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STAR-CRITICAL CONNECTED RAMSEY NUMBERS FOR 2-COLORINGS OF COMPLETE GRAPHS

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Abstract. This paper builds upon Sumner's work by further investigating the concept of connected Ramsey numbers, specifically focusing on star-critical connected Ramsey numbers. We obtain star-critical connected Ramsey numbers for several cases of trees versus complete graphs, stars versus stars, and paths versus paths. The connected Ramsey number for a star versus K_3 is also evaluated. Exact values are also obtained for the connected Ramsey numbers of $K_{1,n}$ versus K_3. This research explores the interplay between connectivity and graph coloring within the context of Ramsey theory.

1. Introduction

In 1978, David Sumner [13] introduced a variation of Ramsey numbers by focusing on 2-colorings of graphs where the subgraphs formed by edges of each color are required to be connected. This paper builds on Sumner's ideas by considering the star-critical analogue of connected Ramsey numbers. Before delving into our main findings, we must provide an overview of essential definitions and background concepts.

A 2-*coloring* of a graph $G = (V, E)$ is a function

$$f : E(G) \longrightarrow \{\text{red}, \text{blue}\}.$$

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A *connected* 2*-coloring* of a graph is a 2-coloring in which the subgraphs spanned by edges in each color are connected. In order for a graph to have a connected 2-coloring, every vertex must have degree at least 2. If G_1 and G_2 are graphs, then the *Ramsey number* $r(G_1, G_2)$ is the least positive integer p such that every 2-coloring of K_p (the complete graph of order p) contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. The existence of $r(G_1, G_2)$ follows from Frank Ramsey's influential theorem [12]. A 2-coloring of $K_{r(G_1,G_2)-1}$ that avoids both a red copy of G_1 and a blue copy of G_2 is termed a *critical coloring* for $r(G_1, G_2)$. An overview of known values and bounds for various Ramsey numbers (and some of their generalizations) can be found in Radziszowski's dynamic survey [11].

The *connected Ramsey number* $r_c(G_1, G_2)$ is the least positive integer p such that every connected 2-coloring of K_p contains a red subgraph that is isomorphic to G_1 or a blue subgraph that is isomorphic to G_2. A result due to Bosák, Rosa, and Znám [1] implies that if $p = r_c(G_1, G_2)$, then for every $n \geq p$, every connected 2-coloring of K_n contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. The restriction to connected 2-colorings is analogous to the way we limit our attention to rainbow triangle-free colorings when defining Gallai-Ramsey numbers (cf. [10]). Since every connected 2-coloring of a graph is a 2-coloring, it follows that

$$r_c(G_1, G_2) \leq r(G_1, G_2),$$

for every pair of graphs G_1 and G_2. When equality holds, we say that (G_1, G_2) is *Ramsey-connected*.

Sumner proved that if G_1 and G_2 are both graphs of order at least 4 that do not contain bridges (edges whose removal disconnects the graph), then (G_1, G_2) is Ramsey-connected (see [13, Theorem 2.1]). At present, the known connected Ramsey numbers where at least one of the graphs G_1 or G_2 has a bridge include paths versus paths [13], certain trees versus complete graphs [2], and various trees versus trees [4].

In 2010, Jonelle Hook [8] introduced the concept of a star-critical Ramsey number in her dissertation (see also [9]). In order to define this concept, we first define the notation $K_n \sqcup K_{1,k}$ to be the graph formed by joining a vertex v to K_n using exactly k edges. For graphs G_1 and G_2, the *star-critical Ramsey number* $r^*(G_1, G_2)$ is then defined to be the least k (where $1 \leq k \leq r(G_1, G_2) - 1$) such that every 2-coloring of $K_{r(G_1,G_2)-1} \sqcup K_{1,k}$ contains a red subgraph that is isomorphic to G_1 or a blue subgraph that is isomorphic to G_2. When $r^*(G_1, G_2) = r(G_1, G_2) - 1$, we say that (G_1, G_2) is *Ramsey-full*. For an overview of the current known star-critical Ramsey numbers, see [3].

