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ON RELATIONSHIP BETWEEN REFORMULATED SOMBOR AND OTHER VERTEX–DEGREE INDICES

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Abstract. Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with $n \geq 2$ vertices and m edges, with vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$, and edge degree sequence $\Delta_e = d(e_1) \geq d(e_2) \geq \cdots \geq d(e_n) = \delta_e$. The reformulated Sombor index is defined as $RS(G) = \sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2}$. We consider a relationship between reformulated Sombor index and some of the vertex–degree-based indices.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with $n \geq 2$ vertices and m edges. If vertices i and j (i.e. edges e_i and e_j) are adjacent in G, we write $i \sim j$ (i.e. $e_i \sim e_j$). Denote with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$ and $\Delta_e = d(e_1) \geq d(e_2) \geq \cdots \geq d(e_n) = \delta_e$, vertex and edge degree sequences, respectively. As usual, $L(G)$ denotes a line graph.

In graph theory, a graph invariant is property of the graph that is preserved by isomorphisms. The graph invariants that assume only numerical values are usually referred to as topological indices in chemical graph theory. Many of them are defined as simple functions of the degree sequence of

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(molecular) graph. These type of indices are also known as the bond incident degree (BID in short) indices [28].

Most of the BID indices can be represented as [10, 15, 25, 28]:

$$TI(G) = \sum_{i\sim j} F(f(d_i), f(d_j))\,, \tag{1.1}$$

where F is a symmetric non–negative real function of d_i and d_j, and f is a positive real valued function of vertex degrees.

If instead $L(G)$ is considered, a class of reformulated bond incident degree indices (that is edge–degree–based topological indices) can be defined as [2]

$$TI_e(G) = ETI(G) = TI(L(G)) = \sum_{e_i\sim e_j} F(f(e_i), f(e_j))\,. \tag{1.2}$$

If F is an additive function, i.e. $F(x, y) = x + y$, then the following identities are valid (see e.g. [7]):

$$TI(G) = \sum_{i\sim j}(f(d_i) + f(d_j)) = \sum_{i=1}^{n} d_i f(d_i)\,, \tag{1.3}$$

and

$$ETI(G) = \sum_{e_i\sim e_j}\Big(f(d(e_i)) + f(d(e_j))\Big) = \sum_{i=1}^{m} d(e_i) f(d(e_i))\,. \tag{1.4}$$

When $f(x) = x$, from (1.3) the first Zagreb index, $M_1(G)$, is obtained [11]

$$M_1(G) = \sum_{i\sim j}(d_i + d_j) = \sum_{i=1}^{n} d_i^2\,.$$

Let us note that additive version of the first Zagreb index was defined in [20].

When $F(x, y) = x \cdot y$ and $f(x) = x$, from (1.1) the second Zagreb index, $M_2(G)$, introduced in [12], is obtained:

$$M_2(G) = \sum_{i\sim j} d_i d_j\,.$$

The corresponding reformulated first and second Zagreb indices, $EM_1(G)$ and $EM_2(G)$, were defined in [18] as

$$EM_1(G) = \sum_{e_i\sim e_j}(d(e_i) + d(e_j)) = \sum_{i=1}^{m} d(e_i)^2 \quad \text{and} \quad EM_2(G) = \sum_{e_i\sim e_j} d(e_i) d(e_j)\,.$$

The forgotten topological index [9] is obtained from (1.3) for $F(x, y) = x + y$ and $f(x) = x^2$, that is

$$F(G) = \sum_{i\sim j}(d_i^2 + d_j^2) = \sum_{i=1}^{n} d_i^3\,.$$

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The corresponding reformulated forgotten index, $EF(G)$ was conceived in [16] as

$$EF(G)=\sum_{e_i\sim e_j}\left(d(e_i)^2+d(e_j)^2\right)=\sum_{i=1}^{m}d(e_i)^3\,.$$

