, $M \cap N$ is a proper subgroup of N that is normal in both M and N (since N is elementary abelian), and hence $M \cap N$ is normal in G; that is, $M \cap N = \{1\}$ and $G = N \rtimes M$. Furthermore, since $N \cong C_p^n$, there is a natural homomorphism $\phi : M \to \mathrm{GL}(n,p)$. Since $\mathrm{Ker}(\phi)$ is a normal subgroup of M that centralizes N, $\mathrm{Ker}(\phi)$ is a normal subgroup of G that is contained in M. By assumption, $\mathrm{Ker}(\phi) = \{1\}$, and so ϕ is faithful. Thus, $M \lesssim \mathrm{GL}(n,p)$ and $G \lesssim \mathrm{AGL}(n,p)$.

Let $x \in G$, and suppose $|x| = p^j \cdot k$, where $\gcd(p, k) = 1$. There exist $s, t \in \mathbb{Z}$ such that $p^j s + kt = 1$, so

$$x = (x^{p^j})^s (x^k)^t \in \langle x^{p^j}, x^k \rangle.$$

Thus, to conclude that M is weakly exponential with respect to G, it suffices to show that there is a conjugate of M containing both $(x^{p^j})^{|G:M|}$ and $(x^k)^{|G:M|}$.

First, consider x^k. Since $|x^k| = p^j$, x^k is contained in a Sylow p-subgroup of $G \lesssim \mathrm{AGL}(n, p)$. By Lemma 4.1,

$$(x^k)^{|G:M|} = (x^k)^{p^n} = 1. \tag{4.2}$$

Next, we work with x^{p^j}, which has order k. Since k is coprime to p and G is solvable, x^{p^j} is contained in a Hall p'-subgroup H of G [4, Theorem 3.13]. Furthermore, since $|M| = |G|/p^n$, M contains a Hall p'-subgroup L of G. By Hall's Theorem [4, Theorem 3.14], H and L are conjugate in G. So, there exists $g \in G$ such that $L^g = H$, and hence

$$x^{p^j} \in H = L^g \subseteq M^g. \tag{4.3}$$

By (4.2) and (4.3), both $(x^{p^j})^{|G:M|}$ and $(x^k)^{|G:M|}$ are in M^g, which completes the proof. ☐

5. Almost all symmetric and alternating groups are wexp-nonsolvable

The purpose of this section is to determine which symmetric and alternating groups are wexp-nonsolvable. Apart from three exceptions (A_5, A_6, and S_6), all symmetric and alternating groups that are not solvable are wexp-nonsolvable.

In the course of proving that a group G is wexp-solvable, we will frequently use the following elementary lemma to help verify that a subgroup of G is weakly exponential with respect to G.

Lemma 5.1. *Let $H \leqslant G$ and let $y \in G$ be such that $|y| = p^k$ for some prime p and some $k \geqslant 0$. If H contains a Sylow p-subgroup of G, then $y \in H^g$ for some $g \in G$.*

Proof. The element y is contained in a Sylow p-subgroup of G, and all such subgroups are conjugate in G. ☐

We begin by considering the cases when $n = 5$ or 6.

Lemma 5.2. *The group A_5 is wexp-solvable.*

Proof. The maximal subgroups of $G = A_5$ are isomorphic to S_3 (index 10, one conjugacy class), D_5 (index 6, one conjugacy class; here, D_n indicates a dihedral group of order $2n$), and A_4 (index 5, one conjugacy class). These groups are all solvable, and hence wexp-solvable by Theorem 1.8. So, by Lemma 3.2, it suffices to verify that each of these subgroups is weakly exponential with respect to G. Note that elements of G have order 1, 2, 3 or 5.

Let M be a maximal subgroup of G and let $x \in G$. If $M \cong S_3$, then $|x^{|G:M|}| \in \{1,3\}$ and M contains a Sylow 3-subgroup of G. If $M \cong D_5$, then $|x^{|G:M|}| \in \{1,5\}$ and M contains a Sylow 5-subgroup of G. Finally, if $M \cong A_4$, then $|x^{|G:M|}| \in \{1,2,3\}$ and M contains both a Sylow 2-subgroup and a Sylow 3-subgroup of G. In all cases, by Lemma 5.1 we have $x^{|G:M|} \in M^g$ for some $g \in G$. Thus, every maximal subgroup of G is weakly exponential with respect to G, as desired. ☐

