www.theoryofgroups.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 2 No. 1 (2013), pp. 187-198.

www.ui.ac.ir

ON THE REPRESENTATION THEORY OF THE ALTERNATING GROUPS

T. CECCHERINI-SILBERSTEIN*, F. SCARABOTTI AND F. TOLLI

Dedicated to our mentor and friend Toni Machì

Communicated by Patrizia Longobardi

Abstract. We present the basic results on the representation theory of the alternating groups $\mathfrak{A}_n$. Our approach is based on Clifford theory.

1. Introduction

The purpose of this note is to present the basic results on the representation theory of the alternating groups $\mathfrak{A}_n$, by using an approach based on Clifford theory.

While the representation theory of the symmetric groups $\mathfrak{S}_n$ is most popular and widely studied, the representation theory of $\mathfrak{A}_n$ is often neglected and only a few monographs provide an exhaustive and fully detailed treatment of it. Among these exceptional monographs we mention the one by Fulton and Harris [6] where some important results are, however, left as (clearly outlined) exercises, and the one by James and Kerber [7] which is implicitly based on Clifford theory and refers to the book by Curtis and Reiner [5] for a description of this theory and the proofs of its main results.

In the study of induced representations, Clifford theory is the tool for relating the representation theory of a group with the representation theory of its normal subgroups. We refer to our research-expository paper [2] (see also our recent monograph [4]) for a fully complete treatment. The present note should be considered as an appendix to our paper with a concrete and important example fully worked out. Indeed, $\mathfrak{A}_n$ is a (normal) subgroup of index two in $\mathfrak{S}_n$, for $n \geq 2$, and Clifford theory, specialized for subgroups of index two, yields a complete and exhaustive description of the representation theory of $\mathfrak{A}_n$.

MSC(2010): Primary: 20C30; Secondary: 20B35, 20E45.

Keywords: Alternating group, irreducible representation, character, conjugacy class, Clifford theory.

Received: 14 January 2013, Accepted: 11 May 2013.

*Corresponding author.

The paper is organized as follows. The first two sections collect some background material (without proofs, but with appropriate references): in Section 2 we recall the basic results of the representation theory of the symmetric groups $\mathfrak{S}_n$ including the Murnaghan-Nakayama rule, and in Section 3 we describe Clifford theory for subgroups of index two. Section 4 constitutes the core of the paper and is fully self-contained. We describe the conjugacy classes and the irreducible representations of $\mathfrak{A}_n$; in particular, we present a detailed proof of the Frobenius formula (Theorem 4.5) for the corresponding characters. Finally, we discuss ambivalence of the groups $\mathfrak{A}_n$, providing a complete analysis.

2. Representation theory and conjugacy classes of the symmetric groups $\mathfrak{S}_n$

In this section we review the main facts on the representation theory and the conjugacy classes of $\mathfrak{S}_n$.

It is a general fact that there is a one-to-one correspondence between the set of conjugacy classes of a finite group G and the *dual* of G, denoted by $\widehat{G}$, a complete set of irreducible pairwise inequivalent representations of G.

When $G = \mathfrak{S}_n$ these two sets are parameterized by the set of partitions of n, that is, by the set of tuples $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ are positive integers satisfying $\lambda_1+\lambda_2+\cdots+\lambda_\ell = n$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition of n. We then write $\lambda \vdash n$ and denote by S^λ the associated irreducible representation of $\mathfrak{S}_n$. For instance, $S^{(n)}$ is the trivial representation, while $S^{(1^n)}$, $(1^n) = (1, 1, \dots, 1)$, is the *alternating* (or *sign*) representation. We shall adopt the following notation: the character of the representation S^λ will be denoted by χ^λ (with λ superscript), while the corresponding conjugacy class (formed by all permutations whose cyclic structure is λ) will be denoted by $\mathcal{C}_\lambda$ (with λ subscript).

Pictorially, one associates with λ a diagram, called the *Young frame of shape* λ, consisting of n boxes with k left-justified rows, the ith row containing exactly λ_i boxes.