The inverse degree indeg index [25], $ISI(G)$, is obtained from (1.1) for $F(x,y)=\frac{xy}{x+y}$ and $f(x)=x$ as

$$ISI(G)=\sum_{i\sim j}\frac{d_id_j}{d_i+d_j}\,.$$

The corresponding reformulated degree indeg index, $ISI_e(G)$, is defined as [3]

$$ISI_e(G)=\sum_{e_i\sim e_j}\frac{d(e_i)d(e_j)}{d(e_i)+d(e_j)}\,.$$

The Albertson index [1] is obtained from (1.1) for $F(x,y)=|x-y|$ and $f(x)=x$, that is

$$Alb(G)=\sum_{i\sim j}|d_i-d_j|\,.$$

For $F(x,y)=|x-y|$ and $f(x)=x$ from (1.2) the corresponding reformulated Albertson index is obtained as

$$Alb_e(G)=\sum_{e_i\sim e_j}|d(e_i)-d(e_j)|\,.$$

The geometric–arithmetic topological index [26] is obtained from (1.1) by the substitutions $F(x,y)=\frac{2\sqrt{xy}}{x+y}$ and $f(x)=x$, that is

$$GA(G)=\sum_{i\sim j}\frac{2\sqrt{d_id_j}}{d_i+d_j}\,.$$

The corresponding reformulated geometric–arithmetic index is obtained from (1.2) by the same substitutions, that is

$$GA_e(G)=\sum_{e_i\sim e_j}\frac{2\sqrt{d(e_i)d(e_j)}}{d(e_i)+d(e_j)}\,.$$

The arithmetic–geometric index [23] is obtained from (1.1) for $F(x,y)=\frac{x+y}{2\sqrt{xy}}$ and $f(x)=x$, as

$$AG(G)=\sum_{i\sim j}\frac{d_i+d_j}{2\sqrt{d_id_j}}\,.$$

Analogously, by the same substitutions, from (1.2) the corresponding reformulated arithmetic–geometric index is obtained as

$$AG_e(G)=\sum_{e_i\sim e_j}\frac{d(e_i)+d(e_j)}{2\sqrt{d(e_i)d(e_j)}}\,.$$

The symmetric division deg index [27] is obtained from (1.1) for $F(x,y)=\frac{x}{y}+\frac{y}{x}$ and $f(x)=x$, as

$$SDD(G)=\sum_{i\sim j}\left(\frac{d_i}{d_j}+\frac{d_j}{d_i}\right)\,.$$

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The reformulated symmetric division deg index [21] is obtained by the same substitution from (1.2), as

$$SDD_e(G) = \sum_{e_i \sim e_j} \left(\frac{d(e_i)}{d(e_j)} + \frac{d(e_j)}{d(e_i)} \right).$$

The Sombor index, defined in [13], can be obtained from (1.1) by the substitutions $F(x,y) = \sqrt{x+y}$ and $f(x) = x^2$, as

$$SO(G) = \sum_{i \sim j} \sqrt{d_i^2 + d_j^2}\,.$$

The corresponding reformulated Sombor index, introduced in [14], is obtained by the same substitution from (1.2), as

$$RS(G) = \sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2}\,.$$

Inspired by the results presented in [14], we further investigate mathematical properties of the reformulated Sombor index and its relationship with above mentioned topological indices.

2. Preliminaries

We first recall one inequality for the real number sequences that will be frequently used in proofs of theorems.

Lemma 2.1. [22] *Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \dots, m$, be two sequences of positive real numbers. Then, for any $r \geq 0$ holds*

$$\sum_{i=1}^{m} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^m x_i\right)^{r+1}}{\left(\sum_{i=1}^m a_i\right)^r}\,. \tag{2.1}$$

Equality holds if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_m}{a_m}$.

Remark 2.2. *Lemma 2.1 is given in its original form as presented in [22]. But, it is not difficult to verify that it is valid for any real r, such that $r \leq -1$ or $r \geq 0$, and when $-1 \leq r \leq 0$ the opposite inequality holds. Equality holds if and only if $r = -1$, or $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_m}{a_m}$.*

Now we prove an auxiliary result that will be used later in the paper.