Lemma 5.3. *The group A_6 is wexp-solvable.*

Proof. Let $G = A_6$. The maximal subgroups of G are isomorphic to S_4 (index 15, two conjugacy classes), $(C_3 \times C_3) \rtimes C_4$ (index 10, one conjugacy class), and A_5 (index 6, two conjugacy classes). Every element x of G is contained in a cyclic group of order 3, 4, or 5. Let M be a maximal subgroup of G. If $M \cong S_4$, then $|x^{|G:M|}| \in \{1,2,4\}$ and M contains a Sylow 2-subgroup of G, so Lemma 5.1 applies in this case. If $M \cong (C_3 \times C_3) \rtimes C_4$, then $|x^{|G:M|}| \in \{1,2,3\}$. This time, M contains a Sylow 3-subgroup of G, but not a Sylow 2-subgroup of G. However, all elements of order 2 are conjugate in G, so $x^{|G:M|} \in M^g$ for some $g \in G$ regardless of the order of $x^{|G:M|}$. Lastly, if $M \cong A_5$, then $|x^{|G:M|}| \in \{1,2,5\}$ and M contains a Sylow 5-subgroup of G. Here, we may proceed as in the prior case to conclude that $M \leqslant_{wexp} G$. ☐

Lemma 5.4. *The group S_5 is minimal wexp-nonsolvable.*

Proof. Let $G = S_5$. This group is not wexp-solvable. Take $x = (1\ 2\ 3)(4\ 5)$ and $M = \mathrm{Sym}(\{1,2,3,4\}) = S_4$. Then, $|x| = 6$ and $|G : M| = 5$, so $|x^{|G:M|}| = 6$. Since S_4 does not contain an element of order 6, M is not weakly exponential with respect to G. The only proper quotient of G is solvable and thus wexp-solvable by Theorem 1.8, so G is minimal wexp-nonsolvable. ☐

Lemma 5.5. *The group S_6 is wexp-solvable.*

Proof. Let $G = S_6$. The maximal subgroups of G are isomorphic to $S_4 \times S_2$ (index 15, two conjugacy classes), S_3 wr S_2 (index 10, one class), S_5 (index 6, two classes), and A_6 (index 2, one class). Furthermore, G contains three classes of elements of order 2, two classes of order 3, two classes of order 4, one class of order 5, and two classes of order 6. We will begin by checking that $M \leqslant_{wexp} G$ for each maximal subgroup M of G. Let $x \in G$.

If $M \cong A_6$, then M is normal in G and hence $M \leqslant_{exp} G$. If $M \cong S_4 \times S_2$, then $|x^{|G:M|}| \in \{1,2,4\}$ and M contains a Sylow 2-subgroup of G, so we may use Lemma 5.1.

Next, if $M \cong S_3$ wr S_2, then $|x^{|G:M|}| \in \{1,2,3\}$ and M contains a Sylow 3-subgroup of G, so Lemma 5.1 applies if $|x^{|G:M|}| \neq 2$. On the other hand, if $|x^{|G:M|}| = 2$, then x is either a 4-cycle or the product of a 4-cycle and a 2-cycle; either way, $x^{|G:M|}$ is a product of two disjoint 2-cycles. Such a product stabilizes a decomposition of $\{1,\ldots,6\}$ into two sets of size 3, so a product of two disjoint 2-cycles is contained in a conjugate of M in G. Thus, $M \leqslant_{wexp} G$.

Finally, consider the case where $M \cong S_5$; note that there are two conjugacy classes of subgroups isomorphic to S_5 in G. This time, $|x^{|G:M|}| \in \{1,2,5\}$ and M contains a Sylow 5-subgroup of G, so, if $|x^{|G:M|}| \in \{1,5\}$, then we are done by Lemma 5.1. If $|x^{|G:M|}| = 2$, then, by similar reasoning as above, $x^{|G:M|}$ is a product of two disjoint 2-cycles. Since all subgroups isomorphic to S_5 in G contain a product of two disjoint transpositions, $x^{|G:M|} \in M^g$ for some $g \in G$ when x is a product of two 2-cycles as well, and hence $M \leqslant_{wexp} G$.

At this point, we know that every maximal subgroup of G is weakly exponential with respect to G. Since the maximal subgroups isomorphic to one of $S_4 \times S_2$, S_3 wr S_2, or A_6 are wexp-solvable (Theorem 1.8 or Lemma 5.3), by Lemma 3.1, any subgroup of G that is contained in a maximal subgroup isomorphic to one of these is weakly exponential with respect to G. The only proper subgroups of G that are not maximal and not contained in a subgroup isomorphic to one of $S_4 \times S_2$, S_3 wr S_2, or A_6 are isomorphic to $C_5 \rtimes C_4$ and have index 36 in G. Let $H \cong C_5 \rtimes C_4$ be such a subgroup. Then, $|x^{|G:H|}| \in \{1,5\}$ and H contains a Sylow 5-subgroup of G. Thus, $H \leqslant_{wexp} G$ by Lemma 5.1, which completes the proof. □

We are now ready to prove the classification of minimal wexp-nonsolvable groups that are symmetric or alternating, starting with symmetric groups.

Proof of Theorem 1.10. Let $G = S_n$. When $n \leqslant 4$, S_n is solvable and hence wexp-solvable by Theorem 1.8. By Lemmas 5.4 and 5.5, the group S_5 is minimal wexp-nonsolvable and the group S_6 is wexp-solvable. So, assume $n \geqslant 7$.