Moreover, the associated *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\ell'}) \vdash n$ is defined by setting $\ell' = \lambda_1$ and $\lambda'_j = |\{i \in \{1, 2, \dots, \ell\} : \lambda_i \geq j\}|$ for all $j = 1, 2, \dots, \ell'$. It follows from the definitions that the Young frames associated with conjugate partitions λ and λ' are one the transposed (that is, obtained by exchanging rows and columns) of the other. For instance, one has $(n)' = (1^n)$. If $\lambda = \lambda'$ one says that λ is *self-conjugate* and the associated Young frame is *symmetric.*

We have (cf. [1, Theorem 10.4.1] or [3, Lemma 3.6.10])

$$S^\lambda \otimes S^{(1^n)} \sim S^{\lambda'}. \tag{2.1}$$

As a consequence, $S^\lambda \otimes S^{(1^n)} \sim S^\lambda$ if and only if λ is self-conjugate.

Let now $\lambda \vdash n$ and $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$. A *rim hook tableau* T of *shape* λ and *content* μ is the Young frame of shape λ filled with the numbers $1, 2, \dots, k$ (with possible repetitions) in such a way that

- μ_j is the number of j's in T, $j = 1, 2, \dots, k$;
- the numbers are weakly increasing along the rows and the columns;

- for $j = 1, 2, \ldots, k$ the diagram T_j consisting of the boxes occupied by j is connected (given two boxes b and b' in T_j there exists a sequence $b_0, b_1, \ldots, b_k$, $k \geq 0$, of boxes of T_j such that $b_0 = b$ and $b_k = b'$, and b_i and b_{i+1} share a common edge, for $i = 0, 1, \ldots, k-1$) and intersects the diagonal of T (that is, the set of boxes of T whose coordinates (i.e. row and column numbers) coincide) at most once.

Denoting by $\langle T_j \rangle$ the number of rows of T_j decreased by one, one defines the *height* $\langle T \rangle$ of T by

$$\langle T \rangle = \sum_{j=1}^{k} \langle T_j \rangle.$$

The following is the celebrated Murnaghan-Nakayama rule (cf. [3, Section 3.5.5]).

Theorem 2.1. *Let $\lambda \vdash n$ and let $\pi \in \mathfrak{S}_n$ have cyclic structure $\mu \vdash n$. Then, for the associated character χ^λ, we have*

$$\chi^\lambda(\pi) = \sum_T (-1)^{\langle T \rangle} \tag{2.2}$$

where the sum runs over all rim hook tableaux T of shape λ and content μ.

Example 2.2. Take $\lambda = (5, 4, 3) \vdash 12$ and suppose that $\pi \in \mathfrak{S}_{12}$ has cycle structure $\mu = (1, 2, 3, 6) \vdash 12$. There are only two rim hook tableaux of shape λ and content μ, namely

$T =$

1	3	3	4	4
2	3	4	4	
2	4	4		

$T' =$

1	2	2	4	4
3	3	4	4	
3	4	4		

Figure 1. The two rim hook tableaux of shape $(5, 4, 3)$ and content $(1, 2, 3, 6)$.

Since $\langle T \rangle = 4$ and $\langle T' \rangle = 3$, we have $\chi^\lambda(\pi) = (-1)^4 + (-1)^3 = 1 - 1 = 0$.

Recall that a conjugacy class $\mathcal{C}$ in a group G is said to be *ambivalent* if $g \in \mathcal{C}$ infers $g^{-1} \in \mathcal{C}$, equivalently, if every element in $\mathcal{C}$ is conjugate to its own inverse. The group G itself is said to be *ambivalent* if every conjugacy class is ambivalent, equivalently, if every group element is conjugate to its inverse. It is an easy exercise to check that $\mathfrak{S}_n$ is ambivalent (cf. [1, Corollary 9.8.22]).

Finally, we recall that an irreducible representation σ of a group G is *self-adjoint* if its associated character χ^σ is real-valued; otherwise, σ is termed *complex*. A self-adjoint representation is called *real* if its matrix coefficients are all real-valued; otherwise, it is termed *quaternionic*. It is well known that all representations of $\mathfrak{S}_n$ are real (see, for instance [3, Corollary 3.4.3]).