Lemma 2.3. *Let $L(G)$ be a connected strictly semiregular bipartite graph with $m \geq 3$ vertices. Then there exists a connected graph G with m edges for which $L(G)$ is line graph, if and only if $L(G) \cong P_3$.*

Proof. Let $L(G) \cong K_{p,q}$, $p \geq 1$, $p < q$, $p + q = m$, be a connected strictly semiregular graph that consists of two partitions, V_p and V_q, of vertices, such that $V_p \cap V_q = \emptyset$. Assume $e_1 \in V_p$ and $e_2, e_3, e_4 \in V_q$. Then we have that $e_1 \sim e_2$, $e_1 \sim e_3$, $e_1 \sim e_4$, and $e_2 \nsim e_3$, $e_2 \nsim e_4$ and $e_3 \nsim e_4$. Let $G = (V, E)$ be a graph for which $L(G)$ is its line graph. In that case, vertices e_1, e_2, e_3 and e_4 in $L(G)$, are edges e_1, e_2, e_3 and e_4 in G. Let $e_1 = \{x_1, x_2\}$, where $x_1, x_2 \in V$. Since edges e_1 and e_2 are

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adjacent in G, then $e_2 = \{x_1, x_3\}$, $x_3 \in V$. Also, since e_1 and e_3 are adjacent in G and $e_2 \cap e_3 = \emptyset$, then $e_3 = \{x_2, x_4\}$, $x_4 \in V$. The edge e_4 is adjacent to e_1, so it has to be incident to either vertex x_1 or x_2. However, since $e_2 \cap e_4 = \emptyset$ and $e_3 \cap e_4 = \emptyset$, such edge does not exist. It follows that the vertex e_4 in $L(G)$ neither exists. Consequently, it follows that the partition V_q contains only two vertices, i.e. $q = 2$. Since $p < q$, it is evident that $p = 1$, meaning that $L(G) \cong K_{1,2} \cong P_3$, that is $G \cong P_4$ and vice versa: if $G \cong P_4$, then $L(G) \cong P_3$. □

3. Main results

In the next theorem we establish a relationship between $RS(G)$, $EM_1(G)$ and $Alb_e(G)$.

Theorem 3.1. *Let G be a connected graph. Then, we have*

$$RS(G) \geq \sqrt{\frac{EM_1(G)^2 + Alb_e(G)^2}{2}}\,. \tag{3.1}$$

Equality holds if and only if $L(G)$ is a regular graph or $L(G) \cong P_3$.

Proof. The following identities are valid

$$RS(G) - \sum_{e_i \sim e_j} \frac{2d(e_i)d(e_j)}{\sqrt{d(e_i)^2 + d(e_j)^2}} = \sum_{e_i \sim e_j} \frac{(d(e_i) - d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}}\,,$$

and

$$RS(G) + \sum_{e_i \sim e_j} \frac{2d(e_i)d(e_j)}{\sqrt{d(e_i)^2 + d(e_j)^2}} = \sum_{e_i \sim e_j} \frac{(d(e_i) + d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}}\,.$$

After summation of the above identities we obtain

$$2RS(G) = \sum_{e_i \sim e_j} \frac{(d(e_i) - d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}} + \sum_{e_i \sim e_j} \frac{(d(e_i) + d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}}\,. \tag{3.2}$$

For $r = 1$, $x_i := |d(e_i) - d(e_j)|$, $a_i := \sqrt{d(e_i)^2 + d(e_j)^2}$, with summation performed over all pairs of adjacent edges in G, the inequality (2.1) becomes

$$\sum_{e_i \sim e_j} \frac{(d(e_i) - d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}} \geq \frac{\left(\sum_{e_i \sim e_j} |d(e_i) - d(e_j)|\right)^2}{\sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2}} = \frac{Alb_e(G)^2}{RS(G)}\,. \tag{3.3}$$

Similarly, for $r = 1$, $x_i := d(e_i) + d(e_j)$, $a_i := \sqrt{d(e_i)^2 + d(e_j)^2}$, with summation performed over all pairs of adjacent edges in G, the inequality (2.1) becomes

$$\sum_{e_i \sim e_j} \frac{(d(e_i) + d(e_j))^2}{\sqrt{d(e_i)^2 + d(e_j)^2}} \geq \frac{\left(\sum_{e_i \sim e_j} (d(e_i) + d(e_j))\right)^2}{\sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2}} = \frac{EM_1(G)^2}{RS(G)}\,. \tag{3.4}$$

Now, from (3.2), (3.3) and (3.4) we obtain

$$RS(G)^2 \geq \frac{EM_1(G)^2 + Alb_e(G)^2}{2}\,, \tag{3.5}$$

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from which we arrive at (3.1).

