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ON THE NUMBER OF CLIQUES AND CYCLES IN GRAPHS

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ABSTRACT. We give a new recursive method to compute the number of cliques and cycles of a graph. This method is related, respectively to the number of disjoint cliques in the complement graph and to the sum of permanent function over all principal minors of the adjacency matrix of the graph. In particular, let G be a graph and let $\overline{G}$ be its complement, then given the chromatic polynomial of $\overline{G}$, we give a recursive method to compute the number of cliques of G. Also given the adjacency matrix A of G we give a recursive method to compute the number of cycles by computing the sum of permanent function of the principal minors of A. In both cases we confront to a new computable parameter which is defined as the number of disjoint cliques in G.

1. Introduction

There are a lot of parameters in graph theory studies which are used mainly as tools to distinguish non-isomorphic graphs and classifying them. The number of cycles and the number of cliques are two main important of these parameters, see for instance [1, 2, 6, 7, 10]. Let G be a graph and let $\overline{G}$ be its complement. In section two, using coefficients of the chromatic polynomial of $\overline{G}$, we give a new approach in computing the number of cliques of G. This approach is related to the number of disjoint cliques of $\overline{G}$. One may interpret this study as a kind of generalization of Whitney's Theorem ([9], p. 222) in which some sort of information about a graph from it's chromatic polynomial are obtained. In section three, using the sum of permanent function of some special sub-matrices of the adjacency matrix a new enumeration method for computing the number of cycles is given. We use the standard definitions and notations of classical references [8] and [9].

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2. **Cliques**

Let G be a graph with the vertex set $V(G)$ of size $|V(G)| = n$ and the edge set $E(G)$ of size $|E(G)| = e$. Also let $\overline{G}$ be the graph complement of G, which is a graph with vertex set $V(G)$ and edge set of size $\overline{e} = |E(\overline{G})| = \binom{n}{2} - e$. We use the notations K_n and $L(G)$, respectively for the complete graph of size n and the line graph of G [9]. By an i-clique and i-cycle, respectively we mean a clique of size i and a cycle of length i. Let $P_i(G)$ denotes the number of partitions of $V(G)$ into i nonempty independent sets. Also let $\lambda_j(G)$ and $C_j(G)$ be respectively, the number of cliques of size j and the number of cycles of length j in G. When S is a subset of $V(G)$, the subgraph induced by S in G is denoted by $[S]$. We also consider the following new parameter. First note that we say two cliques are disjoint if they have no common vertex.

Definition 2.1. *Let r be a positive integer. Then we denote the number of r disjoint cliques of sizes $i_1, i_2, \ldots, i_r$ in G by $\eta_G(i_1, i_2, \ldots, i_r)$.*

These sets of r disjoint cliques are not necessarily a partition of $V(G)$. Also note that always $\eta_G(i) = \lambda_i(G)$, $\lambda_3(G) = C_3(G)$ and $\lambda_2(G) = e$. Also if $i_1 = i_2 = \ldots = i_r = 2$, then $\eta_G(2, 2, \ldots, 2)$ denotes the number of matchings of size r in G. By an i-subset we mean a subset of size i. The following lemma gives a little more description of this parameter:

Lemma 2.2. *let G be a graph with n vertices, then:*

1. $\eta_G(1, 1) = \binom{n}{2}$; $\eta_G(1, 2) = e(n - 2)$; $\eta_G(1, 3) = (n - 3)\lambda_3(G)$.
2. $\eta_G(2, 2) = |E(\overline{L(G)})|$.
3. $\eta_G(2, 3) = \eta_{L(G)}(1, 3) - \sum_{v \in V}(e - d(v))\binom{d(v)}{3}$.
4. $\eta_G(2, 4) = \sum_S \lambda_2([S])\lambda_4(G - S)$, where the sum is taken over all 2-subsets of V.
5. $\eta_G(i, j) = \sum_S \lambda_i([S])\lambda_j(G - S)$, where the sum is taken over all i-subsets of V.
6. $\eta_G(2, 2, 2) = \lambda_3(\overline{L(G)})$; $\eta_G(2, 2, \cdots, 2) = \lambda_r(\overline{L(G)})$, where there are r successive $2's$.
7. $\eta_G(i, j, k) = \sum_S \lambda_i([S])\eta_{G-S}(j, k)$, where the sum is taken over all i-subsets S of $V(G)$ and j, k are different from i.
8. $\eta_G(i_1, \ldots, i_r) = \frac{1}{t}\sum_S \lambda_{i_1}([S])\eta_{G-S}(i_2, \ldots, i_r)$, where the sum is taken over all i_1-subsets S of $V(G)$ and t of the parameters $i_1, \ldots, i_r$ are equal to i_1.