3. Clifford theory for subgroups of index two

In this section we introduce notation and review the main results of the theory of induced representations from subgroups of index two, which we shall use later in the core of the paper. We refer to our research-expository paper [2] where the reader may find complete proofs based on Clifford theory.

Let G be a finite group and $N \trianglelefteq G$ a (normal) subgroup of G of index two. For $\theta \in \widehat{G}$ (resp. $\sigma \in \widehat{N}$) we denote by $\mathrm{Res}^G_N(\theta) \in \widehat{N}$ (resp. $\mathrm{Ind}^G_N(\sigma) \in \widehat{G}$) the corresponding restriction (resp. induction) from G to N (resp. from N to G). Also, we shall write $\sigma \preceq \rho$ to denote that σ is a subrepresentation of ρ.

For $\sigma \in \widehat{N}$ and $g \in G$ we set

$$\widehat{G}(\sigma) = \{\theta \in \widehat{G} : \sigma \preceq \mathrm{Res}^G_N(\theta)\} = \{\theta \in \widehat{G} : \theta \preceq \mathrm{Ind}^G_N(\sigma)\},$$

where the second equality follows from Frobenius' reciprocity (see, for instance, [3, Theorem 1.6.11]).

Also, the *g-conjugate* of σ is the representation ${}^g\sigma \in \widehat{N}$ defined by

$$ {}^g\sigma(n) = \sigma(g^{-1}ng) \tag{3.1}$$

for all $n \in N$.

Finally, the subgroup

$$I_G(\sigma) = \{g \in G : \ {}^g\sigma \sim \sigma\} \leq G$$

is called the *inærtia group* of $\sigma \in \widehat{N}$. Note that $N \leq I_G(\sigma) \leq G$ so that one necessarily has either $I_G(\sigma) = N$ or $I_G(\sigma) = G$.

Given any $h \in G \setminus N$ we have the coset decomposition $G = N \coprod hN$ and we define the (one–dimensional) *alternating representation* $(\varepsilon, \mathbb{C})$ of G (with respect to N) by setting

$$\varepsilon(g) = \begin{cases} 1 & \text{if } g \in N \\ -1 & \text{otherwise.} \end{cases} \tag{3.2}$$

for all $g \in G$. Note that when $G = \mathfrak{S}_n$ and $N = \mathfrak{A}_n$, (3.2) is the usual alternating representation of $\mathfrak{S}_n$. The group $C_2 = \{1, -1\}$ acts on $\widehat{N}$ and $\widehat{G}$ as follows. The non-trivial element -1 acts on $\widehat{N}$ by $\widehat{N} \ni \sigma \mapsto {}^h\sigma \in \widehat{N}$ where, as usual, ${}^h\sigma(n) = \sigma(h^{-1}nh)$ for all $n \in N$. Also, -1 acts on $\widehat{G}$ by $\widehat{G} \ni \theta \mapsto \theta \otimes \varepsilon \in \widehat{G}$. Correspondingly, both $\widehat{N}$ and $\widehat{G}$ are partitioned into their C_2-orbits (moreover, every such an orbit consists of one or two representations).

The following theorem (cf. [2, Theorem 3.1]) yields a very natural bijection between the C_2-orbits on $\widehat{N}$ and those on $\widehat{G}$.

Theorem 3.1. (i) *If $I_G(\sigma) = N$, then, setting $\theta := \mathrm{Ind}^G_N\sigma$, one has that θ is irreducible (so that $\theta \in \widehat{G}$), $\theta \otimes \varepsilon \sim \theta$, $\mathrm{Res}^G_N\theta = \sigma \oplus {}^h\sigma$, and $\sigma \not\sim {}^h\sigma$.*

(ii) *If $I_G(\sigma) = G$, then, for every $\theta \in \widehat{G}(\sigma)$, one has $\mathrm{Ind}^G_N(\sigma) = \theta \oplus (\theta \otimes \varepsilon)$, $\theta \not\sim \theta \otimes \varepsilon$, and $\mathrm{Res}^G_N\theta = \mathrm{Res}^G_N(\theta \otimes \varepsilon) = \sigma$.*

(iii) *Setting*

$$\{\sigma, {}^h\sigma\} \mapsto \widehat{G}(\sigma) \qquad \textit{when } I_G(\sigma) = N$$

and

$$\{\sigma\} \mapsto \widehat{G}(\sigma) \qquad \textit{when } I_G(\sigma) = G,$$

one gets a one–to–one correspondence between the C_2-orbits on $\widehat{N}$ and on $\widehat{G}$. In particular, to each single–element orbit on $\widehat{N}$ (resp. on $\widehat{G}$) there corresponds a two–element orbit on $\widehat{G}$ (resp. on $\widehat{N}$).