Equalities in (3.3) and (3.4) hold, respectively, if and only if

$$\frac{|d(e_i)-d(e_j)|}{\sqrt{d(e_i)^2+d(e_j)^2}} \quad \text{and} \quad \frac{d(e_i)+d(e_j)}{\sqrt{d(e_i)^2+d(e_j)^2}}$$

are constant for every pair of adjacent edges e_i and e_j in G. Suppose e_j and e_k are two edges that are adjacent to edge e_i. In that case the following identities are valid

$$\frac{|d(e_i)-d(e_j)|}{\sqrt{d(e_i)^2+d(e_j)^2}} = \frac{|d(e_i)-d(e_k)|}{\sqrt{d(e_i)^2+d(e_k)^2}}$$

and

$$\frac{d(e_i)+d(e_j)}{\sqrt{d(e_i)^2+d(e_j)^2}} = \frac{d(e_i)+d(e_k)}{\sqrt{d(e_i)^2+d(e_k)^2}}\,.$$

From the above identities we obtain

$$2d(e_i)(d(e_i)^2-d(e_j)d(e_k))(d(e_j)-d(e_k))=0\,.$$

Thus, equality in (3.5) holds if and only if $d(e_j)=d(e_k)$, that is if and only if $L(G)$ is a regular or semiregular bipartite graph. Now, by Lemma 2.3 we conclude that equality in (3.1) holds if and only if $L(G)$ is a regular graph, or $L(G)\cong P_3$. □

Corollary 3.2. *Let G be a connected graph with $m\geq 3$ edges. Then we have*

$$RS(G)\geq \frac{\sqrt{2}}{2}EM_1(G)\,. \tag{3.6}$$

Equality holds if and only if $L(G)$ is a regular graph.

Proof. For any graph G holds that $Alb_e(G)\geq 0$, with equality if and only if $L(G)$ is a regular graph. From this inequality and (3.1), the inequality (3.6) follows. □

Remark 3.3. *In* [14, Theorem 5.1] *it was proven that*

$$RS(G)\geq \frac{1}{2}EM_1(G)\,.$$

Since $\sqrt{2}>1$, the inequality (3.6) is stronger than the above inequality.

Remark 3.4. *It should be pointed out that some new inequalities involving A_α–spectral radius (for connected line graphs) can be obtained by using some of the results of* [17] *and this paper. For instance, one of such inequalities follows from* [17, Corollary 2.3] *and Theorem 3.1.*

Corollary 3.5. *Let G be a connected graph with $m\geq 3$ edges. Then we have*

$$RS(G)\geq \frac{\sqrt{2}(M_1(G)-2m)^2}{2m}\,. \tag{3.7}$$

Equality holds if and only if $L(G)$ is a regular graph.

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Proof. In [6] it was proven that

$$EM_1(G) \geq \frac{(M_1(G)-2m)^2}{m},$$

with equality if and only if $L(G)$ is regular. From the above inequality and (3.6) we arrive at (3.7). □

Remark 3.6. *In* [14, Theorem 4.1 (i)] *it was proven that*

$$RS(G) \geq \frac{\sqrt{2}}{2}(M_1(G)-2m).$$

Since G is connected and $m \geq 3$, the inequality (3.7) is stronger than the above.

Corollary 3.7. *Let G be a graph of size $m \geq 3$. Then we have*

$$RS(G) \geq \frac{2\sqrt{2}m(2m-n)^2}{n^2}. \tag{3.8}$$

Equality holds if and only if G is regular.

Proof. In [8] it was proven that

$$M_1(G) \geq \frac{4m^2}{n},$$

with equality if and only if G is a regular graph. From the above and inequality (3.7) we obtain (3.8). □

Remark 3.8. *In* [14, Theorem 3.1] *it was proven that*

$$RS(G) \geq \frac{\sqrt{2}}{2}m.$$

Since G is connected and $m \geq 3$, the inequality (3.8) is stronger than the above one.