Proof. As defined $\eta_G(1, 1)$ denotes the number of pairs of disjoint vertices which is $\binom{n}{2}$. Also $\eta_G(1, 2)$ is the number of pairs of disjoint edges and vertices. Clearly there are $n - 2$ pairs of this kind when the edge is fixed. Letting the fixed edge go through all e edges then the number of all these pairs is $e(n - 2)$. As defined $\eta_G(1, 3)$ is the number of pairs of a point and a triangle, nothing have a common incident. The number of such pairs for a fixed triangle is clearly $n - 3$. The fixed triangle can be chosen from a set of $\lambda_3(G)$ size, so the value is $(n - 3)\lambda_3(G)$. As defined $\eta_G(2, 2)$ means the number of pairs of non-adjacent edges or equivalently the set of matchings of size 2. These pairs of edges are pairs of disjoint vertices in $L(G)$, the line graph of G, so it is equal to the number of edges in the complement of $L(G)$ or $|E(\overline{L(G)})|$. As defined $\eta_G(2, 3)$ is the number of pairs of an edge and a triangle, without

any common incidence. The set of these pairs is in a one one correspondence with the set of pairs of a vertex and a triangle, without any common incidence in $L(G)$. These kinds of pairs in $L(G)$ may also emerge from another way. In fact each vertex of degree $d(v)$ make $\binom{d(v)}{3}$ triangles in $L(G)$. Reducing the number of pairs involving these latter type of triangles imply the result. As defined $\eta_G(2,4)$ is the number of pairs of an edge and a K_4 subset of G, without any common incidence. Given an edge $T = uv$ the number of K_4 disjoint from T is $\lambda_4(G - \{u, v\})$. Note that if $S = \{u, v\}$ does not induce an edge in G, then $\lambda_2([S]) = 0$. So one can compute the required number as $\sum_S \lambda_2([S])\lambda_4(G - S)$, where the sum is taken over all 2-subsets of V. Other items have the same arguments. We end the proof with a note about the last item in which we divide the summation over t, where t as defined is the number of repetition i_1. Note that each number in computing $\eta_G(i_1, \ldots, i_r)$ is equal to t numbers of the form $\lambda_{i_1}([S])\eta_{G-S}(i_2, \ldots, i_r)$, when S is any one of the t subsets of size i_1 through the r-tuple $(i_2, \ldots, i_r)$. □

Within the above new parameter $\eta_G(i_1, \ldots, i_r)$ we can give the following description of $P_i(G)$.

Theorem 2.3. *let G be a graph with n vertices, then we have:*

1. $P_n(G) = 1$, $P_{n-1}(G) = |E(\overline{G})|$, $P_{n-2}(G) = \eta_{\overline{G}}(2,2) + \lambda_3(\overline{G})$.
2. $P_{n-3}(G) = \eta_{\overline{G}}(2,2,2) + \eta_{\overline{G}}(2,3) + \lambda_4(\overline{G})$.
3. $P_{n-4}(G) = \eta_{\overline{G}}(2,2,2,2) + \eta_{\overline{G}}(2,2,3) + \eta_{\overline{G}}(3,3) + \eta_{\overline{G}}(2,4) + \lambda_5(\overline{G})$.
4. $P_{n-5}(G) = \eta_{\overline{G}}(2,2,2,2,2) + \eta_{\overline{G}}(2,2,2,3) + \eta_{\overline{G}}(2,3,3) + \eta_{\overline{G}}(2,2,4) + \eta_{\overline{G}}(2,5) + \lambda_6(\overline{G})$.