We now turn our attention to the study of the relation between conjugacy classes (resp. characters) of G and those of N. First observe that every conjugacy class $\mathcal{C}$ of G contained in N either is itself a conjugacy class of N or splits into two conjugacy classes of N. Indeed, there is a natural C_2-action on the conjugacy classes of N: for a conjugacy class $\mathcal{C}$ of N set ${}^h\mathcal{C} = h^{-1}\mathcal{C}h$. Clearly, ${}^h\mathcal{C}$ is again a conjugacy class of N:

$$n^{-1}(h^{-1}th)n = h^{-1}(hn^{-1}h^{-1}thnh^{-1})h \in {}^h\mathcal{C}$$

for all $t \in \mathcal{C}$ and $n \in N$. Moreover, $\mathcal{C} \cup {}^h\mathcal{C}$ is a conjugacy class of G. This situation is the analogous to the one described in Theorem 3.1. If $\mathcal{C} \neq {}^h\mathcal{C}$ (and therefore $\mathcal{C} \cap {}^h\mathcal{C} = \emptyset$) we say that $\mathcal{C}$ is a *split* class of N. One also says that $\{\mathcal{C}, {}^h\mathcal{C}\}$ is a *pair of split classes* or that $\mathcal{C} \cup {}^h\mathcal{C}$ *splits* into two N-classes. If $\mathcal{C} = {}^h\mathcal{C}$ (and therefore $\mathcal{C}$ is a conjugacy class of G) we say that $\mathcal{C}$ is *nonsplit*.

Proposition 3.2. (i) *The number of split classes in N is equal to $|\{\sigma \in \widehat{N} : \sigma \not\sim {}^h\sigma\}|$.*

(ii) *The number of nonsplit classes in N is equal to the number of conjugacy classes of G but not in N. Moreover, this number equals $|\{\sigma \in \widehat{N} : \sigma \cong {}^h\sigma\}|$.*

Proof. See [2, Proposition 3.1]. □

Remark 3.3. From Theorem 3.1 we deduce some formulas that relate the characters of σ and θ.

Suppose first that $\theta \ncong \theta \otimes \varepsilon$. We then have $\chi^{\theta\otimes\varepsilon}(nh) = \chi^\theta(nh)\chi^\varepsilon(nh) = -\chi^\theta(nh)$ and $\chi^\sigma(n) = \chi^\theta(n) = \chi^{\theta\otimes\varepsilon}(n)$, for all $n \in N$.

Suppose now that $\theta \sim \theta \otimes \varepsilon$. In this case, we have $\chi^\theta(g) = 0$ if $g \notin N$ and $\chi^\theta(n) = \chi^\sigma(n) + \chi^{{}^h\sigma}(n)$, for all $n \in N$. Moreover, if $\mathcal{C}$ is a nonsplit class of N and $n \in \mathcal{C}$, then $\chi^\sigma(n) = \chi^{{}^h\sigma}(n) = \frac{1}{2}\chi^\theta(n)$. If $\mathcal{C} \cup {}^h\mathcal{C}$ is a pair of split classes and $n_1 \in \mathcal{C}, n_2 \in {}^h\mathcal{C}$, then $\chi^\sigma(n_1) = \chi^{{}^h\sigma}(n_2)$, $\chi^\sigma(n_2) = \chi^{{}^h\sigma}(n_1)$ and $\chi^\sigma(n_1) + \chi^{{}^h\sigma}(n_1) = \chi^\theta(n_1)$. Finally note that there exists a pair of split classes on which χ^σ and $\chi^{{}^h\sigma}$ disagree (because $\chi^\sigma \neq \chi^{{}^h\sigma}$).