In the next theorem we determine a relationship between $RS(G)$, $EM_1(G)$ and $ISI_e(G)$.

Theorem 3.9. *Let G be a connected graph with $m \geq 3$ edges. Then*

$$RS(G) \leq \sqrt{EM_1(G)(EM_1(G)-2ISI_e(G))}. \tag{3.9}$$

Equality holds if and only if $L(G)$ is regular or $L(G) \cong P_3$.

Proof. The following identity is valid

$$EM_1(G) - 2ISI_e(G) = \sum_{e_i \sim e_j} \frac{d(e_i)^2 + d(e_j)^2}{d(e_i)+d(e_j)}. \tag{3.10}$$

For $r=1$, $x_i := \sqrt{d(e_i)^2 + d(e_j)^2}$, $a_i := d(e_i)+d(e_j)$, with summation performed over all adjacent pairs of edges e_i and e_j in G, the inequality (2.1) becomes

$$\sum_{e_i \sim e_j} \frac{d(e_i)^2 + d(e_j)^2}{d(e_i)+d(e_j)} \geq \frac{\left(\sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2}\right)^2}{\sum_{e_i \sim e_j}(d(e_i)+d(e_j))} = \frac{RS(G)^2}{EM_1(G)}. \tag{3.11}$$

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Now, from the above inequality and identity (3.10), we obtain

$$EM_1(G) - 2ISI_e(G) \geq \frac{RS(G)^2}{EM_1(G)}\,,$$

from which (3.9) immediately follows.

Equality in (3.11) holds if and only if $\frac{\sqrt{d(e_i)^2+d(e_j)^2}}{d(e_i)+d(e_j)}$ is a constant for every pair of adjacent edges e_i and e_j in G. Suppose e_j and e_k are two edges adjacent to edge e_i. Then, the following is valid

$$\frac{\sqrt{d(e_i)^2 + d(e_j)^2}}{d(e_i) + d(e_j)} = \frac{\sqrt{d(e_i)^2 + d(e_k)^2}}{d(e_i) + d(e_k)}\,,$$

that is

$$2d(e_i)(d(e_j) - d(e_k))(d(e_i)^2 - d(e_j)d(e_k)) = 0\,.$$

Thus, we conclude that equality in (3.11) holds if and only if $d(e_j) = d(e_k)$, that is if and only if $L(G)$ is a regular or semiregular bipartite graph. By Lemma 2.3 equality in (3.9) holds if and only if $L(G)$ is regular or $L(G) \cong P_3$. □

Corollary 3.10. *Let G be a connected graph with $m \geq 3$ edges. Then*

$$RS(G) \leq \sqrt{2}(EM_1(G) - 2ISI_e(G))\,. \tag{3.12}$$

Equality holds if and only if $L(G)$ is regular.

Proof. For every i and j, $1 \leq i, j \leq m$, holds

$$(d(e_i) + d(e_j))^2 \geq 4d(e_i)d(e_j)\,,$$

that is

$$d(e_i) + d(e_j) \geq \frac{4d(e_i)d(e_j)}{d(e_i) + d(e_j)}\,. \tag{3.13}$$

Summing up the above inequality over all pairs of adjacent edges e_i and e_j in G, yields

$$EM_1(G) \geq 4ISI_e(G)\,,$$

that is

$$EM_1(G) \leq 2(EM_1(G) - 2ISI_e(G))\,.$$

From the above and (3.9) we arrive at (3.12).

Equality in (3.13) holds if and only if $d(e_i) = d(e_j)$ for every pair of adjacent edges e_i and e_j in G, which implies that equality in (3.12) holds if and only if $L(G)$ is a regular graph. □

Remark 3.11. *The inequality (3.12) was proven in* [14, Theorem 6.2]. *The inequality (3.9) is stronger than (3.12).*

In the next theorem we establish a relationship between $RS(G)$, $M_1(G)$ and $EF(G)$.

Theorem 3.12. *Let G be a connected graph with $m \geq 3$ edges. Then*

$$RS(G) \leq \sqrt{\frac{1}{2}(M_1(G)-2m)EF(G)}\,. \tag{3.14}$$

Equality holds if and only if $L(G)$ is regular or $L(G) \cong P_3$.