Now we introduce the concept of the *star-critical connected Ramsey number* $r_c^*(G_1, G_2)$, defined to be the least k (where $2 \leq k \leq r_c(G_1, G_2) - 1$) such that every connected 2-coloring of $K_{r_c(G_1,G_2)-1} \sqcup K_{1,k}$ contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. The assumption that $k \geq 2$ is due to the fact that no connected 2-coloring of $K_{r_c(G_1,G_2)-1} \sqcup K_{1,1}$ exists. When $r_c^*(G_1, G_2) = r_c(G_1, G_2) - 1$, we say that (G_1, G_2) is *connected Ramsey-full*.

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Denote by P_m the path of order m, and by $K_{1,m}$ the star of order $m+1$. In Section 2, we focus on trees versus complete graphs, proving that

$$\begin{aligned} r_c^*(P_m, K_3) &= m-1, \quad \text{for all } m \geq 4, \\ r_c^*(P_5, K_n) &= n+1, \quad \text{for all } n \geq 3, \\ r_c^*(K_{1,3}, K_n) &= 2, \quad \text{for all } n \geq 3, \\ r_c(K_{1,m}, K_3) &= 2m-1, \quad \text{for all } m \geq 4, \text{ and} \\ r_c^*(K_{1,m}, K_3) &= 2m-2, \quad \text{for all } m \geq 4. \end{aligned}$$

In Section 3, we turn our attention to trees versus trees. In the case of paths versus paths, we prove that

$$\begin{aligned} r_c^*(P_5, P_5) &= 3, \\ r_c^*(P_5, P_6) &= 3, \text{ and} \\ r_c^*(P_6, P_6) &= 5. \end{aligned}$$

For stars versus stars, we prove that

$$r_c^*(K_{1,m}, K_{1,n}) = \begin{cases} 2 & \text{if } m \text{ or } n \text{ is odd} \\ m+n-2 & \text{if } m \text{ and } n \text{ are even,} \end{cases}$$

where $m, n \geq 3$. The paper is concluded by offering some conjectures regarding connected and star-critical connected Ramsey numbers.

2. Trees Versus Complete Graphs

In this section, we determine several connected star-critical Ramsey numbers, and one new connected Ramsey number, involving trees versus complete graphs. First, we consider star-critical connected Ramsey numbers for paths versus complete graphs. In the next two theorems, we prove that (P_m, K_3) and (P_5, K_n) are connected Ramsey-full.

Theorem 2.1. *For all $m \geq 4$, $r_c^*(P_m, K_3) = m-1$.*

Proof. It is known that $r_c(P_m, K_3) = m$ [2]. So, we need only provide a connected 2-coloring of $K_{m-1} \sqcup K_{1,m-2}$ that avoids a red P_m and a blue K_3. Label the vertices in a K_{m-1} by $x_1, x_2, \ldots, x_{m-1}$ and color all of the edges red in the complete subgraph induced by $\{x_1, x_2, \ldots, x_{m-3}\}$.

Next, color each edge of the form $x_{m-2}x_i$ $(1 \leq i \leq m-3)$ blue, each edge of the form $x_{m-1}x_i$ $(1 \leq i \leq m-4)$ blue, and edges $x_{m-1}x_{m-3}$ and $x_{m-1}x_{m-2}$ red. To this connected 2-coloring of K_{m-1}, introduce vertex v, joining it via blue edges to $x_1, x_2, \ldots, x_{m-3}$ and via a red edge to x_{m-1} (vx_{m-2} is the missing edge). For example, Figure 1 shows the $m = 10$ case. The longest red path in this