We may express n as $n = k + \ell$, where $\gcd(k,n) = \gcd(\ell,n) = 1$ and $\ell \geqslant k \geqslant 3$. Explicitly, when n is odd, we can take $k = \frac{n-1}{2}$ and $\ell = \frac{n+1}{2}$; when n is divisible by 4, we may set $k = \frac{n}{2} - 1$ and $\ell = \frac{n}{2} + 1$; and when $n \equiv 2 \pmod 4$, we can let $k = \frac{n}{2} - 2$ and $\ell = \frac{n}{2} + 2$

Take $x = (1\ 2\ \ldots\ k)(k+1\ \ldots\ n)$, the product of a disjoint k-cycle with a disjoint ℓ-cycle, and let $H = S_{n-1}$. Since $|G:H| = n$, $x^{|G:H|}$ is still the product of a disjoint k-cycle with a disjoint ℓ-cycle, and hence $x^{|G:H|}$ does not fix any points in $\{1,\ldots,n\}$. On the other hand, the conjugates of H in G are the stabilizers of points in the natural action, so $x^{|G:H|}$ cannot be contained in any conjugate of H. Thus, H is not weakly exponential with respect to G, and, when $n \geqslant 7$, S_n is wexp-nonsolvable. The only proper quotient of S_n is C_2 when $n \geqslant 7$, which is solvable and wexp-solvable, and so S_n is indeed minimal wexp-nonsolvable exactly when $n = 5$ or $n \geqslant 7$. □

To end this section, we prove the classification of the minimal wexp-nonsolvable alternating groups.

Proof of Theorem 1.11. The groups A_5 and A_6 are wexp-solvable by Lemmas 5.2 and 5.3, respectively. Assume $n \geqslant 7$, and let $G = A_n$.

If n is even, then the choice of $x \in S_n$ in the proof of Theorem 1.10 is the product of two odd cycles, and hence $x \in A_n$. If we choose $H = A_{n-1}$, then $|G:H| = n$. The proof that $x^{|G:H|} \notin H^g$ for any

$g \in G$ is analogous to the proof for S_n, and H is not weakly exponential with respect to G. Thus, G is wexp-nonsolvable when n is even.

If n is odd, then $n-4$ is odd and $\gcd(n-4, n) = 1$. Consider

$$x = (1\ 2\ \dots\ n-4)(n-3\ n-2)(n-1\ n).$$

If we choose $H = A_{n-1}$, then $|G : H| = n$, and the conjugates of H in G are the stabilizers of a point in the natural action. On the other hand, $x^{|G:H|}$ is still the product of an $(n-4)$-cycle with two disjoint 2-cycles, so $x^{|G:H|}$ fixes no point in the natural action of A_n on $\{1, \dots, n\}$ and is not in any conjugate of H in G. Thus, H is not weakly exponential with respect to G, and G is wexp-nonsolvable. Since G is simple, it is in fact minimal wexp-nonsolvable. □

6. Weakly exponential-solvable groups isomorphic to PSL$(2, q)$

6.1. **Structure of PSL$(2, q)$.** The maximal subgroups of PSL$(2, q)$ were classified by Dickson [1], where q is a prime power. Here, we split the result into cases depending on q (and exclude some small groups, which exhibit exceptional behavior). In what follows, the dihedral group D_n has order $2n$, and, unless otherwise stated, all maximal subgroups in a particular class are conjugate.

Theorem 6.1. *Let $q = p^d \geqslant 4$ be a prime power, $q \neq 5, 7, 9, 11$, and let $e = \gcd(p-1, 2)$. Then, the maximal subgroups of $G = \mathrm{PSL}(2, q)$ are listed in Tables 1 (which applies to all q), 2 ($q = p$ prime), 3 (q even), 4 ($q > p$, q odd).*

TABLE 1. Maximal subgroups of $G = \mathrm{PSL}(2, q)$, all $q \geqslant 4$, $q \neq 5, 7, 9, 11$.

Structure of M	Conditions on q	$\lvert G : M\rvert$	# Conj. Classes
$C_p^d \rtimes C_{(q-1)/e}$	all q	$q+1$	1
$D_{(q-1)/e}$	all q	$\frac{q(q+1)}{e}$	1
$D_{(q+1)/e}$	all q	$\frac{q(q-1)}{e}$	1

TABLE 2. Other maximal subgroups of $G = \mathrm{PSL}(2,p)$, p prime, $p \geqslant 13$.

Structure of M	Conditions on p	$\|G:M\|$	# Conj. Classes
A_5	$p \equiv \pm 1 \pmod{10}$	$\frac{(p-1)p(p+1)}{120}$	2
A_4	$p \equiv \pm 3 \pmod 8$, $p \not\equiv \pm 1 \pmod{10}$	$\frac{(p-1)p(p+1)}{24}$	1
S_4	$p \equiv \pm 1 \pmod 8$	$\frac{(p-1)p(p+1)}{48}$	2

TABLE 3. Other maximal subgroups of $G = \mathrm{PSL}(2,q)$, $q > 2$ even.