Proof. For $r=1$, $x_i := \sqrt{d(e_i)^2+d(e_j)^2}$, $a_i := 1$, with summation performed over all pairs of adjacent edges e_i and e_j in G, the inequality (2.1) becomes

$$\sum_{e_i \sim e_j} (d(e_i)^2+d(e_j)^2) \geq \frac{\left(\sum_{e_i\sim e_j} \sqrt{d(e_i)^2+d(e_j)^2}\right)^2}{\sum_{e_i\sim e_j} 1}\,, \tag{3.15}$$

that is

$$EF(G) \geq \frac{RS(G)^2}{\frac{1}{2}(M_1(G)-2m)}\,,$$

from which we obtain (3.14).

Equality in (3.15) holds if and only if $\sqrt{d(e_i)^2+d(e_j)^2}$ is a constant for every pair of adjacent edges e_i and e_j in G. Suppose e_j and e_k are two edges that are adjacent to edge e_i in G. Then, the following is valid

$$\sqrt{d(e_i)^2+d(e_j)^2} = \sqrt{d(e_i)^2+d(e_k)^2}\,,$$

that is $d(e_j)=d(e_k)$. This means that equality in (3.15) holds if and only if $L(G)$ is a regular or semiregular bipartite graph. Now, by Lemma 2.3, equality in (3.14) holds if and only if $L(G)$ is regular or $L(G) \cong P_3$. □

Corollary 3.13. *Let G be a connected graph with $m \geq 2$ edges. Then*

$$RS(G) \leq \frac{\sqrt{2}EF(G)}{2\delta_e}\,. \tag{3.16}$$

Equality holds if and only if $L(G)$ is regular.

Proof. Since

$$EF(G) = \sum_{e_i\sim e_j} (d(e_i)^2+d(e_j)^2) \geq 2\delta_e^2 \sum_{e_i\sim e_j} 1 = \delta_e^2(M_1(G)-2m)\,,$$

then the following is valid

$$M_1(G)-2m \leq \frac{EF(G)}{\delta_e^2}\,.$$

Now, from the above inequality and inequality (3.14) we obtain (3.16). □

Remark 3.14. *Since $\delta_e \geq 2(\delta-1)$ $(\delta \geq 2)$ then from (3.16) we obtain*

$$RS(G) \leq \frac{\sqrt{2}EF(G)}{4(\delta-1)}\,.$$

Equality holds if and only if G is regular. The above inequality was proven in [14, Theorem 8.1]. *The inequality (3.16) is stronger than the above one.*

Corollary 3.15. *Let G be a connected graph of order $n \geq 2$ and size m. Then we have*

$$RS(G) \leq \sqrt{\frac{1}{2}(2m(\Delta+\delta-1)-n\Delta\delta)EF(G)}\,.$$

Equality holds if and only if G is regular or semiregular bipartite graph, or $G \cong P_4$.

Proof. In [5] it was proven that

$$M_1(G) \leq 2m(\Delta+\delta)-n\Delta\delta\,,$$

with equality if and only if $d_i \in \{\Delta, \delta\}$, for $i = 1, 2, \ldots, n$. From the above and inequality (3.14) we obtain the required result. □

In the next theorem we determine a relationship between $RS(G)$, $EM_2(G)$, $EF(G)$, $AG_e(G)$ and $GA_e(G)$.

Theorem 3.16. *Let G be a connected graph of size $m \geq 3$. Then*

$$RS(G) \leq \sqrt{\frac{1}{2}(EF(G)+2EM_2(G))(2AG_e(G)-GA_e(G))}\,. \tag{3.17}$$

Equality holds if and only if $L(G)$ is regular.

