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THE PROBABILITY THAT TWO ELEMENTS OF A GROUP HAVE THE SAME CENTRALIZERS

SAMAN RAHIMIRAD AND MOHAMMAD ZARRIN

ABSTRACT. In this paper we provide some bounds for the probability, denoted by $\mathcal{PC}(G)$, that two randomly chosen elements in a given finite group have the same centralizers. In particular, among other results, we give the following best possible bounds for $\mathcal{PC}(G)$, depending only on $|G : Z(G)|$ and the smallest prime divisor of $|G|$.

1. Introduction

The concept of the probability in group theory has been studied by many authors. For example, P. Neumann in [5], considered two combinatorial problems; Wilson in [9] studied the probability of generating a nilpotent subgroup of a finite group, denoted by $\pi_n(G)$, and proved that if $\pi_n(G)$ tends to 0 as the index of the Fitting subgroup of G tends to infinity.

One of the most studied probabilities is the probability that two elements of a finite group G commute which is denoted by $d(G)$ (or $\pi_a(G)$) and is called the commutativity degree of G. Clearly

$$\pi_a(G) \leq \pi_n(G).$$

Many authors, have tried to give formulas of $d(G)$ for some particular finite groups G. Also, Tarnauceanua in [7], gave a generalization of the commutativity degree and studied the concept of the subgroup commutativity degree of a finite group G, (see also [8]).

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*Corresponding author.

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We introduce and study the concept of the same-cent degree of a finite group G. Assume that S is a non-empty subset of the finite group G. Put

$$C_n(S) = \{(x_1, x_2, \ldots, x_n) \in S \times \cdots \times S \mid \ \forall\, 1 \leq i \leq j \leq n \ C_G(x_i) = C_G(x_j)\},$$

where $C_G(a)$ is the centralizer of a in G. Now, we define $\mathcal{PC}^n(S)$ to be the probability that $(n+1)$ elements $x_1, x_2, \ldots, x_{n+1} \in S$ have the same centralizers, where x_i, $1 \leq i \leq n+1$, are chosen independently and uniformly at random from S and we call it the n-th same-cent degree of a finite group G. In fact, we have

$$\mathcal{PC}^n(S) = \frac{|C_{n+1}(S)|}{|S|^{n+1}}.$$

In this paper we study the probability $\mathcal{PC}^1(G)$, simply denoted by $\mathcal{PC}(G)$, that two randomly chosen elements in a given finite group have the same centralizers. In particular, we give the following best possible upper and lower bounds for $\mathcal{PC}(G)$, depending only on $|G : Z(G)|$ and the smallest prime divisor of $|G|$. We also explore the influence of $\mathcal{PC}(G)$ on the structure of groups.

Our main results are as follows:

Theorem 1.1. *Let G be a finite group with center $Z(G)$ and p be the smallest prime divisor of $|G|$. Then*

$$\frac{1}{|G : Z(G)|} \leq \mathcal{PC}(G) \leq \frac{2}{|G : Z(G)|^2} - (1 + \frac{1}{p})\frac{1}{|G : Z(G)|} + \frac{1}{p}.$$

Moreover, if $|Cent(G)| = |G : Z(G)|$. Then

$$\mathcal{PC}(G) = \frac{1}{|G : Z(G)|},$$

where $Cent(G) = \{C_G(g) \mid g \in G\}$.

In fact, as a result, we show that if G is a non-abelian finite group and p is the smallest prime divisor of $|G|$, Then $\mathcal{PC}(G) < \frac{1}{p}$, see Corollary 3.3.

Theorem 1.2. *Let G be a finite non-abelian group. Then*

$$\pi_a(G) - \mathcal{PC}(G) \geq \frac{2}{|G : Z(G)|} - \frac{2}{|G : Z(G)|^2}.$$

Moreover, the equality holds for $AC-$groups.