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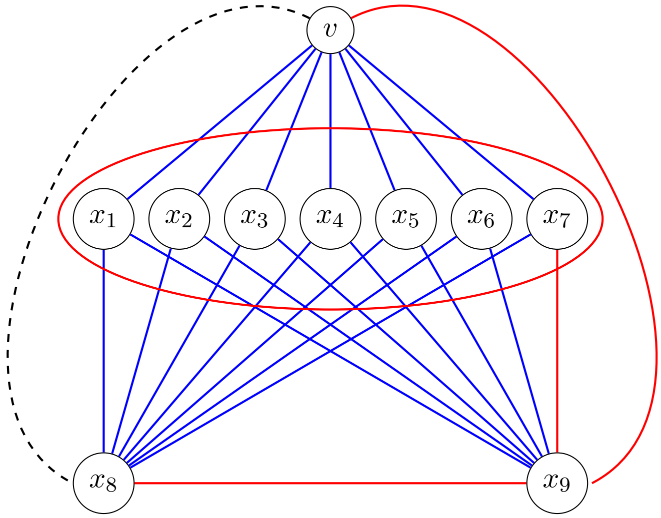

FIGURE 1. A connected 2-coloring of $K_9 \sqcup K_{1,8}$ that avoids a red P_m and a blue K_3.

connected 2-coloring of $K_{m-1} \sqcup K_{1,m-2}$ consists of $m-1$ vertices and the subgraph spanned by the blue edges is bipartite. So, no red P_m or blue K_3 exists, and it follows that $r_c^*(P_m, K_3) = m - 1$. □

In the proof of the following theorem, a broom graph is used in the construction of the lower bound. The tree resulting from connecting the central vertex of a star $K_{1,n}$ with the end vertex of the path P_{l-1} via an edge is symbolized as $B_{k,l}$ and is commonly referred to as a *broom*.

Theorem 2.2. *For all $n \geq 3$, $r_c^*(P_5, K_n) = n + 1$.*

Proof. It is known that $r_c(P_5, K_n) = n + 2$ [2]. We will first provide a connected 2-coloring of $K_{n+1} \sqcup K_{1,n}$ that avoids a red P_5 and a blue K_n. Start with a red $B_{n-2,3}$ in which the vertices in the P_2 are labelled x_1 and x_2, the center of the $K_{1,n-2}$ is labelled x_3, the leaves of the $K_{1,n-2}$ are labelled $x_4, x_5, \ldots, x_{n+1}$, and the edge joining the path and star is x_2x_3. Color all other edges blue. The longest red path in the resulting connected 2-coloring of K_{n+1} has four vertices, and no blue K_n exists since vertex x_2 has blue degree $n-2$ and vertex x_3 has blue degree 1, leaving only $n-1$ vertices with blue degree $n-1$.

With this critical coloring for $r_c(P_5, K_n)$, introduce a vertex v, joining it to vertex x_2 via a red edge and to vertices $x_3, x_4, \ldots, x_{n+1}$ with blue edges. Here, vx_1 is the missing edge. Figure 2 shows this construction when $n = 7$. The resulting connected 2-coloring of $K_{n+1} \sqcup K_{1,n}$ has a longest red path having four vertices. To see that it also avoids a blue K_n, note that vertex x_2 has blue degree $n-2$, x_3 has blue degree 2, and v has blue degree $n-1$. So, x_2 and x_3 cannot be contained in a blue K_n.

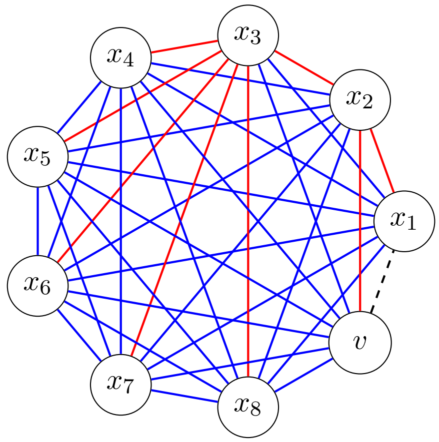

FIGURE 2. A connected 2-coloring of $K_8 \sqcup K_{1,7}$ that avoids a red P_5 and a blue K_7.