Proof. As defined $P_n(G)$ is the number of partitions of G into n independent sets, which is clearly equal to 1. As defined $P_{n-1}(G)$ is the number of partitions of G into $n - 1$ independent sets and so only one of these sets have 2 non-adjacent vertices. This latter set correspond to an edge in $\overline{G}$. So $P_{n-1}(G)$ is equal to the number of edges in $\overline{G}$ or $|E(\overline{G})|$. As defined $P_{n-2}(G)$ is the number of partitions of G into $n - 2$ independent sets. So either one of these sets have three independent vertices and the other sets have one vertex or two of these sets each have two non-adjacent vertices and the other sets have one vertex. Equivalent cases in $\overline{G}$ happen when either we are looking for the number of triangles or for the number of pairs of disjoint edges which is clearly equal to in $\eta_{\overline{G}}(2,2) + \lambda_3(\overline{G})$. As defined $P_{n-3}(G)$ is the number of partitions of G into $n - 3$ independent vertices. So three cases may happen for these sets: *i*. Three sets each have 2 independent vertices, others have one vertex. *ii*. One set have three, another one has two, others have one vertex. *iii*. One set have four, others have one vertex. As above, by considering the related parameters in $\overline{G}$, the required equality is obtained. Alike arguments work for the proof of other items. □

The above Theorem gives a way to write down similar equations for each $P_{n-r}(G)$, $(r \in \mathbb{Z})$ in terms of $\eta_{\overline{G}}(i_1, \cdots, i_{r'})$; $r' \leq r$ and $\lambda_{r+1}(\overline{G})$. In what follows an expression of the chromatic polynomial in terms of $P_i(G)$ is given.

Theorem 2.4. *([9], p. 220) Let k be an integer and let $\chi(G) = c$ be the chromatic number of the graph G. Then $P_i(G)$'s are related by the following presentation of the chromatic polynomial:*

$$\chi(G, k) = \sum_{i=c}^{n} \binom{k}{i} i! P_i(G)$$
$$= \frac{k!}{(k-c)!}[P_c(G) + (k - c)P_{c+1}(G) + \cdots + (k - c)(k - (c+1)) \ldots (k - (n - 1))P_n(G)].$$

As it is seen all integers $k = 0, 1, \ldots, c-1$ are roots of this polynomial. So if we divide the chromatic polynomial by $k(k-1)(k-2)\cdots(k-(c-1))$ a polynomial of degree $(n-c)$ appears. We call this polynomial *the reduced chromatic polynomial of G* and denote it by $\chi_{\mathcal{R}}(G, k)$. If we arrange this latter polynomial by factoring the common coefficients of k^i then we have:

$$\chi_{\mathcal{R}}(G, k) = k^{n-c}(P_n(G)) + k^{n-(c+1)}((-(c + \cdots + (n-1))P_n(G) + P_{n-1}(G))$$
$$+ k^{(n-(c+2))}((\sum \Pi_{i,j=c,i\neq j}^{n-1} ij)P_n(G) - ((c + \cdots + (n-2))P_{n-1}(G) + P_{n-2}(G)) + \cdots.$$

Now, if we have the reduced chromatic polynomial of a graph as follows:

$$f(k) = \sum_{i=0}^{n-c} \alpha_i k^i,$$

then a one-one correspondence between the factors of the reduced polynomial and $f(k)$, gives a system of $n-c$ linear equations with triangular coefficient matrix (as bellow). This system can be solved easily by classical methods to evaluate unknown variables $p_i(G)'s$.

$$
\begin{cases}
\alpha_{n-c} &= p_n(G) \\[2mm]
\alpha_{n-(c+1)} &= (-(c + (c+1)\cdots + (n-1))P_n(G) + P_{n-1}(G) \\[2mm]
\alpha_{n-(c+2)} &= (\sum \Pi_{i,j=c,i\neq j}^{n-1} ij)P_n(G) - (c + (c+1)\cdots + (n-2))P_{n-1}(G) + P_{n-2}(G) \\[2mm]
\alpha_{n-(c+3)} &= (\sum \Pi_{i,j,k=c,i\neq j\neq k}^{n-1} ijk)P_n(G) + (\sum \Pi_{i,j=c,i\neq j}^{n-2} ij)P_{n-1}(G) \\[2mm]
& \quad -(c + (c+1)\cdots + (n-3))P_{n-2}(G) + P_{n-3}(G) \\[2mm]
\alpha_{n-(c+4)} &= (\sum \Pi_{i,j,k,l=c,i\neq j\neq k\neq l}^{n-1} ijkl)P_n(G) - (\sum \Pi_{i,j,k=c,i\neq j\neq k}^{n-2} ijk)P_{n-1}(G) \\[2mm]
& \quad +(\sum \Pi_{i,j=c,i\neq j}^{n-3} ij)P_{n-2}(G) - (c + (c+1)\cdots + (n-4))P_{n-3}(G) + P_{n-4}(G) \\[2mm]
\cdots &= \cdots
\end{cases}
$$