2. Some properties of $\mathcal{PC}(G)$

First of all, note that for a non-abelian group, since for any element $x \in G \setminus Z(G)$, $(1, x) \notin C_2(G)$, we have

$$\mathcal{PC}(G) \lneqq \pi_a(G).$$

Therefore, $\mathcal{PC}(G) = \pi_a(G)$ if and only if G is an abelian group. Moreover, Gustafson in [3] proved that $\pi_a(G) \leq \frac{5}{8}$, for any non-abelian finite group G. In this paper we show that $\mathcal{PC}(G) < \frac{1}{2} = \frac{4}{8}$, for any non-abelian finite group G, see Corollary 3.3.

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

Lemma 2.1. *Let H and K be two finite groups. Then*

$$\mathcal{PC}(H \times K) = \mathcal{PC}(H)\mathcal{PC}(K).$$

Proof. It is straightforward. □

Corollary 2.2. *Let G be a Hamiltonian group. Then $\mathcal{PC}(G) = \frac{1}{4}$.*

Proof. It is enough to note that $\mathcal{PC}(Q_8) = \frac{1}{4}$. □

Lemma 2.3. *Let G be a non-abelian finite group, and N be a normal subgroup of G where $|N \cap G'| = 1$. Then*

$$\mathcal{PC}(G) = \frac{1}{|N|^2}\mathcal{PC}(\frac{G}{N}).$$

Proof. First, we claim that:

$$\frac{C_G(x)N}{N} = C_{\frac{G}{N}}(xN).$$

Clearly for each element $x \in G$ we have

$$\frac{C_G(x)N}{N} \subseteq C_{\frac{G}{N}}(xN).$$

Let $gN \in C_{\frac{G}{N}}(xN)$. Therefore, $g^{-1}x^{-1}gx \in N$, so $gx = xg$ and $g \in C_G(x)$. Hence $\frac{C_G(x)N}{N} = C_{\frac{G}{N}}(xN)$. It is easy to see that if $C_G(x) = C_G(y)$, then $C_{\frac{G}{N}}(xN) = C_{\frac{G}{N}}(yN)$.

Now Suppose $xN, yN \in \frac{G}{N}$ such that $C_{\frac{G}{N}}(xN) = C_{\frac{G}{N}}(yN)$. Now if $gN \in C_{\frac{G}{N}}(xN) = C_{\frac{G}{N}}(yN)$, then $gyN = ygN$ and so, by assumption $|N \cap G'| = 1$, it implies that $g \in C_G(y)$. Therefore $C_G(x) \subseteq C_G(y)$ and similarly, $C_G(y) \subseteq C_G(x)$. Hence $C_G(x) = C_G(y)$. It follows that $|C_2(G)| = |C_2(\frac{G}{N})|$. Thus $\mathcal{PC}(G) = \frac{1}{|N|^2}\mathcal{PC}(\frac{G}{N})$. □

Note that if $|N \cap G'| \neq 1$, then Lemma 2.3 does not necessarily hold. For example consider the Dihedral group of order 16, $G = D_{16}$, and $N = Z(G)$.

Here we are going to find the $\mathcal{PC}(G)$, for a few of famous groups, like Dihedral groups and p-groups, see Proposition 3.9.

Proposition 2.4. *(A). Let D_{2n} be the Dihedral group of order $2n$ and $n \geq 3$. Then*

$$\mathcal{PC}(D_{2n}) = \begin{cases} \frac{n^2-n+2}{4n^2} & \text{if } n \text{ is odd} \\ \\ \frac{n^2-2n+8}{4n^2} & \text{if } n \text{ is even} \end{cases}$$

(B). Let SD_{2^n} be the semi-dihedral group of order 2^n. Then

$$\mathcal{PC}(SD_{2^n}) = \frac{2^{2n-2} - 2^{n+1} + 2^n + 8}{2^{2n}}.$$

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

Proof. (A). Let

$$G = D_{2n} = \langle a, b \ : \ a^n = b^2 = 1, \ bab = a^{-1} \rangle.$$

So $C_{D_{2n}}(a) = \langle a \rangle$.

