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THREE NEW CLASSES OF BINOMIAL FIBONACCI SUMS

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Abstract. In this paper, we introduce three new classes of binomial sums involving Fibonacci (Lucas) numbers and weighted binomial coefficients. One particular result is linked to a problem proposal recently published in the journal The Fibonacci Quarterly.

1. Introduction and motivation

As usual, we will use the notation F_n for the nth Fibonacci number and L_n for the nth Lucas number, respectively. Both number sequences are defined, for $n \in \mathbb{Z}$, through the same recurrence relation $x_n = x_{n-1} + x_{n-2}, n \geq 2$, with initial values $F_0 = 0, F_1 = 1$, and $L_0 = 2, L_1 = 1$, respectively. They possess the explicit formulas (Binet forms)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

where $\alpha = (1+\sqrt{5})/2$ is the golden section and $\beta = -1/\alpha$. For negative subscripts one checks easily that $F_{-n} = (-1)^{n-1} F_n$ and $L_{-n} = (-1)^n L_n$. For more information about these famous sequences we refer, among others, to the books by Koshy [17] and Vajda [22]. In addition, one can consult the On-Line Encyclopedia of Integer Sequences [25] where these sequences are listed under the ids A000045 and A000032, respectively.

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The literature on Fibonacci numbers is very rich. Dozens of articles and problem proposals dealing with binomial sum identities involving these sequences exist. Classical articles on the topic are [7, 8, 12, 13, 19, 24], among others. Newer contributions include [14, 20, 15, 16] and recent articles are [1, 2, 3, 4, 5, 6, 18, 21].

This note is motivated by the problem proposal [10] where the author asked to prove the identities

$$\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{k+1}=\frac{F_{2n+1}+L_{2n+1}}{n+1}\quad\text{and}\quad\sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)}=\frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)}.$$

A solution with a slight generalization was provided by Ventas in [23]. Here, we introduce some generalized variants of this proposal which should be regarded as attractive complements. More precisely, we present three presumably new classes of Fibonacci (Lucas) binomial sums possessing the same structure. Our results follow from three recently published polynomial identities derived by Dattoli et al. [9]. For $x \in \mathbb{C}$, they are given by

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+1}x^{k+1}(1+x)^{n-k}=\frac{(1+x)^{n+1}-1}{n+1}, \tag{1.1}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}x^{k+2}(1+x)^{n-k}=\frac{(1+x)^{n+2}-(n+2)x-1}{(n+1)(n+2)}, \tag{1.2}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(k+1)(k+2)}x^{k+2}(1+x)^{n-k}=\frac{(n+1)x(1+x)^{n+1}-(1+x)^{n+1}+1}{(n+1)(n+2)}. \tag{1.3}$$

These identities are variants of those found in H. W. Gould's classic [11, pp. 5–6, Identities 1.37–1.41].

In the course of derivation we will make use of the following known results.

Lemma 1.1. *For any integer s, we have*

$$(-1)^s+\alpha^{2s}=\alpha^s L_s, \qquad \textit{and} \qquad (-1)^s+\beta^{2s}=\beta^s L_s. \tag{1.4}$$

Lemma 1.2. *Let r and s be any integers. Then the following identities hold* [12]

$$L_{r+s}-L_r\alpha^s=-\beta^r F_s\sqrt{5}, \tag{1.5}$$

$$L_{r+s}-L_r\beta^s=\alpha^r F_s\sqrt{5}, \tag{1.6}$$

$$F_{r+s}-F_r\alpha^s=\beta^r F_s, \tag{1.7}$$

$$F_{r+s}-F_r\beta^s=\alpha^r F_s. \tag{1.8}$$

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2. First set of results

Theorem 2.1. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{s(k+1)+t}F_r^{k+1}F_s^{n-k}F_{rn-s(k+1)-rk-t} = \frac{1}{n+1}\Big((-1)^{t+1}F_s^{n+1}F_{r(n+1)-t}-F_tF_{r+s}^{n+1}\Big) \tag{2.1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{s(k+1)+1+t}F_r^{k+1}F_s^{n-k}L_{rn-s(k+1)-rk-t} = \frac{1}{n+1}\Big((-1)^{t}F_s^{n+1}L_{r(n+1)-t}-L_tF_{r+s}^{n+1}\Big). \tag{2.2}$$

Proof. Set $x=-F_r\alpha^s/F_{r+s}$ in (1.1), use (1.7) and multiply through by α^t, to obtain