Main Conclusion: If the chromatic polynomial of a graph is given, then at first step one can find all $P_i(G), c \leq i \leq n$ from the above system of equations. At the second step one can now compute recursively, the number of i-cliques in $\overline{G}$ or $\lambda_i(\overline{G})$ via Theorem 2.3 and Lemma 2.2. As for example given $P_{n-2}(\overline{G})$, then one can compute $\lambda_3(G)$ from the equation given in item one of Theorem 2.3. To sum up, for evaluating $\lambda_i(G)$ the chromatic polynomial of $\overline{G}$ should be evaluated firstly.

Note that when we have a graph G in hand, the recursive computation:

$$\chi(\overline{G}, k) = \chi(\overline{G} - e, k) - \chi(\overline{G}.e, k)$$

is a last resort as a constructive method to get this polynomial ([9], p. 221). For an extensive literature one is refereed to [5].

Example 1. Let $G = C_5$, we try to find the number of cliques or $\lambda_i(G)$ in G by above method. Note that in this case $\overline{G} = C_5$, so we have $\chi(\overline{G}, k) = (k-1)^5 - (k-1)$. Also the chromatic number c of $\overline{G}$ is $c = 3$. The reduced chromatic polynomial of $\overline{G}$ is:

$$\chi_{\mathcal{R}}(\overline{G}, k) = ((k-1)^5 - (k-1))/k(k-1)(k-2) = k^2 - 2k + 2.$$

Therefore:

$$\alpha_2 = 1 = P_5(\overline{G}); \ \alpha_1 = -2 = -7P_5(\overline{G}) + P_4(\overline{G})$$

and

$$\alpha_0 = 2 = 12P_5(\overline{G}) - 3P_4(\overline{G}) + P_3(\overline{G}).$$

Hence $P_5(\overline{G}) = 1$, $P_4(\overline{G}) = 5$, $P_3(\overline{G}) = 5$. Invoking Theorem 2.3 imply that:

$$P_3(\overline{G}) = \eta_G(2,2) + \lambda_3(G).$$

Note that $L(G) = G = C_5$ as well and so by Lemma 2.2 we have:

$$\eta_G(2,2) = |E(\overline{L(G)})| = |E((\overline{G}))| = 5.$$

Consequently $\lambda_3(G) = 0$, as required. In general if in the above system of equations we have $n = c + i$, then up to $P_c(G)$ can be computed from the reduced chromatic polynomial of G and so up to $\lambda_{i+1}(\overline{G})$ can be computed from Theorem 2.3 and Lemma 2.2.

3. Cycles

The following two lemmas show that the enumeration of cliques in any graph can be translated to the enumeration of cycles. This helps us to replace $\lambda_i(G)$ in Lemma 2.2 by a function of $C_j(G)$, $j \leq i$ if needed. In other words Lemma 2.2 helps one to compute $C_j(G)$ whenever the other parameters are in hand.

Lemma 3.1. *Let G be a graph of order n. Then we have:*

1. $G = K_n$ if and only if $C_3(G) = \binom{n}{3}$.
2. $G = K_n$ if and only if $C_n(G) = (n-1)!/2$.
3. $G = K_n$ if and only if for all i-subsets S of $V(G)$, where $(3 \leq i \leq n)$, we have $0 \neq C_i(G) = \binom{n}{i}C_i([S])$.

Proof. Any n points can be arranged in $n!$ cases. Sure any such arrangement of n vertices of K_4 is a cycle. Also any cycle of K_n appears just $2n$ times in all $n!$ possible arrangement of vertices. So in Part 2 there are $n!/2n$ cycles of length n. Part 3 shows that all subsets of size i should have the same non-zero number of cycles of length i.   □

Lemma 3.2. *Let G be a graph of order n, and i, j two positive integers such that $j \leq i \leq n$, also let $\lfloor x \rfloor$ be the greatest integer not greater than x, then we have:*

1. $\lambda_i(G) = \sum_S \lfloor C_3([S])/\binom{i}{3} \rfloor$, where the sum is taken over all i-subsets S of $V(\Gamma)$.