Structure of M	Conditions on q	$\|G:M\|$	# Conj. Classes
$\mathrm{PGL}(2,q_0) \cong \mathrm{PSL}(2,q_0)$	$q = q_0^r$, r prime, $q_0 > 2$	$\frac{(q-1)q(q+1)}{(q_0-1)q_0(q_0+1)}$	1

TABLE 4. Other maximal subgroups of $G = \mathrm{PSL}(2,q)$, $q = p^d \geqslant 25$, q odd.

Structure of M	Conditions on q	$\|G:M\|$	# Conj. Classes
A_5	$p \equiv \pm 3 \pmod{10}$ and $q = p^2$	$\frac{(q-1)q(q+1)}{120}$	2
$\mathrm{PGL}(2,q_0)$	$q = q_0^2$	$\frac{(q-1)q(q+1)}{2(q_0-1)q_0(q_0+1)}$	2
$\mathrm{PSL}(2,q_0)$	$q = q_0^r$, r prime and odd	$\frac{(q-1)q(q+1)}{(q_0-1)q_0(q_0+1)}$	1

Lemma 6.2. *Let $p \geqslant 13$ be a prime. All maximal subgroups of $\mathrm{PSL}(2,p)$ are wexp-solvable groups.*

Proof. By Theorem 6.1, all maximal subgroups of $\mathrm{PSL}(2,p)$, $p \geqslant 13$, are either solvable or isomorphic to A_5. In either case, these groups are wexp-solvable. □

We also highlight some well-known facts about conjugacy classes of subgroups and elements of $\mathrm{PSL}(2,q)$ [1].

Lemma 6.3. *Let p be prime, $d \geqslant 1$, and $q = p^d \geqslant 4$.*

(1) If $e = \gcd(p-1,2)$, then every element of $\mathrm{PSL}(2,q)$ is contained in a cyclic subgroup isomorphic to $C_{(q-1)/e}$, C_p, or $C_{(q+1)/e}$. Moreover, all cyclic groups of a given order are conjugate in $\mathrm{PSL}(2,q)$.

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(2) If p is odd, then $\mathrm{PSL}(2,p)$ *contains a unique conjugacy class of elements of order* 2*, a unique conjugacy class of elements of order* 3*, and at most one conjugacy class of elements of order* 4*.*

Because of Lemma 6.3(1), when checking whether a subgroup of $\mathrm{PSL}(2,q)$ is weakly exponential, it is enough to focus on elements of order p, $(q-1)/e$, or $(q+1)/e$.

The following result applies for all prime powers q such that $q \geqslant 4$, $q \neq 5, 7, 9, 11$. As in Theorem 6.1, we omit these cases due to the exceptional behavior of maximal subgroups.

Proposition 6.4. *Let p be prime, $d \geqslant 1$, $q = p^d \geqslant 4$, $q \neq 5, 7, 9, 11$, and let* $e = \gcd(p-1, 2)$*. If* $G = \mathrm{PSL}(2,q)$ *and $M \leqslant G$ is isomorphic to one of $C_p^d \rtimes C_{(q-1)/e}$, $D_{(q-1)/e}$, or $D_{(q+1)/e}$, then $M \leqslant_{wexp} G$.*

Proof. By Lemma 6.3, every element of G is contained in a cyclic subgroup isomorphic to $C_{(q-1)/e}$, C_p, or $C_{(q+1)/e}$, so we need only check elements x with orders p, $(q-1)/e$, or $(q+1)/e$ in G for each M.

Suppose first that $|x| = p$. If $M \cong C_p^d \rtimes C_{(p-1)/e}$, then x is in a conjugate of M, and we are done. Otherwise, $|G : M|$ is divisible by p, and so the result holds in this case.

Next, suppose $|x| = (q-1)/e$. If $M \cong C_p^d \rtimes C_{(p-1)/e}$ or $M \cong D_{(q-1)/e}$, then x is in a conjugate of M, and we are done. Otherwise, $|G : M|$ is divisible by $(q-1)/e$, and so the result holds in this case.

Finally, suppose $|x| = (q+1)/e$. If $M \cong D_{(q+1)/e}$, then x is in a conjugate of M, and we are done. Otherwise, $|G : M|$ is divisible by $(q+1)/e$, proving the result. □

6.2. **The primes for which** $\mathrm{PSL}(2,p)$ **is wexp-solvable.**

The purpose of this section is to prove Theorem 1.12 in the case when $q = p$. We will need a few preliminary results to do so.