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{r(n-k)+1}F_r^{k+1}F_s^{n-k}\alpha^{k(s+r)-rn+s+t} = \frac{1}{n+1}\Big((-1)^{r(n+1)}F_s^{n+1}\alpha^{-r(n+1)+t}-\alpha^tF_{r+s}^{n+1}\Big).$$

Similarly, setting $x=-F_r\beta^s/F_{r+s}$ in (1.1), using (1.8) and multiplying through by β^t, yields

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{r(n-k)+1}F_r^{k+1}F_s^{n-k}\beta^{k(s+r)-rn+s+t} = \frac{1}{n+1}\Big((-1)^{r(n+1)}F_s^{n+1}\beta^{-r(n+1)+t}-\beta^tF_{r+s}^{n+1}\Big).$$

The results follow by combining these identities according to the Binet forms while applying $F_{-n}=(-1)^{n-1}F_n$ and $L_{-n}=(-1)^nL_n$. □

Theorem 2.1 contains many interesting identities as special cases which are presented as two corollaries.

Corollary 2.2. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^kF_{n-2k-1+t} = \frac{1}{n+1}\Big(F_{n+1+t}-F_t\Big), \tag{2.3}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}F_{n-3k-2-t} = \frac{1}{n+1}\Big((-1)^{t+1}F_t2^{n+1}-F_{n+1-t}\Big), \tag{2.4}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^kF_{2n-3k-1+t} = \frac{1}{n+1}\Big(F_{2n+2+t}-F_t2^{n+1}\Big), \tag{2.5}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}F_{2n-4k-2-t} = \frac{1}{n+1}\Big((-1)^{t+1}F_t3^{n+1}-F_{2n+2-t}\Big), \tag{2.6}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^kF_{2n-k+1+t} = \frac{1}{n+1}\Big(F_{2n+2+t}-F_t\Big), \tag{2.7}$$

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$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{n+k+1}2^{n-k}F_{n+2k+3+t}=\frac{1}{n+1}\Big((-2)^{n+1}F_{n+1+t}-F_t\Big), \tag{2.8}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}2^{n-k}F_{2n+k+3+t}=\frac{1}{n+1}\Big(2^{n+1}F_{2n+2+t}-F_t\Big), \tag{2.9}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}3^{n-k}F_{2(n+k+2)+t}=\frac{1}{n+1}\Big(3^{n+1}F_{2n+2+t}-F_t\Big). \tag{2.10}$$

Corollary 2.3. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}L_{n-2k-1+t}=\frac{1}{n+1}\Big(L_{n+1+t}-L_t\Big), \tag{2.11}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}L_{n-3k-2-t}=\frac{1}{n+1}\Big((-1)^{t}L_t2^{n+1}-L_{n+1-t}\Big), \tag{2.12}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}L_{2n-3k-1+t}=\frac{1}{n+1}\Big(L_{2n+2+t}-L_t2^{n+1}\Big), \tag{2.13}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}L_{2n-4k-2-t}=\frac{1}{n+1}\Big((-1)^{t}L_t3^{n+1}-L_{2n+2-t}\Big), \tag{2.14}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}L_{2n-k+1+t}=\frac{1}{n+1}\Big(L_{2n+2+t}-L_t\Big), \tag{2.15}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{n+k+1}2^{n-k}L_{n+2k+3+t}=\frac{1}{n+1}\Big((-2)^{n+1}L_{n+1+t}-L_t\Big), \tag{2.16}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}2^{n-k}L_{2n+k+3+t}=\frac{1}{n+1}\Big(2^{n+1}L_{2n+2+t}-L_t\Big), \tag{2.17}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}3^{n-k}L_{2(n+k+2)+t}=\frac{1}{n+1}\Big(3^{n+1}L_{2n+2+t}-L_t\Big). \tag{2.18}$$

Theorem 2.4. *If s is an even integer and t is any integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}L_s^{n-k}F_{s(n+k+2)+t}=\frac{1}{n+1}\Big(L_s^{n+1}F_{s(n+1)+t}-F_t\Big) \tag{2.19}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^{k}L_s^{n-k}L_{s(n+k+2)+t}=\frac{1}{n+1}\Big(L_s^{n+1}L_{s(n+1)+t}-L_t\Big). \tag{2.20}$$

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Proof. Let s be even. Set $x=\alpha^{2s}$ and $x=\beta^{2s}$, respectively, in (1.1), and use Lemma 1.1. Multiply through the resulting equations by α^t and β^t, respectively, and combine according to the Binet forms. □

Remark 2.5. *Note that when $s=2$, Theorem 2.4 gives (2.10) and (2.18), respectively.*

Working with $x=-F_{r+s}/(\alpha^s F_r)$ and $x=-F_{r+s}/(\beta^s F_r)$, and using the same arguments we get the next results.