Also, if v were to be contained in a blue K_n, then all of the blue edges incident with v would have to be included. As one of these edges joins to x_3, this is not possible. Thus, $r_c^*(P_5, K_n) = n + 1$. □

Now we consider some cases of stars versus complete graphs.

Theorem 2.3. *For all $n \geq 3$, $r_c^*(K_{1,3}, K_n) = 2$.*

Proof. It was shown in [2] that $r_c(K_{1,3}, K_n) = 2n$, for all $n \geq 3$. The only connected graphs of order $2n-1$ with a maximum vertex degree of 2 are C_{2n-1} and P_{2n-1}. So, every connected 2-coloring of K_{2n-1} that lacks a red $K_{1,3}$ either contains a red C_{2n-1} or a red P_{2n-1}, and all remaining edges are blue. In the case where the subgraph spanned by the red edges is a P_{2n-1} given by $x_1x_2 \cdots x_{2n-1}$, the subgraph induced by $\{x_1, x_3, \dots, x_{2n-1}\}$ is a blue K_n. So, every critical coloring for $r_c(K_{1,3}, K_n)$ contains a red C_{2n-1}. Introducing a vertex v and joining it to the existing K_{2n-1} with any red edge results in a red $K_{1,3}$. In order to have a connected coloring, at least two edges (one in each color) must be joined between v and the K_{2n-1}, from which it follows that $r_c^*(K_{1,3}, K_n) = 2$. □

The final case of a tree versus a complete graph that we will consider requires that we also determine the corresponding connected Ramsey number.

Theorem 2.4. *For all $m \geq 4$,*

$$r_c(K_{1,m}, K_3) = 2m - 1 \quad \textit{and} \quad r_c^*(K_{1,m}, K_3) = 2m - 2.$$

Proof. To establish the lower bounds, begin with the connected 2-coloring of $K_{2(m-1)}$ formed by joining two red K_{m-1}-subgraphs with a single red edge. Color all additional edges blue. The maximum red degree of any vertex in the resulting K_{2m-2} is $m-1$ and the blue subgraph is bipartite. It follows that $r_c(K_{1,m}, K_3) \geq 2m - 1$. Now introduce a vertex v and join it to all of the vertices in one of the red K_{m-1}-subgraphs with blue edges and to $m-2$ vertices in the other red K_{m-1}-subgraph (all vertices except for the one vertex that has red degree $m-1$) with red edges (see Figure 3 for the case $m = 6$). The resulting connected 2-coloring of $K_{2m-2} \sqcup K_{1,2m-3}$ lacks a red $K_{1,m}$ and a blue K_3.

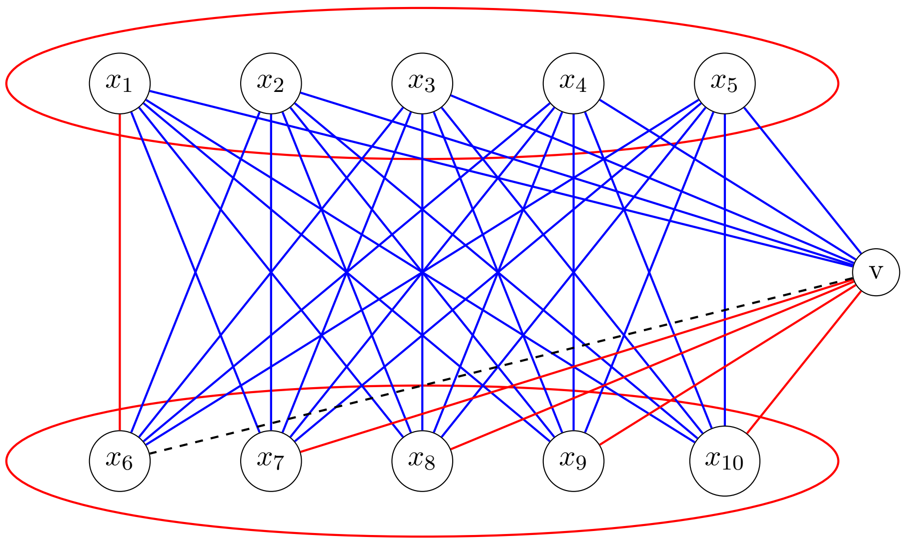

FIGURE 3. A connected 2-coloring of $K_{10} \sqcup K_{1,9}$ that avoids a red $K_{1,m}$ and a blue K_3.