Proposition 6.5. *Let $p \geqslant 13$ be prime. The group* $G = \mathrm{PSL}(2,p)$ *is wexp-solvable if and only if the maximal subgroups isomorphic to one of A_4, S_4, or A_5 are weakly exponential with respect to G.*

Proof. By Lemma 6.2, every maximal subgroup of G is wexp-solvable, so, by Lemma 3.2, G is wexp-solvable if and only if all maximal subgroups are weakly exponential with respect to G. That we need only check subgroups isomorphic to A_4, A_5, or S_4 follows from Theorem 6.1 and Proposition 6.4. □

Lemma 6.6. *Let $p \geqslant 13$ be prime, let* $G = \mathrm{PSL}(2,p)$*, and let M be a maximal subgroup of G isomorphic to one of A_5, A_4, or S_4.*

(1) If $a \in G$ and $|a| = p$, then $a^{|G:M|} = 1$.

(2) There exist unique even integers k and ℓ such that

  (i) $2|M| = k\ell$, and

  (ii) $(p-1)/k$ and $(p+1)/\ell$ are coprime integers.

(3) With k and ℓ as in part (2), let $x, y \in G$ such that $|x| = (p-1)/2$ and $|y| = (p+1)/2$. Then, $|x^{|G:M|}| = k/2$ and $|y^{|G:M|}| = \ell/2$.

Proof. Note that $|G : M| = (p-1)p(p+1)/(2|M|)$ and the only primes that could divide $|M|$ are 2, 3, and 5.

(1) Since $p \geqslant 13$, p is coprime to $2|M|$, $p-1$, and $p+1$. Thus, $\gcd(p, |G : M|) = p$, and so $a^{|G:M|} = 1$ whenever $|a| = p$.

(2) Certainly, $(p-1)(p+1)/(2|M|)$ is an integer, and the integers $p-1$ and $p+1$ have no odd prime factors in common. Moreover, exactly one of $p-1$ or $p+1$ is congruent to $2 \pmod 4$, and the other is congruent to $0 \pmod 4$. Without loss of generality, assume that $p-1 \equiv 2 \pmod 4$. Factor $2|M|$ as $2|M| = 2^e m$, where $e \geq 3$ and m is odd. Take $k = 2\gcd(p-1, m)$ and $\ell = 2^{e-1}\gcd(p+1, m)$. Then, k and ℓ have all of the required properties. Note that these integers are unique, since k must be congruent to $2 \pmod 4$, ℓ must be congruent to $0 \pmod 4$, and $p-1$ and $p+1$ share no odd prime factors.

(3) We may factor $|G : M|$ as

$$|G : M| = \frac{p-1}{k} \cdot p \cdot \frac{p+1}{\ell},$$

where $(p-1)/k$ and $(p+1)/\ell$ are coprime. Then,

$$\gcd\left(\frac{p-1}{2}, |G : M|\right) = \frac{p-1}{k} \quad \text{and} \quad \gcd\left(\frac{p+1}{2}, |G : M|\right) = \frac{p+1}{\ell},$$

which implies that $|x^{|G:M|}| = k/2$ and $|y^{|G:M|}| = \ell/2$. □

Lemma 6.7. *Let $p \geqslant 13$ be prime. If $p \equiv \pm 1 \pmod{10}$, $G = \mathrm{PSL}(2, p)$, and $M \cong A_5$ is a subgroup of G, then there exists an element $x \in G$ of order $(p-1)/2$ or $(p+1)/2$ such that the order of $x^{|G:M|}$ is at least 6. In particular, M is not weakly exponential with respect to G, and G is wexp-nonsolvable.*

Proof. Assume $p \equiv \pm 1 \pmod{10}$. By Lemma 6.3, it suffices to consider elements of G of order p, $(p-1)/2$, and $(p+1)/2$. From Lemma 6.6, we know that $a^{|G:M|} \in M$ whenever $|a| = p$. For elements of other orders, we will consider cases based on the value of $p \pmod{12}$. The tables below summarize the possibilities; the notation is as in Lemma 6.6.

TABLE 5. $M \cong A_5$, $p \equiv 1 \pmod{10}$

$p \pmod{12}$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	60	2	30	1
5	20	6	10	3
7	30	4	15	2
11	10	12	5	6

TABLE 6. $M \cong A_5$, $p \equiv 9 \pmod{10}$

$p \pmod{12}$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	12	10	6	5
5	4	30	2	15
7	6	20	3	10
11	2	60	1	30

Since A_5 contains no elements of order 6, 15, 10, or 30, we see that in each case either $x^{|G:M|}$ or $y^{|G:M|}$ fails to be in any conjugate of M. Therefore, M is not weakly exponential with respect to G, and G is wexp-nonsolvable. □

Lemma 6.8. *Let* $p \geqslant 13$ *be prime. Let* $p \equiv \pm 3 \pmod{10}$ *and* $p \equiv \pm 3 \pmod 8$*. Then, a maximal subgroup* $M \cong A_4$ *is weakly exponential with respect to* $G = \mathrm{PSL}(2,p)$ *if and only if* $p \equiv \pm 5 \pmod{24}$*. In particular, if* $p \not\equiv \pm 5 \pmod{24}$*, then there exists an element* $x \in G$ *of order* $(p-1)/2$ *or* $(p+1)/2$ *such that the order of* $x^{|G:M|}$ *is* 6*, and a maximal subgroup* $M \cong A_4$ *is weakly exponential with respect to* G *if and only if*

$$p \equiv 43, 53, 67, 77 \pmod{120}.$$

Proof. As in Lemma 6.7, it suffices to consider elements of G of order $(p-1)/2$ or $(p+1)/2$. We will consider cases depending on the value of $p \pmod 3$, and use the notation of Lemma 6.6. Tables 7 and 8 list the possibilities.