If n is an odd integer, then $Z(G) = \{1\}$. Therefore, one can see that

$$C_G(a) = C_G(a^2) = \cdots = C_G(a^{n-1}) = \langle a \rangle,$$

and for all $i = 1, 2, \ldots, n$ we have $C_G(a^i b) = \{1, a^i b\}$. Therefore if $i \neq j$ then $C_G(a^i b) \neq C_G(a^j b)$. Consequently $|C_2(D_{2n})| = (n-1)^2 + n + 1 = n^2 - n + 2$ and so

$$\mathcal{PC}(D_{2n}) = \frac{|C_2(D_{2n})|}{|D_{2n}|^2} = \frac{n^2 - n + 2}{4n^2}.$$

If n be an even integer, then $Z(G) = \{1, a^{\frac{n}{2}}\}$. So we can see that $C_G(a^i) = \langle a \rangle$, for all $i \in \{1, \ldots, n-1\} \setminus \{\frac{n}{2}\}$, and for all $1 \leq i \leq n$ we have

$$C_G(a^i b) = \{1, a^i b, a^{\frac{n}{2}}, a^{\frac{n}{2}+i} b\}.$$

Therefore $C_G(a^i b) = C_G(a^{\frac{n}{2}+i} b)$. It follows that

$$|C_2(G)| = (n-2)^2 + 4 \times \frac{n}{2} + 4 = n^2 - 2n + 8.$$

Hence $\mathcal{PC}(G) = \frac{n^2 - 2n + 8}{4n^2}$, as wanted.

(B). Let

$$G = SD_{2^n} = \langle a, b \ : \ a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{2^{n-2}-1} \rangle.$$

We know that $Z(SD_{2^n}) = \{1, a^{2^{n-2}}\}$. So by an argument similar to Part (A), the result follows. □

Corollary 2.5. *For any integer $n \geq 3$, $\mathcal{PC}(D_{2n}) \leq \frac{1}{4}$ and*

$$\lim_{n \to \infty} (D_{2n}) = \frac{1}{4}.$$

3. Proof of the main theorems

To prove our main result, we need the following lemmas and definition.

We define an equivalence relation $\mathcal{R}$ on a finite group G, as below:

$$\forall \ x, y \in G \quad x\mathcal{R}y \ if \ and \ only \ if \ C_G(x) = C_G(y).$$

It is interesting that the commutativity relation among all elements of G is not equivalent relation but the same-centralizers elements relation, i.e. $\mathcal{R}$, among elements of G is equivalent relation. Therefore the equivalence classes of $\mathcal{R}$ are like:

$$\beta(x) = \{y \in G \mid C_G(y) = C_G(x)\}.$$

Clearly, if $x \in Z(G)$ then $\beta(x) = Z(G)$, and if $x \notin Z(G)$ then $\beta(x) \cap Z(G) = \varnothing$.

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

We recall that $Cent(G) = \{C_G(g) \mid g \in G\}$. For more information regarding the influence of $|Cent(G)|$ on groups see [2, 12, 13]. If the $\beta(x_1)$, $\beta(x_2)$,...,$\beta(x_t)$ are the distinct class of equivalence relation $\mathcal{R}$, then obviously $t = |Cent(G)|$. Since $G = \bigsqcup_{i=1}^{t} \beta(x_i)$ and $x_i Z(G) \subseteq \beta(x_i)$ then we have

$$|G| = \sum_{i=1}^{t} |\beta(x_i)| \geq \sum_{i=1}^{t} |Z(G)| = t|Z(G)| = |Cent(G)||Z(G)|.$$

And so $|Cent(G)| \leq |G : Z(G)|$, see of [11, Theorem 2.2] and also [10]. By the above notation we have

$$\mathcal{PC}(G) = \frac{\sum_{i=1}^{|Cent(G)|} |\beta(x_i)|^2}{|G|^2}.$$

Lemma 3.1. *Let G ba a finite group. Then $|\beta(x)| = |\beta(y)|$ for all $x, y \in G$, if and only if $|Cent(G)| = |G : Z(G)|$.*

Proof. For every element x in G we have $xZ(G) \subseteq \beta(x)$. Since $|\beta(x)| = |\beta(1)| = |Z(G)| = |xZ(G)|$, we have $\beta(x) = xZ(G)$.