The theorem will now follow from proving that every connected 2-coloring of K_{2m-1} contains a red $K_{1,m}$ or a blue K_3. So, consider a connected 2-coloring of K_{2m-1}. We break the remainder of the proof into two cases.

Case 1: Suppose that there exists a vertex x with red degree less than $m-1$. If such a vertex exists, then its blue degree is at least $2m-2-(m-2) = m$. Assume that $xy_1, xy_2, \ldots, xy_m$ are blue edges. If any edge in the subgraph induced by $\{y_1, y_2, \ldots, y_m\}$ is blue (say, y_iy_j is blue), then a blue K_3 is formed (induced by the subgraph $\{x, y_i, y_j\}$). So, the subgraph induced by $\{y_1, y_2, \ldots, y_m,\}$ is a red K_m. In order for the coloring to be connected, there exists some vertex $z \notin \{x, y_1, y_2, \ldots, y_m\}$ such that zy_k is red for some $1 \leq k \leq m$. In this case, y_k has red degree at least m, forming the center vertex for a red $K_{1,m}$.

Case 2: Suppose that every vertex has red degree at least $m-1$. If no red $K_{1,m}$ exists, then every vertex must have red degree $m-1$ and blue degree $m-1$. Let x be some vertex and assume that its blue neighborhood is $N_B(x) = \{y_1, y_2, \ldots, y_{m-1}\}$ and its red neighborhood is $N_R(x) = \{z_1, z_2, \ldots, z_{m-1}\}$. If the subgraph induced by $N_B(x)$ contains any blue edge, then a blue K_3 is formed. So, assume that $N_B(x)$ induces a red K_{m-1}. Each vertex in $N_B(x)$ must then join to exactly one vertex in $N_R(x)$ via a red edge. Without loss of generality, assume that y_1, z_1 is red and y_1z_i is blue for all $2 \leq i \leq m-1$. If any edge in the subgraph induced by $N_R(x) - \{z_1\}$ is blue, then a blue K_3 is formed. It follows that $N_R(x) - \{z_1\}$ induces a red K_{m-2}. At this stage, we have the coloring given in Figure 4.

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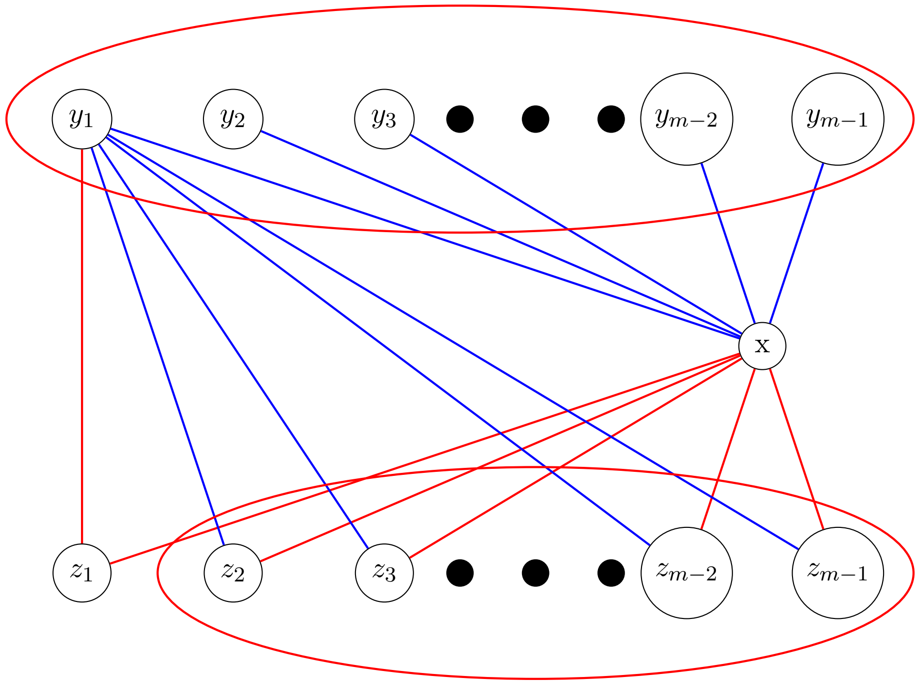