TABLE 7. $M \cong A_4$, $p \equiv 3 \pmod 8$

$p \pmod 3$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	6	4	3	2
2	2	12	1	6

TABLE 8. $M \cong A_4$, $p \equiv 5 \pmod 8$

$p \pmod 3$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	12	2	6	1
2	4	6	2	3

Now, A_4 contains no element of order 6, so M is not weakly exponential with respect to G when either $p \equiv 3 \pmod 8$ and $p \equiv 2 \pmod 3$, or $p \equiv 5 \pmod 8$ and $p \equiv 1 \pmod 3$. In the other two cases, one of $x^{|G:M|}$ or $y^{|G:M|}$ equals 2, and the other equals 3. By Lemma 6.3(2), all elements of order 2 are conjugate in G, and likewise for elements of order 3. Since A_4 contains both elements of order 2 and elements of order 3, this means that some conjugate of M contains $x^{|G:M|}$, and some conjugate of M contains $y^{|G:M|}$.

We conclude that if $p \equiv 3 \pmod 8$, then M is weakly exponential with respect to G whenever $p \equiv \pm 3 \pmod{10}$ and $p \equiv 1 \pmod 3$. This is equivalent to having $p \equiv 43, 67 \pmod{120}$. Similarly, if $p \equiv 5 \pmod 8$, then M has the desired property when $p \equiv 53, 77 \pmod{120}$. ☐

Lemma 6.9. *Let* $p \geqslant 13$ *be prime. Let* $p \equiv \pm 3 \pmod{10}$ *and* $p \equiv \pm 1 \pmod 8$*. Then, a maximal subgroup* $M \cong S_4$ *is weakly exponential with respect to* $G = \mathrm{PSL}(2,p)$ *if and only if* $p \equiv \pm 7 \pmod{24}$*. In particular, if* $p \not\equiv \pm 7 \pmod{24}$*, then there exists an element* $x \in G$ *of order* $(p-1)/2$ *or* $(p+1)/2$ *such that the order of* $x^{|G:M|}$ *is* 12*, and a maximal subgroup* $M \cong S_4$ *is weakly exponential with respect to* G *if and only if*

$$p \equiv 7, 17, 103, 113 \pmod{120}.$$

Proof. Just as in the previous lemma, we consider cases depending on the residue of p modulo 3 and use Lemmas 6.3 and 6.6. The relevant data is shown in the tables below.

TABLE 9. $M \cong S_4$, $p \equiv 1 \pmod 8$

$p \pmod 3$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	24	2	12	1
2	8	6	4	3

TABLE 10. $M \cong S_4$, $p \equiv 7 \pmod 8$

$p \pmod 3$	k	ℓ	$\lvert x^{\lvert G:M\rvert}\rvert$	$\lvert y^{\lvert G:M\rvert}\rvert$
1	6	8	3	4
2	2	24	1	12

The group S_4 contains no element of order 12, but does contain elements of order 3 and elements of order 4. Moreover, in G, all elements of order 3 are conjugate, as are all elements of order 4. Arguing as in Lemma 6.8, we conclude that M is weakly exponential with respect to G if and only if either $p \equiv 1 \pmod 8$ and $p \equiv 2 \mod 3$, or $p \equiv 7 \pmod 8$ and $p \equiv 1 \pmod 3$. Equivalently, $p \equiv 7, 17 \pmod{24}$. □

We are now ready to prove Theorem 1.12 in the case when q is prime.

Theorem 6.10. *Let p be a prime. Then,* $\mathrm{PSL}(2,p)$ *is a wexp-solvable group if and only if $p = 2, 3, 5$ or*

$$p \equiv 7, 17, 43, 53, 67, 77, 103, 113 \pmod{120}.$$

Proof. First, when $p = 2$, 3, or 5, $\mathrm{PSL}(2,p)$ is isomorphic to (respectively) S_3, A_4, or A_5, each of which is wexp-solvable by Theorem 1.8 or Lemma 5.2.