So $|\beta(x)| = |\beta(y)|$, for all $x, y \in G$, if and only if $\beta(x) = xZ(G)$, for all $x \in G$, if and only if the number of distinct equivalence classes of relation $\mathcal{R}$ be equal to the number of distinct cosets of $Z(G)$, if and only if $|Cent(G)| = |G : Z(G)|$. □

Lemma 3.2. *Let G be a finite group. Then*

$$\mathcal{PC}(G) = \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} |\beta(x)|}{|G|^2}$$

.

Proof. Clearly,

$$\begin{aligned}\mathcal{PC}(G) &= \frac{\sum_{x \in G} |\beta(x)|}{|G|^2} \\ &= \frac{\sum_{x \in Z(G)} |\beta(x)| + \sum_{x \notin Z(G)} |\beta(x)|}{|G|^2} \\ &= \frac{\sum_{x \in Z(G)} |Z(G)| + \sum_{x \notin Z(G)} |\beta(x)|}{|G|^2} \\ &= \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} |\beta(x)|}{|G|^2},\end{aligned}$$

as desired. □

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

Proof of Theorem 1.1. For every element x in G we have $xZ(G) \subseteq \beta(x)$. So $|Z(G)| \leq |\beta(x)|$, for all $x \in G$. Hence

$$\begin{aligned}
\mathcal{PC}(G) &= \frac{\sum\limits_{x \in G} |\beta(x)|}{|G|^2} \\
&\geq \frac{\sum\limits_{x \in G} |Z(G)|}{|G|^2} \\
&= \frac{|G||Z(G)|}{|G|^2} \\
&= \frac{1}{|G : Z(G)|}
\end{aligned}$$

which is the lower bound.

On the other hand, since for all element $x \notin Z(G)$ we have $\beta(x) \cup Z(G) \subseteq C_G(x)$ and $\beta(x) \cap Z(G) = \varnothing$. Therefore, $|\beta(x)| + |Z(G)| \leq |C_G(x)|$, so $|\beta(x)| \leq |C_G(x)| - |Z(G)|$. By using Lemma (3.2),

$$\begin{aligned}
\mathcal{PC}(G) &= \frac{|Z(G)|^2 + \sum\limits_{x \notin Z(G)} |\beta(x)|}{|G|^2} \\
&\leq \frac{|Z(G)|^2 + \sum\limits_{x \notin Z(G)} (|C_G(x)| - |Z(G)|)}{|G|^2} \\
&= \frac{1}{|G : Z(G)|^2} + \frac{1}{|G|^2} \sum_{x \notin Z(G)} |C_G(x)| - \frac{1}{|G|^2} \sum_{x \notin Z(G)} |Z(G)| \\
&= \frac{1}{|G : Z(G)|^2} + \frac{1}{|G|} \sum_{x \notin Z(G)} \frac{1}{|G : C_G(x)|} - \frac{(|G| - |Z(G)|)|Z(G)|}{|G|^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|G : Z(G)|^2} + \frac{1}{|G|} \sum_{x \notin Z(G)} \frac{1}{p} - \frac{|G||Z(G)|}{|G|^2} + \frac{|Z(G)|^2}{|G|^2} \\
&= \frac{2}{|G : Z(G)|^2} + \frac{1}{|G|} \frac{|G| - |Z(G)|}{p} - \frac{1}{|G : Z(G)|} \\
&= \frac{2}{|G : Z(G)|^2} + \frac{1}{p} - \frac{1}{p} \frac{1}{|G : Z(G)|} - \frac{1}{|G : Z(G)|} \\
&= \frac{2}{|G : Z(G)|^2} + \frac{1}{p} - (1 + \frac{1}{p}) \frac{1}{|G : Z(G)|}.
\end{aligned}$$

and this gives the upper bound.

Now if $|Cent(G)| = |G : Z(G)|$. Then according to Lemma 3.1, $|\beta(x)| = |\beta(1)| = |Z(G)|$ for all

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

$x \in G$. Therefore

$$\mathcal{PC}(G) = \frac{\sum_{x \in G} |\beta(x)|}{|G|^2} = \frac{\sum_{x \in G} |Z(G)|}{|G|^2} = \frac{|G||Z(G)|}{|G|^2} = \frac{1}{|G:Z(G)|}.$$

Both bounds are the best possible, for instance consider the Dihedral group of order 8, $G = D_8$.