4. Representation theory of $\mathfrak{A}_n$

4.1. **Conjugacy classes of $\mathfrak{A}_n$.** For $\mu = (\mu_1, \mu_2, \dots, \mu_m) \vdash n$ we have that $\mathcal{C}_\mu \subseteq \mathfrak{A}_n$ if and only if $n - m$ is even. Indeed, any cycle $(a_1, a_2, \dots, a_k)$ of length k may be written as the product of $k - 1$ transpositions:

$$(a_1, a_2, \dots, a_k) = (a_1, a_2)(a_1, a_3) \cdots (a_1, a_k)$$

and therefore any $\pi \in \mathcal{C}_\mu$ may be written as the product of $(\mu_1 - 1) + (\mu_2 - 1) + \cdots + (\mu_n - 1) = n - m$ transpositions.

Lemma 4.1. *Let $\mathcal{C}_\mu$ be the conjugacy class of $\mathfrak{S}_n$ associated with the partition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$. Suppose that $n - m$ is even. Then $\mathcal{C}_\mu$ splits into two $\mathfrak{A}_n$ classes if and only if $\mu_1 > \mu_2 > \cdots > \mu_m$ and $\mu_1, \mu_2, \dots, \mu_m$ are odd numbers.*

Proof. Take $\pi \in \mathcal{C}_\mu$ and let $\mathcal{C}$ be the $\mathfrak{A}_n$-conjugacy class of π. Then $\mathcal{C}_\mu$ does not split in $\mathfrak{A}_n$ (that is $\mathcal{C} = \mathcal{C}_\mu$) if and only if for any $\sigma \in \mathfrak{S}_n$ we have $\sigma^{-1}\pi\sigma \in \mathcal{C}$. This is equivalent to the following condition: there exists $\theta \in \mathfrak{S}_n \setminus \mathfrak{A}_n$ such that: $\theta^{-1}\pi\theta = \pi$. Indeed, if $\sigma^{-1}\pi\sigma \in \mathcal{C}$ with $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$, then there exists $\rho \in \mathfrak{A}_n$ such that $\sigma^{-1}\pi\sigma = \rho^{-1}\pi\rho$ and we can take $\theta = \sigma\rho^{-1}$.

Let $1 \leq i \leq m-1$. Supposing that $\mu_i = \mu_{i+1}$ is odd, then, denoting by $(a_1, a_2, \ldots, a_{\mu_i})$ and $(b_1, b_2, \ldots, b_{\mu_{i+1}})$ the corresponding cycles in π, we can take $\theta = (a_1, b_1)(a_2, b_2)\cdots(a_{\mu_i}, b_{\mu_{i+1}})$. On the other hand, if μ_i is even then the corresponding cycle $(a_1, a_2, \ldots, a_{\mu_i})$ is odd and we can take $\theta = (a_1, a_2, \ldots, a_{\mu_i})$. By the above arguments we deduce the necessary condition.

Conversely, suppose that $\mu_1, \mu_2, \ldots, \mu_m$ are distinct odd numbers. Then $\theta^{-1}\pi\theta = \pi$ if and only if any cycle of θ is also a cycle of π or is trivial. Since these are only even permutations, there exist no $\theta \in \mathfrak{S}_n \setminus \mathfrak{A}_n$ satisfying $\theta^{-1}\pi\theta = \pi$, and the sufficient condition follows as well. □

We will denote by Split_n the set of all partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vdash n$ such that $\mu_1, \mu_2, \ldots, \mu_n$ are odd and distinct.

4.2. **The irreducible representations of $\mathfrak{A}_n$.** For any pair $\{\lambda, \lambda'\}$ with $\lambda \vdash n$, $\lambda \not\sim \lambda'$, choose a representative and denote by $\mathcal{N}_n$ the set of these representatives. On the other hand, we denote by $\mathcal{S}I_n$ the set of all self-conjugate partitions of n. From Theorem 3.1, Lemma 4.1 and Proposition 3.2 we then deduce the following:

Theorem 4.2. *For $\lambda \in \mathcal{N}_n$ set $\widetilde{S}^\lambda = \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n} S^\lambda$, and for $\lambda \in \mathcal{S}I_n$ denote by $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{A}_n} S^\lambda = S^\lambda_- \oplus S^\lambda_+$ the corresponding $\mathfrak{A}_n$-irreducible decomposition. Then*

$$\{\widetilde{S}^\lambda : \lambda \in \mathcal{N}_n\} \cup \{S^\lambda_-, S^\lambda_+ : \lambda \in \mathcal{S}I_n\}$$

is a complete set of irreducible pairwise inequivalent representations of $\mathfrak{A}_n$. Moreover, $|\mathcal{S}I_n| = |\mathrm{Split}_n|$.