FIGURE 4. A connected 2-coloring of K_{2m-1} in which every vertex has red degree $m-1$ and blue degree $m-1$.

If y_iz_1 is red for all i such that $2 \leq i \leq m-1$, then a red $K_{1,m}$ is formed with center vertex z_1 and leaves $y_1, y_2, \ldots, y_{m-1}, x$. So, assume that there exists y_i, with $2 \leq i \leq m-1$, such that y_iz_j is red for some $2 \leq j \leq m-1$. Then the edges

$$y_iz_1, y_iz_2, \ldots, y_iz_{j-1}, y_iz_{j+1}, \ldots, y_iz_{m-1}$$

must all be blue. If a blue K_3 is to be avoided, then the subgraph induced by $\{z_1, z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_{m-1}\}$ is a red K_{m-2}. If edge z_1z_j is red, then z_1 is the center vertex of a red $K_{1,n}$ with leaves $y_1, z_2, z_3, \ldots, z_{m-1}, x$. So, z_1z_j must be blue.

Now consider a vertex y_k, where $3 \leq k \leq m-1$. If both y_kz_1 and y_kz_j are blue, then $\{y_k, z_1, z_j\}$ induces a blue K_3. If y_kz_1 is red, then a red $K_{1,m}$ is formed with center vertex z_1 and leaves $y_1, y_k, z_3, z_4, \ldots, z_{m-1}, x$. If y_kz_j is red, then a red $K_{1,m}$ is formed with center vertex z_j and leaves $y_i, y_k, z_3, z_4, \ldots, z_{m-1}, x$. □

3. Trees Versus Trees

In this section, we consider star-critical connected Ramsey numbers for trees versus trees. We begin with some cases of paths versus paths. In 1967, Gerencsér and Gyárfás [5] proved that

$$r(P_m, P_n) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1,$$

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for all $n \geq m \geq 2$. The connected version of this Ramsey number was considered by Sumner [13], who showed that for all $n \geq m \geq 5$,

$$r_c(P_m, P_n) = \begin{cases} r(P_m, P_n) - 1 & \text{if } m \text{ is odd} \\ r(P_m, P_n) - 2 & \text{if } m \text{ is even} \end{cases}$$
$$= \begin{cases} n + \lfloor m/2 \rfloor - 2 & \text{if } m \text{ is odd} \\ n + \lfloor m/2 \rfloor - 3 & \text{if } m \text{ is even.} \end{cases}$$

In the following theorem, we consider the star-critical connected Ramsey number for paths when $n = m = 5$.

Theorem 3.1. $r_c^*(P_5, P_5) = 3$.

Proof. Note that $r_c(P_5, P_5) = 5$ [13]. Up to a reordering of the vertices, only one connected 2-coloring of K_4 is possible (see Lemma 9 of [4]). The subgraph spanned by edges in each color in such a critical coloring form a P_4. Without loss of generality assume that $abcd$ is a red P_4 and $bdac$ is a blue P_4 (see Figure 5).