Corollary 3.3. *Let G be a non-abelian finite group, and p be the smallest prime divisor of $|G|$. Then $\mathcal{PC}(G) < \frac{1}{p}$.*

□

Proof. Suppose, by contrary, that $\frac{1}{p} \leq \mathcal{PC}(G)$. Theorem 1.1 follows that

$$\begin{aligned} \frac{1}{p} &\leq \frac{2}{|G:Z(G)|^2} - (1+\frac{1}{p})\frac{1}{|G:Z(G)|} + \frac{1}{p} \\ &\Rightarrow (\frac{p+1}{p})\frac{1}{|G:Z(G)|} \leq \frac{2}{|G:Z(G)|^2} \\ &\Rightarrow \frac{|G:Z(G)|}{2} \leq \frac{p}{p+1} = 1 - \frac{1}{p+1} \\ &\Rightarrow |G:Z(G)| \leq 2 - \frac{2}{p+1} \\ &\Rightarrow |G:Z(G)| < 2. \end{aligned}$$

Hence $|G:Z(G)| = 1$, which is a contradiction. □

Note that Corollary 3.3 does not apply to $\pi_a(G)$. For example $\pi_a(Q_8) = \frac{5}{8} \not< \frac{1}{2}$.

Corollary 3.4. *If G be a non-abelian finite group, then $\mathcal{PC}(G) < \frac{1}{2}$.*

Lemma 3.5. *If G is a capable group and $|G'| = n$, then*

$$\mathcal{PC}(G) \geq \frac{1}{n^{2\log_2 n}}.$$

Proof. The [6, Corollary 2] and Theorem 1.1 give the result. □

Remark 3.6. For any $x \in G$, $\beta(x) \cup Z(G) \subseteq C_G(x)$, but is not necessarily subgroup of G. For example let $G = S_4$, the symmetric group of degree 4, and $x = (12)$. Then we know that $Z(G) = 1$ and

$$C_G(12) = C_G(34) = \{(1), (12), (34), (12)(34)\}.$$

So $(12), (34) \in \beta((12))$. But

$$\beta((12)(34)) = \{(1), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\} \neq C_G(12).$$

Therefore $(12)(34) \notin \beta(x) \cup Z(G)$.

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

Here we show that for some classes of groups, $\beta(x) \cup Z(G)$ is a subgroup.

A group G is called an AC-group if the centralizer of every non-central element is abelian. If G is an AC-group, then by using [1, Lemma 3.6] we have $\beta(x) = C_G(x) \setminus Z(G)$ for any $x \in G \setminus Z(G)$. So $\beta(x) \cup Z(G) = C_G(x)$, for all $x \in G$, which is a subgroup of G.

A group is called an F-group, if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$, (see [4]).

Lemma 3.7. *Let G be a F-group. Then $\beta(x) \cup Z(G)$ is a subgroup of G, for all element $x \in G$.*

Proof. Let $g_1, g_2 \in \beta(x) \cup Z(G)$. If $g_1 \in \beta(x)$ and $g_2 \in Z(G)$, then $C_G(g_1) = C_G(x)$. Since $g_2 \in Z(G)$, we have $C_G(g_1g_2) = C_G(g_1) = C_G(x)$, so $g_1g_2 \in \beta(x) \cup Z(G)$. If $g_1, g_2 \in Z(G)$, then $g_1g_2 \in \beta(x) \cup Z(G)$. If $g_1, g_2 \in \beta(x)$, then $C_G(g_1) = C_G(g_2) = C_G(x)$. In this case $C_G(x) \leq C_G(g_1g_2)$, and hence $C_G(x) = C_G(g_1g_2)$, therefore $g_1g_2 \in \beta(x) \cup Z(G)$. This shows that $\beta(x) \cup Z(G)$ is a subgroup of G. □

The following question would be interesting, as a new class of groups:

Question 3.8. *Set $F(x) = \beta(x) \cup Z(G)$. Is it possible to charactrize the class of groups in which $F(x)$ is a subgroup of G, for all element $x \in G$?*

Proof of Theorem 1.2. We know that $|\beta(x)| \leq |C_G(x)| - |Z(G)|$. Put $c(G) = \{(x, y) \in G \times G \mid xy = yx\}$. Then $\pi_a(G) = \frac{|c(G)|}{|G|^2}$. Therefore,

$$|c(G)| = \sum_{x \in G} |C_G(x)| = \sum_{x \notin Z(G)} |C_G(x)| + \sum_{x \in Z(G)} |C_G(x)|,$$

so

$$\sum_{x \notin Z(G)} |C_G(x)| = |c(G)| - |G||Z(G)|.$$

Now Lemma 3.2 follows that:

$$\begin{aligned}
\mathcal{PC}(G) &\leq \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} (|C_G(x)| - |Z(G)|)}{|G|^2} \\
&= \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} |C_G(x)| - \sum_{x \notin Z(G)} |Z(G)|}{|G|^2} \\
&= \frac{|Z(G)|^2 + |c(G)| - |G||Z(G)| - (|G| - |Z(G)|)(|Z(G)|)}{|G|^2} \\
&= \frac{2|Z(G)|^2 - 2|G||Z(G)| + |c(G)|}{|G|^2} \\
&= \frac{2}{|G : Z(G)|^2} - \frac{2}{|G : Z(G)|} + \pi_a(G).
\end{aligned}$$

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Hence

$$\pi_a(G) - \mathcal{PC}(G) \geq \frac{2}{|G:Z(G)|} - \frac{2}{|G:Z(G)|^2}.$$

Now, we show that the above equality holds for AC-groups. That is,

$$\pi_a(G) - \mathcal{PC}(G) = \frac{2}{|G:Z(G)|} - \frac{2}{|G:Z(G)|^2}.$$

Assume that, G be an AC-group then $\beta(x) = C_G(x) \setminus Z(G)$, for any $x \in G \setminus Z(G)$. By Lemma 3.2, one can follow that

$$\mathcal{PC}(G) = \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} (|C_G(x)| - |Z(G)|)}{|G|^2}$$
$$= \frac{2}{|G:Z(G)|^2} - \frac{2}{|G:Z(G)|} + \pi_a(G).$$

Hence,

$$\pi_a(G) - \mathcal{PC}(G) = \frac{2}{|G:Z(G)|} - \frac{2}{|G:Z(G)|^2}.$$

In what follows, we are going to find $\mathcal{PC}(G)$, for some finite p-groups.

Proposition 3.9. *Let p be a prime number and G be a non-abelian finite group.*

(1) *If $|\frac{G}{Z(G)}| = p^2$. Then*

$$\mathcal{PC}(G) = \frac{p^3 - p^2 - p + 2}{p^4}.$$

(2) *If $|\frac{G}{Z(G)}| = p^3$. Then*

$$\mathcal{PC}(G) \in \{\frac{p^4 - p^3 - p + 2}{p^6},\ \frac{2p^4 - 2p^3 - p^2 + 2}{p^6}\}.$$

□

Proof. (1). Suppose $x \in G \setminus Z(G)$, since $Z(G) \subsetneq Z(C_G(x)) \subseteq C_G(x) \subsetneq G$ and $|\frac{G}{Z(G)}| = p^2$ so $Z(C_G(x)) = C_G(x)$. Therefore G is an AC-group. Hence

$$|\beta(x) = |C_G(x)| - |Z(G)| = \frac{|G|}{p} - \frac{|G|}{p^2}.$$

Now Lemma 3.2 follows that

$$\mathcal{PC}(G) = \frac{|Z(G)|^2 + \sum_{x \notin Z(G)} |\beta(x)|}{|G|^2}$$
$$= \frac{|Z(G)|^2 + (|G| - |Z(G)|)(\frac{|G|}{p} - \frac{|G|}{p^2})}{|G|^2}$$
$$= \frac{1}{p} - \frac{1}{p^2} - \frac{1}{p^3} + \frac{2}{p^4}$$
$$= \frac{p^3 - p^2 - p + 2}{p^4}.$$

http://dx.doi.org/10.22108/ijgt.2024.142171.1912

(2). It is easy to see that G is an AC-group, so for any $x \in G\setminus Z(G)$ we have $|\beta(x)| = |C_G(x)|-|Z(G)|$. On the other hand for any $x \in G \setminus Z(G)$ since $Z(G) < C_G(x)$, so $|G : C_G(x)| = p \text{ or } p^2$.