The following lemma provides a natural and useful bijection between $\mathcal{S}I_n$ and Split_n (thus recovering the last assertion of the previous theorem).

Lemma 4.3. *The map*

$$\begin{array}{ccc} \mathcal{S}I_n & \to & \mathrm{Split}_n \\ \lambda & \mapsto & \mu = \mu(\lambda) \end{array} \tag{4.1}$$

that associates with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathcal{S}I_n$ the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ defined by setting $k = \max\{i \in \{1, 2, \ldots, \ell\} : 2\lambda_i - 2i + 1 > 0\}$ and $\mu_i = 2\lambda_i - 2i + 1$ for $i = 1, 2, \ldots, k$, is a bijection.

Proof. There is a nice pictorial description of this correspondence, from which the lemma easily follows. Denote by r_i the set of boxes in the ith row of the diagram of λ, $i = 1, 2, \ldots, \lambda_1$, and by c_j the set of boxes in the jth column, $j = 1, 2, \ldots, \lambda_1$. Then we have:

$$\begin{aligned} 2\lambda_1 - 1 &= |c_1 \cup r_1| \\ 2\lambda_2 - 3 &= |(c_2 \cup r_2) \setminus (c_1 \cup r_1)| \\ 2\lambda_3 - 5 &= |(c_3 \cup r_3) \setminus ((c_2 \cup r_2) \cup (c_1 \cup r_1))\,| \\ \cdots &= \cdots \end{aligned}$$

Note that k can be interpreted as the maximum $i \in \{1, 2, \ldots, \lambda_1\}$ such that $r_i \cap c_i \neq \emptyset$. We then write 1 inside the boxes of $c_1 \cup r_1$, 2 inside those of $(c_2 \cup r_2) \setminus (c_1 \cup r_1)$, 3 inside those of $|(c_3 \cup r_3) \setminus [(c_2 \cup$

$r_2) \cup (c_1 \cup r_1)]$, and so on. This way, the number of boxes labeled with i is exactly $2\lambda_i - 2i + 1$. In particular, there is exactly one box labeled by k and there are no boxes labeled by $k+1, k+2, \ldots$ $\square$

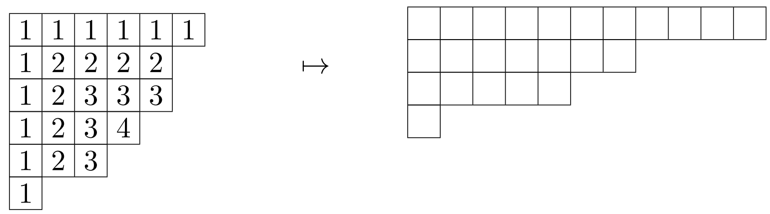

Figure 2. With $\lambda = (6, 5, 5, 4, 3, 1) \in \mathcal{SI}_{24}$ one associates $\mu = (11, 7, 5, 1) \in \mathrm{Split}_{24}$.

Remark 4.4. There is an alternative description of the bijection presented in the previous lemma. Given the Young frame corresponding to a partition $\lambda \vdash n$, we denote by $H^\lambda_{i,j}$ the set consisting of the box b of coordinates (i, j) together with all boxes that are either to the right (but in the same row) or lower down (but in the same column) with respect to box b. We call $H^\lambda_{i,j}$ the (i, j)-*hook* associated with λ and we denote by $h^\lambda_{i,j}$ its *length*, i.e. the number of boxes in it. Then the correspondence in (4.1) can be rewritten as:

$$(\lambda_1, \lambda_2, \ldots, \lambda_\ell) \mapsto (h^\lambda_{1,1}, h^\lambda_{2,2}, \ldots, h^\lambda_{k,k}).$$