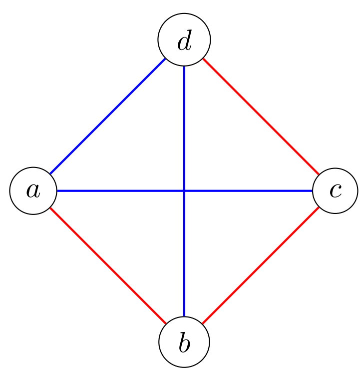

FIGURE 5. The only connected 2-coloring of K_4.

Introduce a vertex v, join it to a with a blue edge, and join it to b with a red edge. In the resulting $K_4 \sqcup K_{1,2}$, the subgraph spanned by edges in each of the colors is isomorphic to the broom $B_{2,3}$, which does not contain P_5 as a subgraph. It follows that $r_c^*(P_5, P_5) \geq 3$.

To prove the reverse inequality, start with the K_4 given in Figure 5 and join v to this K_4 with three edges. Now v cannot join to a or d with a red edge without producing a red P_5 and v cannot join to b or c with a blue edge without producing a blue P_5. So, v must join to either a and d with blue edges or to b and c with red edges. In the first case, $bdvac$ is a blue P_5, and in the second case, $abvcd$ is a red P_5. It follows that $r_c^*(P_5, P_5) \leq 3$, from which the theorem follows. □

Theorem 3.2. $r_c^*(P_5, P_6) = 3$.

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Proof. Observe that $r_c(P_5, P_6) = 6$ [13]. Start with a connected 2-coloring of K_5 and note that it must have a removable vertex (see [2, Theorem 2.1]). Removing such a vertex leaves a K_4 colored as in Figure 5 since only one connected 2-coloring of K_4 exists (see [4, Lemma 9]). The vertex removed to form this K_4 must have been incident with edges in both colors in order for the coloring to have been connected. Considering all of the possibilities for how this vertex joins to the K_4, we find that all connected 2-colorings of K_5 (up to isomorphism) contain both a red P_5 and a blue P_5, except for the coloring shown in the first image in Figure 6. This coloring contains a red P_4 and a blue P_5.

FIGURE 6. Connected 2-colorings of K_5 and $K_5 \sqcup K_{1,2}$ that avoid a red P_5 and a blue P_6.

A new vertex v can be joined to b with a red edge and to d with a blue edge (see the second image in Figure 6) without producing a red P_5 or a blue P_6. It follows that $r_c^*(P_5, P_6) \geq 3$. To prove the reverse inequality, consider a connected 2-coloring of $K_5 \sqcup K_{1,3}$. Then the K_5 must be colored as in the first image in Figure 6. The vertex v can only join to one of b or c without producing a red P_5. Also, v can only join to d without producing a blue P_6. It follows that $r_c^*(P_5, P_6) \leq 3$. □

Theorem 3.3. $r_c^*(P_6, P_6) = 5$.

Proof. Given that $r_c(P_6, P_6) = 6$ [13], we need only provide a connected 2-coloring of $K_5 \sqcup K_{1,4}$ that avoids a monochromatic P_6. The coloring in Figure 7 accomplishes this. Note that if a red path was formed that used every vertex, then its endpoints would have to be b and e. Then at most one of the vertices v and a could be included in the path. Similarly, if a blue path was formed that used every vertex, then its endpoints would have to be c and d. Once again, at most one of the vertices v and a could be included in the path. It follows that $r_c^*(P_6, P_6) = 5$. □

In 1972, Harary [7] proved that

$$r(K_{1,m}, K_{1,n}) = \begin{cases} m+n & \text{if } m \text{ or } n \text{ is odd} \\ m+n-1 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

The corresponding connected Ramsey number was recently determined in [4], where it was shown that for all $m, n \geq 3$, $(K_{1,m}, K_{1,n})$ is Ramsey-connected. In order to determine the star-critical connected

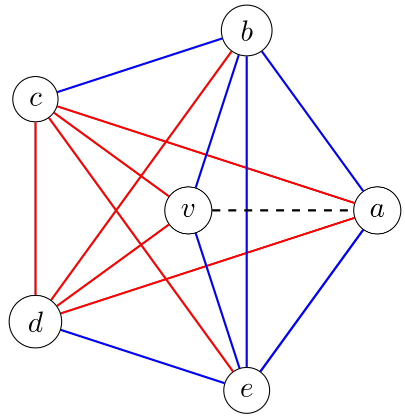

FIGURE 7. A connected 2-coloring of $K_5 \sqcup K_{1,4}$ that avoids a red P_6 and a blue P_6.