If $|G : C_G(x)| = p^2$, for all $x \in G \setminus Z(G)$, then

$$\begin{aligned}\mathcal{PC}(G) &= \frac{|Z(G)|^2 + \sum\limits_{x \notin Z(G)} |\beta(x)|}{|G|^2} \\ &= \frac{|Z(G)|^2 + (|G| - |Z(G)|)(\frac{|G|}{p^2} - \frac{|G|}{p^3})}{|G|^2} \\ &= \frac{2}{p^6} + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^5} \\ &= \frac{p^4 - p^3 - p + 2}{p^6}\end{aligned}$$

If $|G : C_G(x)| = p$, for some $x \in G \setminus Z(G)$. We claim that there is exactly one such centralizer, namely $C_G(x)$. suppose, by contrary,that there exists an elements $x \neq y \in G \setminus Z(G)$ that $|G : C_G(y)| = p$ and $C_G(x) \neq C_G(y)$. Since

$$|C_G(x)| \lneq |C_G(x)C_G(y)| \leq |G|,$$

so $G = C_G(x)C_G(y)$. Now as G is an AC-group, it is well-known that $C_G(x) \cap C_G(y) = Z(G)$. Consequently

$$|G| = |C_G(x)C_G(y)| = \frac{|C_G(x)||C_G(y)|}{|Z(G)|}.$$

It follows that $\frac{|G|}{|Z(G)|} = p^2$, which is a contradiction and the claim is proven. Hence

$$\begin{aligned}\mathcal{PC}(G) &= \frac{|Z(G)|^2 + (|C_G(x)| - |Z(G)|)^2 + \sum\limits_{g \in G \setminus C_G(x)} (|C_G(g)| - |Z(G)|)}{|G|^2} \\ &= \frac{|Z(G)|^2 + (|C_G(x)| - |Z(G)|)^2 + (|G| - |C_G(x)|)(|C_G(g)| - |Z(G)|)}{|G|^2} \\ &= \frac{\frac{|G|^2}{p^6} + (\frac{|G|}{p} - \frac{|G|}{p^3})^2 + (|G| - \frac{|G|}{p})(\frac{|G|}{p^2} - \frac{|G|}{p^3})}{|G|^2} \\ &= \frac{2p^4 - 2p^3 - p^2 + 2}{p^6}.\end{aligned}$$

□

Corollary 3.10. *Let p be a prime number and G be a $p-$group of order p^n.*

(i) *If $n \in \{1, 2\}$, then $\mathcal{PC}(G) = 1$;*
(ii) *If $n = 3$, then $\mathcal{PC}(G) \in \{1, \frac{p^3-p^2-p+2}{p^4}\}$;*
(iii) *If $n = 4$, then $\mathcal{PC}(G) \in \{1, \frac{p^4-p^3-p+2}{p^6}, \frac{2p^4-2p^3-p^2+2}{p^6}\}$.*

Finally, it is worth pondering the influence of the function $\mathcal{PC}(G)$ on the structure of groups, like solvability or nilpotentcy.

Question 3.11. *1. Is it true that if $\frac{1}{20} \leq \mathcal{PC}(G)$, then G is solvable?*
2. Is it true that if $\frac{2}{9} \leq \mathcal{PC}(G)$, then G is nilpotent?

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Saman Rahimirad
Department of Mathematics, University of Kurdistan, P.O. Box 416, Sanandaj, Kurdistan, Iran
Email: s.rahimirad70@gmail.com

Mohammad zarrin
Department of Mathematics, Texas State University, 601 University Drive, San Marcos, TX, 78666, USA
Email: m.zarrin@txstate.edu

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