For $\lambda \vdash n$ we now denote by χ^λ the character of the irreducible representation S^λ of $\mathfrak{S}_n$, and, when $\lambda \in \mathcal{SI}_n$, we denote by χ^λ_+ and χ^λ_- the characters of S^λ_+ and S^λ_-, respectively. By Remark 3.3 we know that the values of the characters of $\mathfrak{A}_n$ are completely determined by those of $\mathfrak{S}_n$ when the corresponding partition either is in $\mathcal{N}_n$ or is in $\mathcal{SI}_n$ but the conjugacy class at which the character is evaluated is non-split. In order to determine the values of $\chi^\lambda_\pm$ on pairs of split conjugacy classes we need the following deep result of Frobenius. For its formulation, since any character χ takes a constant value on each conjugacy class $\mathcal{C}$, we denote this value by $\chi(\mathcal{C})$. For its proof, we combine the presentations in [7, Theorem 2.5.13] and [6, Exercise 5.4].

Theorem 4.5 (Frobenius)**.** *Let $\lambda \in \mathcal{SI}_n$ and denote by χ^λ (resp. χ^λ_+ and χ^λ_-) the corresponding characters of $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$). Let also $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathrm{Split}_n$ and denote by $\mathcal{C}_\mu$ (resp. $\mathcal{C}^+_\mu$ and $\mathcal{C}^-_\mu$) the corresponding conjugacy classes in $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$). Then*

(i) *if μ does not correspond to λ under the map* (4.1) *then*

$$\chi^\lambda_+(\mathcal{C}^+_\mu) = \chi^\lambda_+(\mathcal{C}^-_\mu) = \chi^\lambda_-(\mathcal{C}^+_\mu) = \chi^\lambda_-(\mathcal{C}^-_\mu) = \frac{1}{2}\chi^\lambda(\mathcal{C}_\mu);$$

(ii) *if μ corresponds to λ then*

$$\chi^\lambda_+(\mathcal{C}^+_\mu) = \chi^\lambda_-(\mathcal{C}^-_\mu) = \frac{1}{2}\left[(-1)^m + \sqrt{(-1)^m \mu_1\mu_2\cdots\mu_k}\right] \tag{4.2}$$

and

$$\chi^\lambda_+(\mathcal{C}^-_\mu) = \chi^\lambda_-(\mathcal{C}^+_\mu) = \frac{1}{2}\left[(-1)^m - \sqrt{(-1)^m \mu_1\mu_2\cdots\mu_k}\right] \tag{4.3}$$

where $m = m(\mu) = \sum_{j=1}^k (\mu_j - 1)/2 \equiv (n-k)/2$.

Proof. In the following, we denote by $\nu = (\nu_1, \nu_2, \dots, \nu_h) \in \mathrm{Split}_n$ a general partition of n whose associated conjugacy class is split, and by $\mu = \mu(\lambda)$ the partition corresponding to λ. We shall first prove (ii) under the assumption of (i), and then we prove (i). We divide the proof into several steps.

Step 0. Under our assumptions, from the Murnaghan-Nakayama rule (Theorem 2.1) one immediately deduces the following special case:

$$\chi^\lambda(\mathcal{C}_\mu) = (-1)^{(n-k)/2}. \tag{4.4}$$

Step 1. From Clifford theory (see Remark 3.3), we know that $\chi^\lambda_+(\mathcal{C}^+_\nu) = \chi^\lambda_-(\mathcal{C}^-_\nu)$ (resp. $\chi^\lambda_+(\mathcal{C}^-_\nu) = \chi^\lambda_-(\mathcal{C}^+_\nu)$) and that, denoting by x_ν (resp. y_ν) this value, one has $x_\nu + y_\nu = \chi^\lambda(\mathcal{C}_\nu)$.