Ramsey number for stars, we need the following well-known result concerning the factorization of complete graphs of even order (e.g., see [6, Theorem 9.7]).

Lemma 3.4. [6] *For every $k \in \mathbb{N}$, the complete graph K_{2k} factors into $k-1$ spanning cycles and a 1-factor (i.e., a perfect matching).*

In the following theorem, the connected star-critical connected Ramsey number for stars is determined.

Theorem 3.5. *For all $m, n \geq 3$,*

$$r_c^*(K_{1,m}, K_{1,n}) = \begin{cases} 2 & \text{if } m \text{ or } n \text{ is odd} \\ m+n-2 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. First, we handle the case where m or n is odd, so that

$$r_c(K_{1,m}, K_{1,n}) = m + n.$$

Consider a connected 2-coloring of K_{m+n-1} that avoids a red $K_{1,m}$ and a blue $K_{1,n}$. Then each vertex is incident with exactly $m-1$ red edges and exactly $n-1$ blue edges. Joining a vertex v with either color edge to any vertex in this critical coloring necessarily forms a red $K_{1,m}$ or a blue $K_{1,n}$. In order for the resulting coloring to be connected, one edge of each color must be added. It follows that $r_c^*(K_{1,m}, K_{1,n}) = 2$ in this case.

Now consider the case where m and n are even, so that

$$r_c(K_{1,m}, K_{1,n}) = m + n - 1.$$

The complete graph K_{m+n-2} has even order, so let $m+n-2 = 2k$. By Lemma 3.4, K_{m+n-2} can be decomposed into $k-1$ spanning cycles a a single 1-factor. Color $\frac{m-2}{2}$ of the spanning cycles red and the other $\frac{n-2}{2}$ spanning cycles blue. The 1-factor has size k and we color $\frac{n-2}{2}$ of its edges red and

$\frac{m}{2}$ of its edges blue. Let A denote the set of vertices incident with the red edges in the 1-factor and let B denote the set of vertices incident with the blue edges in the 1-factor. At this point, we have a critical coloring for $r_c(K_{1,m}, K_{1,n})$ in which $n-2$ vertices have red degree $m-1$ and blue degree $n-2$ (those in the set A) and m vertices have red degree $m-2$ and blue degree $n-1$ (those in the set B). Introduce a vertex v and join it to all of the vertices in A with blue edges and $m-1$ of the vertices in B with red edges. The result is a connected 2-coloring of $K_{m+n-2} \sqcup K_{1,m+n-3}$ that avoids a red $K_{1,m}$ and a blue $K_{1,n}$. It follows that $r_c^*(K_{1,m}, K_{1,n}) = m + n - 2$. □

4. Conclusion

We conclude the paper by stating some conjectures regarding the evaluation of certain connected Ramsey numbers and star-critical connected Ramsey numbers. Building off of the conjecture stated in [2] concerning the evaluation of $r_c(P_m, K_n)$, we offer the following extension.

Conjecture 4.1. *For all $m \geq 4$ and $n \geq 3$, $r_c(P_m, K_n) = m + n - 3$ and $r_c^*(P_m, K_n) = m + n - 4$.*

Based on the results in [2] along with Theorem 2.4 of this paper, we give the following conjecture.

Conjecture 4.2. *For all $m \geq 4$ and $n \geq 3$, $r_c(K_{1,m}, K_n) = m + n - 5$.*

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