The proof that $\mathrm{PSL}(2,7)$ is wexp-solvable is similar to that for A_5. The group $\mathrm{PSL}(2,7)$ contains two classes of maximal subgroups isomorphic to S_4 (index 7) and one conjugacy class of subgroups isomorphic to $C_7 \rtimes C_3$ (index 8). There is a single conjugacy class of elements of order 2, order 3, and order 4, and, while there are two conjugacy classes of elements of order 7, elements from both conjugacy classes are contained in a single $C_7 \rtimes C_3$ maximal subgroup. Again, the result follows from inspection.

Next, $\mathrm{PSL}(2,11)$ is not wexp-solvable: let x be an element of order 6 and let M be a maximal subgroup isomorphic to A_5 (index 11). Then, $x^{\lvert G:M\rvert}$ has order 6, but A_5 does not contain elements of order 6.

Finally, let $p \geqslant 13$ be prime. By Proposition 6.5, it suffices to check whether maximal subgroups isomorphic to one of A_4, A_5, or S_4 are weakly exponential. By Lemmas 6.7, 6.8, 6.9, all such maximal subgroups are weakly exponential when $p \equiv \pm 3 \pmod{10}$ and when $p \equiv \pm 5, \pm 7 \pmod{24}$, i.e., exactly when

$$p \equiv 7, 17, 43, 53, 67, 77, 103, 113 \pmod{120},$$

as desired. □

Remark 6.11. Lemmas 6.7, 6.8, 6.9, and the proof of Theorem 6.10 show that, if $G = \mathrm{PSL}(2,p)$ is wexp-nonsolvable, then there exists $x \in G$ of order $(p \pm 1)/2$ and $H \leqslant G$ such that H does not contain an element of order $\lvert x^{\lvert G:H\rvert}\rvert$.

6.3. **The groups PSL$(2,q)$ that are wexp-solvable.** This subsection is dedicated to the proof of Theorem 1.12. Throughout this subsection, q will be a prime power.

Proposition 6.12. *Let $q = q_0^2$, where $q_0 \geqslant 4$. Then, $G = \mathrm{PSL}(2,q)$ is not wexp-solvable.*

Proof. By Theorem 6.1, G contains a maximal subgroup M isomorphic to $\mathrm{PGL}(2,q_0)$. If $e = \gcd(q-1,2)$, then $|G:M| = q_0(q+1)/e$.

Let x be an element of order $(q-1)/e$ in G. Since $\gcd(|G:M|,(q-1)/e) = 1$, the element $x^{|G:M|}$ has order $(q-1)/e$. Since the maximum order of an element in $M \cong \mathrm{PGL}(2,q_0)$ is q_0+1 and

$$q_0 + 1 < \frac{q_0^2 - 1}{e} = \frac{q-1}{e},$$

we see that $x^{|G:M|}$ is not in any conjugate of M in G. Thus, M is not weakly exponential, and hence G is not wexp-solvable. □

Lemma 6.13. *Let $q = q_0^n$, where n is an odd integer. Let $G = \mathrm{PSL}(2,q)$, H be a subgroup of G isomorphic to $\mathrm{PSL}(2,q_0)$, and $e = \gcd(q-1,2)$.*

(1) $|G:H| = k \cdot \ell \cdot m$, where $k = (q-1)/(q_0-1)$, $\ell = q/q_0$, $m = (q+1)/(q_0+1)$, and k, ℓ, and m are pairwise coprime.

(2) If x is an element of order p in G, then $x^{|G:H|} = 1$.

(3) If x is an element of order $(q-1)/e$ in G, then $x^{|G:H|}$ has order $(q_0-1)/e$ and is contained in a conjugate of H in G.

(4) If x is an element of order $(q+1)/e$ in G, then $x^{|G:H|}$ has order $(q_0+1)/e$ and is contained in a conjugate of H in G.

(5) H is weakly exponential with respect to G.

Proof. First, since $q = q_0^n$, n odd, we have $|G:H| = k \cdot \ell \cdot m$ as in the statement of (1), where k, ℓ, and m are integers. It is immediate that $\gcd(k,\ell) = \gcd(\ell,m) = 1$. Since $\gcd(q-1,q+1) = e \leqslant 2$ and $q_0 \pm 1$ is even if and only if $e = 2$, we see that $\gcd(k,m) = 1$ as well, proving (1).

For (2), if x is an element of order p, then p divides ℓ, and so

$$x^{|G:H|} = x^{\ell} = 1.$$

For (3), if x is an element of order $(q-1)/e$, then, by (1), we see that $|x^{|G:H|}| = |x^k|$, and so $x^{|G:H|}$ has order

$$\frac{\frac{q-1}{e}}{\frac{q-1}{q_0-1}} = \frac{q_0-1}{e}.$$

By Lemma 6.3(1), all cyclic subgroups of order $(q_0-1)/e$ are conjugate in G. Since H contains a cyclic subgroup of order $(q_0-1)/e$, we see that $x^{|G:H|}$ is contained in some conjugate of H, as desired.

The proof for (4) is analogous to the proof of (3).