2. $\lambda_i(G) = \sum_S \lfloor C_j([S])/C_j(K_i) \rfloor$, where the sum is taken over all i-subsets S of $V(\Gamma)$.

Proof. If a subset S of $V(G)$ induces a clique of size i, then we have $\lfloor C_j([S])/C_j(K_i) \rfloor = 1$, otherwise $\lfloor C_j([S])/C_j(K_i) \rfloor = 0$. The number of all cliques of size i can be computed if S runs over all subsets of size i in $V(G)$. When $j = 3$ then $C_j(K_i) = \binom{i}{3}$.   □

The following theorem is our main observation in this section. First note that a principal minor of a matrix is a sub-matrix obtained by taking a subset of rows and the same subset of columns. Principal minors of the adjacency matrix of a graph are really the adjacency matrices of all of it's subgraphs (see also [4], p. 45). Within the following theorem one can compute the number of all cycles of length i of a given graph by computing the sum of permanents of all principal minors of the adjacency matrix. As before we consider G to be a graph of order n and e edges. Also let A be the adjacency matrix of G, which is an n-square matrix. When r is a positive integer ($1 \le r \le n$), we denote by A_r any one of the different r-square principal minors of A obtained as the adjacency matrices of any induced subgraphs of order r. In other words the A_r's are all r-squares submatrices obtained by crossing any set of r rows of A with the same set of columns. For example A_1's are the diagonal matrices of A which are all zero and so on.

Theorem 3.3. *Within the above assumptions on G with adjacency matrix A we have:*

1. $\sum Per(A_1) = 0$, where the sum is taken over all 1-square principal minors A_1.
2. $\sum Per(A_2) = e$, where the sum is taken over all 2-square principle minors A_2.
3. $\sum Per(A_3) = 2C_3(G)$, where the sum is taken over all 3-square principle minors A_3.
4. $\sum Per(A_4) = \eta_G(2,2) + 2C_4(G)$, where the sum is as above.
5. $\sum Per(A_5) = 2\eta_G(2,3) + 2C_5(G)$, where the sum is as above.
6. $\sum Per(A_6) = \eta_G(2,2,2) + 4\eta_G(3,3) + 2\eta_G(2,4) + 2C_6(G)$, where the sum is as above.

Proof. Note that for a graph with adjacency matrix B the value $Per(B)$ counts the number of $\{1,2\}$-factors T of the graph, such that each T is counted 2^r times, where r is the number of cycles of T (see also [3]). Also note that the matrices A_i's, as above, are the adjacency matrices of all induced subgraphs $[S]$ of G when S ranges over all i-subsets of G ($1 \le i \le n$) and so the value of $Per(A_i)$ counts all their $\{1,2\}$-factors. Consider for example A_6 which is the adjacency matrix of an induced subgraph $[S]$ when S is a subset of $V(G)$ of size 6. The set of all $\{1,2\}$-factors of this graph consists of either a perfect matching of size 3 or a pair of disjoint 3 cycles or a pair of disjoint edge and 4-cycle or ultimately one 6-cycle. Multiplying each of these possible $\{1,2\}$-factors with 2^r, where r is the number of cycles in the given $\{1,2\}$-factor, yields the result.   □

Main Conclusion. When the adjacency matrix of a graph is in hand, the above Theorem plus Lemmas 2.2 and 3.2 gives a constructive recursive method to compute the number of all cycles.

Example 2. Consider the graph C_5 in example 1 and consider its adjacency matrix which is a matrix of size 5×5. One can compute easily the following summations:

$$\sum Per(A_3) = 0, \ \sum Per(A_4) = 5 \text{ and } \sum Per(A_5) = 2.$$

So by Theorem 3.3 we have

$$0 = 2C_3(G), \ 5 = \eta_G(2,2) + 2C_4(G) \ \text{and} \ 2 = 2\eta_G(2,3) + 2C_5(G).$$

Invoking Lemma 2.2 and note that $L(G) = G$, then we have:

$$\eta_G(2,2) = |E(\overline{L(G)})| = 5 \ \text{and} \ \eta_G(2,3) = \eta_{L(G)}(1,3) - \sum_{v \in V}(e - d(v))\binom{d(v)}{3} = 0.$$

Consequently:

$$C_3(G) = 0, \ C_4(G) = 0 \text{ and } C_5(G) = 1.$$

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