Proof. The following identity is valid

$$2AG_e(G)-GA_e(G) = \sum_{e_i\sim e_j} \frac{d(e_i)^2+d(e_j)^2}{(d(e_i)+d(e_j))\sqrt{d(e_i)d(e_j)}}\,. \tag{3.18}$$

By the arithmetic–geometric mean inequality, AM-GM, (see e.g. [19]) the following is valid

$$\sqrt{d(e_i)d(e_j)} \leq \frac{1}{2}(d(e_i)+d(e_j))\,. \tag{3.19}$$

From the above and identity (3.18), we obtain

$$2AG_e(G)-GA_e(G) \geq 2\sum_{e_i\sim e_j} \frac{d(e_i)^2+d(e_j)^2}{(d(e_i)+d(e_j))^2}\,. \tag{3.20}$$

For $r = 1$, $x_i := \sqrt{d(e_i)^2+d(e_j)^2}$, $a_i := (d(e_i)+d(e_j))^2$, with summation performed over all pairs of adjacent edges e_i and e_j in G, the inequality (2.1) becomes

$$\sum_{e_i\sim e_j} \frac{d(e_i)^2+d(e_j)^2}{(d(e_i)+d(e_j))^2} \geq \frac{\left(\sum_{e_i\sim e_j}\sqrt{d(e_i)^2+d(e_j)^2}\right)^2}{\sum_{e_i\sim e_j}(d(e_i)+d(e_j))^2}\,,$$

that is

$$\sum_{e_i\sim e_j} \frac{d(e_i)^2+d(e_j)^2}{(d(e_i)+d(e_j))^2} \geq \frac{RS(G)^2}{EF(G)+2EM_2(G)}\,.$$

Now, from the above and inequality (3.20) we obtain

$$2AG_e(G)-GA_e(G) \geq \frac{2RS(G)^2}{EF(G)+2EM_2(G)}\,,$$

from which (3.17) immediately follows.

Equality in (3.19) holds if and only if $d(e_i) = d(e_j)$ for every pair of adjacent edges e_i and e_j in G, which implies that equality in (3.17) holds if and only if $L(G)$ is a regular graph. □

In the next theorem we establish a relationship between $RS(G)$, $EM_2(G)$ and $SDD_e(G)$.

Theorem 3.17. *Let G be a connected graph of size $m \geq 2$. Then we have*

$$RS(G) \leq \sqrt{EM_2(G) SDD_e(G)}\,. \tag{3.21}$$

Equality holds if and only if $L(G)$ is regular or $L(G) \cong P_3$.

Proof. The following identity is valid

$$SDD_e(G) = \sum_{e_i \sim e_j} \left(\frac{d(e_i)}{d(e_j)} + \frac{d(e_j)}{d(e_i)} \right) = \sum_{e_i \sim e_j} \frac{d(e_i)^2 + d(e_j)^2}{d(e_i)d(e_j)}\,. \tag{3.22}$$

On the other hand, for $r = 1$, $x_i := \sqrt{d(e_i)^2 + d(e_j)^2}$, $a_i := d(e_i)d(e_j)$, with summation performed over all pairs of adjacent edges e_i and e_j in G, the inequality (2.1) becomes

$$\sum_{e_i \sim e_j} \frac{d(e_i)^2 + d(e_j)^2}{d(e_i)d(e_j)} \geq \frac{\left(\sum_{e_i \sim e_j} \sqrt{d(e_i)^2 + d(e_j)^2} \right)^2}{\sum_{e_i \sim e_j} d(e_i)d(e_j)} = \frac{RS(G)^2}{EM_2(G)}\,. \tag{3.23}$$

From the above and identity (3.22) we obtain

$$SDD_e(G) \geq \frac{RS(G)^2}{EM_2(G)}\,,$$

from which (3.21) immediately follows.

Equality in (3.23) holds if and only if the expression $\frac{1}{d(e_i)^2} + \frac{1}{d(e_j)^2}$ is a constant for every pair of adjacent edges e_i and e_j in G. Suppose e_j and e_k are two edges adjacent to edge e_i in G. Then, we have that

$$\frac{1}{d(e_i)^2} + \frac{1}{d(e_j)^2} = \frac{1}{d(e_i)^2} + \frac{1}{d(e_k)^2}\,,$$

that is $d(e_j) = d(e_k)$. Thus, the equality in (3.23) holds if and only if $L(G)$ is a regular or semiregular bipartite graph. By Lemma 2.3, equality in (3.21) holds if and only if $L(G)$ is regular or $L(G) \cong P_3$. □

References

[1] M. O. Albertson, The irregularity of graphs, *Ars Combin.*, **46** (1997) 219–225.