Theorem 2.6. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^k F_{r+s}^{k+1}F_s^{n-k}F_{s(k+1)+(r+s)(n-k)-t} = \frac{1}{n+1}\Big(F_s^{n+1}F_{(r+s)(n+1)-t}+(-1)^{(s+1)(n+1)+t}F_tF_r^{n+1}\Big) \tag{2.21}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^k F_{r+s}^{k+1}F_s^{n-k}L_{s(k+1)+(r+s)(n-k)-t} = \frac{1}{n+1}\Big(F_s^{n+1}L_{(r+s)(n+1)-t}+(-1)^{(s+1)(n+1)+t+1}L_tF_r^{n+1}\Big). \tag{2.22}$$

3. Results from identities (1.2) and (1.3)

The results for the other two classes of sums are presented without proofs as the ideas are clear.

Theorem 3.1. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^{r(n-k)}F_r^{k+2}F_s^{n-k}F_{s(k+2)-r(n-k)+t} = \frac{1}{(n+1)(n+2)}\Big((-1)^{t+1}F_s^{n+2}F_{r(n+2)-t}-F_tF_{r+s}^{n+2}\Big)+\frac{1}{n+1}F_rF_{s+t}F_{r+s}^{n+1} \tag{3.1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^{r(n-k)}F_r^{k+2}F_s^{n-k}L_{s(k+2)-r(n-k)+t} = \frac{1}{(n+1)(n+2)}\Big((-1)^{t}F_s^{n+2}L_{r(n+2)-t}-L_tF_{r+s}^{n+2}\Big)+\frac{1}{n+1}F_rL_{s+t}F_{r+s}^{n+1}. \tag{3.2}$$

Theorem 3.2. *If s is an even integer and t is any integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^k L_s^{n-k}F_{2s(k+2)+s(n-k)+t} = \frac{1}{(n+1)(n+2)}\Big(L_s^{n+2}F_{s(n+2)+t}-F_t\Big)-\frac{1}{n+1}F_{2s+t} \tag{3.3}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^k L_s^{n-k}L_{2s(k+2)+s(n-k)+t} = \frac{1}{(n+1)(n+2)}\Big(L_s^{n+2}L_{s(n+2)+t}-L_t\Big)-\frac{1}{n+1}L_{2s+t}. \tag{3.4}$$

Theorem 3.3. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)(k+2)}(-1)^{r(n-k)}F_r^{k+2}F_s^{n-k}F_{s(k+2)-r(n-k)+t}$$

(3.5)
$$= -\frac{1}{(n+1)(n+2)}\Big((-1)^{t+1}F_s^{n+1}F_{r+s}F_{r(n+1)-t} - F_tF_{r+s}^{n+2}\Big) + \frac{1}{n+2}F_s^{n+1}F_r(-1)^{s+t}F_{r(n+1)-s-t}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)(k+2)}(-1)^{r(n-k)}F_r^{k+2}F_s^{n-k}L_{s(k+2)-r(n-k)+t}$$

(3.6)
$$= -\frac{1}{(n+1)(n+2)}\Big((-1)^{t}F_s^{n+1}F_{r+s}L_{r(n+1)-t} - L_tF_{r+s}^{n+2}\Big) + \frac{1}{n+2}F_s^{n+1}F_r(-1)^{s+t+1}L_{r(n+1)-s-t}.$$

Theorem 3.4. *If s is an even integer and t is any integer, then*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)(k+2)}(-1)^{k}L_s^{n-k}F_{s(n+k+4)+t}\\ = -\frac{1}{(n+1)(n+2)}\Big(L_s^{n+1}F_{s(n+1)+t} - F_t\Big) + \frac{1}{n+2}L_s^{n+1}F_{s(n+3)+t}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)(k+2)}(-1)^{k}L_s^{n-k}L_{s(n+k+4)+t}\\ = -\frac{1}{(n+1)(n+2)}\Big(L_s^{n+1}L_{s(n+1)+t} - L_t\Big) + \frac{1}{n+2}L_s^{n+1}L_{s(n+3)+t}.\end{aligned} \tag{3.8}$$

4. Additional sum relations

In [9] the following sum relation is also proved:

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}x^k(1+x)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{(k+1)(k+2)}. \tag{4.1}$$

This relation immediately yields

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}F_{2n-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{F_k}{(k+1)(k+2)},$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{k+2}L_{2n-k} = \sum_{k=0}^{n}\binom{n}{k}\frac{L_k}{(k+1)(k+2)},$$

and hence

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{F_{2n-k}+L_{2n-k}}{k+2} = \sum_{k=0}^{n}\binom{n}{k}\frac{F_k+L_k}{(k+1)(k+2)} = \frac{F_{2n+2}+L_{2n+2}-2}{(n+1)(n+2)},$$

which provides a nice addendum to problem proposal [10]. More generally, we have sum relations of the following form.