To prove (5), by Lemma 6.3(1), it suffices to check elements of order $(q-1)/e$, p, and $(q+1)/e$ in G. If x is an element of one of these orders, then, by the previous parts of the lemma, $x^{|G:H|}$ is contained in a conjugate of H, showing that $H \leqslant_{wexp} G$. □

Proposition 6.14. *Let $q = p^d$, where d is odd, and let $G = \mathrm{PSL}(2, q)$. Then, G is wexp-solvable if and only if $\mathrm{PSL}(2, p)$ is wexp-solvable.*

Proof. Assume $\mathrm{PSL}(2, p)$ is wexp-solvable. We proceed by induction on the number of (not necessarily distinct) primes in a factorization of d. First, let d be an odd prime. Then, by Theorem 6.1, and Proposition 6.4, it suffices to consider maximal subgroups of $G = \mathrm{PSL}(2, p^d)$ isomorphic to $\mathrm{PSL}(2, p)$, which by Lemma 6.13(5) are weakly exponential with respect to G. Thus, by Lemma 3.2, $\mathrm{PSL}(2, q)$ is wexp-solvable when $q = p^d$, d an odd prime.

Now, assume the result is true when d is a product of $n-1$ primes, where $n \geqslant 2$. So, assume d is the product of n primes. Again, by Theorem 6.1, and Proposition 6.4, it suffices to consider maximal subgroups of $G = \mathrm{PSL}(2, p^d)$ isomorphic to $\mathrm{PSL}(2, q_0)$, where $q = q_0^r$, r prime. Since $q_0 = p^{d/r}$ and d/r is the product of $n-1$ primes, $\mathrm{PSL}(2, q_0)$ is wexp-solvable. Moreover, $\mathrm{PSL}(2, q_0)$ is weakly exponential with respect to G by Lemma 6.13(5). Hence, G is itself wexp-solvable. Therefore, if d is odd, $\mathrm{PSL}(2, p^d)$ is wexp-solvable if $\mathrm{PSL}(2, p)$ is.

Conversely, suppose that p is such that $\mathrm{PSL}(2, p)$ is not wexp-solvable, and let $H \leqslant G$ be isomorphic to $\mathrm{PSL}(2, p)$. By Remark 6.11, we may assume that there exists $h \in H$ of order either $(p-1)/2$ or $(p+1)/2$ and $L \leqslant H$ such that L does not contain an element of order $|h^{|H:L|}|$. By Lemma 6.3, h is contained in $\langle x \rangle$, where the order of x is $(q-1)/2$ or $(q+1)/2$, respectively. By Lemma 6.13(3),(4), $x^{|G:H|}$ has order $(p-1)/2$ or $(p+1)/2$, respectively, and hence $\langle x^{|G:H|} \rangle = \langle h \rangle$. Consequently, $h = (x^a)^{|G:H|}$ for some integer a. This means that

$$(x^a)^{|G:L|} = ((x^a)^{|G:H|})^{|H:L|} = h^{|H:L|},$$

and, since L does not contain any elements of this order, $(x^a)^{|G:L|}$ is not in any conjugate of L in G. Therefore, L is not weakly exponential with respect to G, and G is not wexp-solvable, completing the proof. □

We now have enough results to prove Theorem 1.12.

Proof of Theorem 1.12. Let p be a prime, $q = p^d$, and $G = \mathrm{PSL}(2, q)$. If $d = 1$, then the result follows from Theorem 6.10, so we assume $d \geqslant 2$. Suppose first that d is even. If $p = 2$ and $d = 2$, then $G \cong A_5$ and is wexp-solvable by Lemma 5.2, and, if $p = 3$ and $d = 2$, then $G \cong A_6$ and is wexp-solvable by Lemma 5.3. Otherwise, $q \geqslant 16$ and G is not wexp-solvable by Proposition 6.12. Finally, if d is odd, then G is wexp-solvable if and only if $\mathrm{PSL}(2, p)$ is wexp-solvable by Proposition 6.14, completing the proof. □

REFERENCES

[1] L. E. Dickson, *Linear groups: With an exposition of the Galois field theory*, With an introduction by W. Magnus. Dover Publications, Inc., New York, 1958.

[2] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.2*, 2022. https://www.gap-system.org.

[3] M. Garonzi, L.-C. Kappe and E. Swartz, On integers that are covering numbers of groups, *Exp. Math.*, **31** no. 2 (2022) 425–443.

[4] I. M. Isaacs, *Finite group theory*, Graduate Studies in Mathematics, **92**, American Mathematical Society, Providence, RI, 2008.

[5] L. Sabatini, Products of subgroups, subnormality, and relative orders of elements, *Ars Math. Contemp.*, **24** no. 1 (2024) 9 pp.

[6] N. J. Werner, A note on finite nilpotent groups. *Amer. Math. Monthly*, **131** no. 9 (2024) 803–805. https://doi.org/10.1080/00029890.2024.2379751

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