[2] A. Ali, Z. Iqbal and Z. Iqbal, Two physicochemical properties of benzenoid chains: solvent accessible molecular volume and molar refraction, *Canadian Journal of Physics*, **97 5** (2018) 524–528.

[3] M. Bhanumathi, K. E. J. Rani and S. Balachandran, The edge version of inverse sum indeg index of connected graph, *Int. J. Math.*, **7** (2016) 8–12.

[4] B. Bollobas and P. Erdos, Graphs of extremal weights, *Ars. Comb.*, **50** (1998) 225–233.

http://dx.doi.org/10.22108/toc.2024.136304.2036

[5] K. C. Das, Maximizing the sum of the squares of the degrees of a graph, *Discrete Math.*, **285** no. 1–3 (2004) 57–66.

[6] N. De, Some bounds of reformulated Zagreb indices, *Appl. Math. Sci*, **6** no. 101 (2012) 5005–5012.

[7] T. Došlić, T. Reti and D. Vukičević, On the vertex degree indices of connected graphs, *Chem. Phys. Lett.*, **512** (2011) 283–286.

[8] C. S. Edwards, The largest vertex degree sum for a triangle in a graph, *Bull. London Math. Soc.*, **9** no. 2 (1977) 203–208.

[9] B. Furtula and I. Gutman, A forgotten topological index, *J. Math. Chem.*, **53** no. 4 (2015) 1184–1190.

[10] I. Gutman, Degree–based topological indices, *Croat. Chem. Acta*, **86** (2013) 351–361.

[11] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, **17** (1972) 535–538.

[12] I. Gutman, B. Ruščić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.*, **62** (1975) 3399–3405.

[13] I. Gutman, Geometric approach to degree–based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, **86** no. 1 (2021) 11–16.

[14] N. Harish, B. Sarveshkumar and B. Chaluvaraju, The reformulated Sombor index of a graph, *Trans. Combin.*, **13** no. 1 (2024) 1–16.

[15] B. Hollas, The covariance of topological indices thta depend on the degree of a vertex, *MATCH Commun. Math. Comput. Chem.*, **54** (2005) 177–187.

[16] V. R. Kulli, On k edge index and coindex of graphs, *Int. J. Fazzy Math. Arch.*, **10** no. 2 (2016) 111–116.

[17] Z. Lin, On A_α–eigenvalues of graphs and topological indices, *Contrib. Math.*, **5** (2022) 17–24.

[18] A. Miličević, S. Nikolić and N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers.*, **8** (2004) 393–399.

[19] D. S. Mitrinović and P. M. Vasić, *Analytic inequalities*, Springer Verlag, Berlin–Heidelberg–New York, 1970.

[20] S. Nikolić, G. Kovačević, A. Miličević and N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta*, **76** (2003) 113–124.

[21] K. Pattabiraman and T. Suganya, Edge version of some degree based topological descriptors of graphs, *J. Math. Nanosci.*, **8** no. 1 (2018) 1–12.

[22] J. Radon, Über die absolut additiven Mengenfunktionen, *Sitzungsber. Acad. Wissen. Wien*, **122** (1913) 1295–1438.

[23] V. S. Shegehalli and R. Kanabur, Arithmetic–geometric indices of path graph, *J. Comput. Math. Sci.*, **6** no. 1 (2015) 19–24.

[24] R. Todeschini and V. Consonni, *Handbook of molecular descriptors*, Wiley – VCH, Weinheim, 2000.

[25] D. Vukičević and M. Gašperov, Bond additive modeling 1. Adriatic indices, *Croat. Chem. Acta*, **83** (2010) 243–260.

[26] D. Vukičević and B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end vertex degree of edges, *J. Math. Chem.*, **46** (2009) 1369–1376.

[27] D. Vukičević, Bond additive modelling 2. Mathematical properties of max–min rodeg index, *Croat. Chem. Acta*, **83** (2010) 261–273.

http://dx.doi.org/10.22108/toc.2024.136304.2036

[28] D. Vukičević and J. Durdević, Bond additive modeling 10. Upper and lower bounds of bond incident degree indices of catacondensed fluoranthenes, *Chem. Phys. Lett.*, **515** (2011) 186–189.

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