Theorem 4.1. *If r, s and t are any integers ($r \neq 0$ and $r+s \neq 0$) and n is a non-negative integer, then*

$$\begin{aligned} &F_{r+s}^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^{r(n-k)} F_r^k F_s^{n-k} F_{sk-r(n-k)+t} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)(k+2)} (-1)^k \Big(\frac{F_r}{F_{r+s}}\Big)^k F_{sk+t} \\ &= \frac{(-1)^{t+1}}{(n+1)(n+2)} \Big(\Big(\frac{F_s}{F_r}\Big)^2 \Big(\frac{F_s}{F_{r+s}}\Big)^n F_{2s+r(n+2)-t} - \Big(\frac{F_{r+s}}{F_r}\Big)^2 F_{2s-t} \Big) \\ &\qquad + \frac{1}{n+1} \frac{F_{r+s}}{F_r} F_{t-s} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &F_{r+s}^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^{r(n-k)} F_r^k F_s^{n-k} L_{sk-r(n-k)+t} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(k+1)(k+2)} (-1)^k \Big(\frac{F_r}{F_{r+s}}\Big)^k L_{sk+t} \\ &= \frac{(-1)^{t}}{(n+1)(n+2)} \Big(\Big(\frac{F_s}{F_r}\Big)^2 \Big(\frac{F_s}{F_{r+s}}\Big)^n L_{2s+r(n+2)-t} - \Big(\frac{F_{r+s}}{F_r}\Big)^2 L_{2s-t} \Big) \\ &\qquad + \frac{1}{n+1} \frac{F_{r+s}}{F_r} L_{t-s}. \end{aligned} \tag{4.3}$$

In particular,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^{k+1} F_{n-2k} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k F_k}{(k+1)(k+2)} = \frac{1-F_{n+4}}{(n+1)(n+2)} + \frac{1}{n+1} \tag{4.4}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^{k} L_{n-2k} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k L_k}{(k+1)(k+2)} = \frac{L_{n+4}-3}{(n+1)(n+2)} - \frac{1}{n+1}. \tag{4.5}$$

Theorem 4.2. *If s is an even integer and t is any integer, then*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^k L_s^{n-k} F_{s(n+k)+t} = \sum_{k=0}^{n} \binom{n}{k} \frac{F_{2sk+t}}{(k+1)(k+2)} \\ &= \frac{1}{(n+1)(n+2)} \Big(L_s^{n+2} F_{s(n-2)+t} + (-1)^t F_{4s-t} \Big) + \frac{(-1)^t}{n+1} F_{2s-t} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} (-1)^k L_s^{n-k} L_{s(n+k)+t} = \sum_{k=0}^{n} \binom{n}{k} \frac{L_{2sk+t}}{(k+1)(k+2)} \\ &= \frac{1}{(n+1)(n+2)} \Big(L_s^{n+2} L_{s(n-2)+t} - (-1)^t L_{4s-t} \Big) - \frac{(-1)^t}{n+1} L_{2s-t}. \end{aligned} \tag{4.7}$$

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In particular,

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^k 3^{n-k}F_{2(n+k)} = \sum_{k=0}^{n}\binom{n}{k}\frac{F_{4k}}{(k+1)(k+2)} = \frac{1}{(n+1)(n+2)}\Big(3^{n+2}F_{2(n-2)}+21\Big)+\frac{3}{n+1} \tag{4.8}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}(-1)^k 3^{n-k}L_{2(n+k)} = \sum_{k=0}^{n}\binom{n}{k}\frac{L_{4k}}{(k+1)(k+2)} = \frac{1}{(n+1)(n+2)}\Big(3^{n+2}L_{2(n-2)}-47\Big)-\frac{7}{n+1}. \tag{4.9}$$

5. Conclusion

Motivated by the author's recent problem proposal, closed forms for three new classes of binomial sums with Fibonacci and Lucas numbers were derived. In addition, a few sum relations connected with the subject were discussed. Extensions of the results presented in this note to gibonacci or even to Horadam sequences should be possible with little effort. This exercise is left to the interested reader.

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