Step 2. Let $\pi = (a_1, a_2, \dots, a_{\nu_1})(b_1, b_2, \dots, b_{\nu_2}) \cdots (c_1, c_2, \dots, c_{\nu_k})$ denote the cyclic decomposition of $\pi \in \mathcal{C}_\nu$ and set

$$\xi = ((a_2, a_{\nu_1})(a_3, a_{\nu_1-1}) \cdots \ (a_{(\nu_1+1)/2}, a_{(\nu_1+3)/2})) \left((b_2, b_{\nu_2})(b_3, b_{\nu_2-1}) \cdots (b_{(\nu_2+1)/2}, b_{(\nu_2+3)/2})\right) \cdots \left((c_2, c_{\nu_k})(c_3, c_{\nu_k-1}) \cdots (c_{(\nu_k+1)/2}, c_{(\nu_k+3)/2})\right).$$

Then $\xi^{-1}\pi\xi = \pi^{-1}$. Moreover, $\xi \in \mathfrak{A}_n$ if and only if $m = m(\nu)$ is even. Therefore, π is $\mathfrak{A}_n$-conjugate to π^{-1} if and only if m is even (cf. the end of the proof of Lemma 4.1).

Step 3. Suppose m is even. Fix $\pi \in \mathcal{C}^+_\nu$; then $\pi^{-1} \in \mathcal{C}^+_\nu$ and therefore $x_\nu = \chi^\lambda_+(\pi) = \overline{\chi^\lambda_+(\pi^{-1})} = \overline{x_\nu}$. Similarly, by taking $\pi \in \mathcal{C}^-_\nu$, one deduces $y_\nu = \overline{y_\nu}$. It follows that $x_\nu, y_\nu \in \mathbb{R}$.

Suppose now that m is odd. Fix $\pi \in \mathcal{C}^+_\nu$; then $\pi^{-1} \in \mathcal{C}^-_\nu$ and therefore $x_\nu = \chi^\lambda_+(\pi) = \overline{\chi^\lambda_+(\pi^{-1})} = \overline{y_\nu}$.

Step 4. Assume, for the moment that $x_\nu = y_\nu$ if (and only if) $\nu \neq \mu$. Let us set $\theta = \chi^\lambda_+ - \chi^\lambda_-$. Then $\langle \theta, \theta \rangle = 2$ (recall that characters corresponding to inequivalent irreducible representations are orthogonal). We thus have

$$\begin{aligned}
\langle \theta, \theta \rangle &= \frac{1}{|\mathfrak{A}_n|} \sum_{g \in \mathfrak{A}_n} \theta(g)\overline{\theta(g)} \\
&= \frac{2}{n!} \left(\sum_{g \in \mathcal{C}_\mu} \theta(g)\overline{\theta(g)} + \sum_{\substack{\nu \vdash n \\ \nu \neq \mu}} \sum_{g \in \mathcal{C}_\nu} \theta(g)\overline{\theta(g)} \right) \\
\text{(by our assumption)} \quad &= \frac{2}{n!} \sum_{g \in \mathcal{C}_\mu} \theta(g)\overline{\theta(g)} \\
&= \frac{2}{n!} \left(\sum_{g \in \mathcal{C}^+_\mu} \theta(g)\overline{\theta(g)} + \sum_{g \in \mathcal{C}^-_\mu} \theta(g)\overline{\theta(g)} \right) \\
&= \frac{4}{n!} \sum_{g \in \mathcal{C}^+_\mu} |x_\mu - y_\mu|^2 \\
&= \frac{4|\mathcal{C}^+_\mu|}{n!} |x_\mu - y_\mu|^2.
\end{aligned}$$

Since $|\mathcal{C}_\mu^+| = \frac{n!}{2\mu_1\mu_2\cdots\mu_k}$, we deduce that $|x_\mu - y_\mu|^2 = \mu_1\mu_2\cdots\mu_k$. Collecting together our information so far, we get the systems:

$$\begin{cases} x_\mu + y_\mu = 1 & \text{by virtue of (4.4)} \\ x_\mu, y_\mu \in \mathbb{R} & \text{by Step. 3} \\ |x_\mu - y_\mu|^2 = \mu_1\mu_2\cdots\mu_k \end{cases}$$

if m is even, and

$$\begin{cases} x_\mu + y_\mu = -1 & \text{by virtue of (4.4)} \\ x_\mu = \overline{y_\mu} & \text{by Step. 3} \\ |x_\mu - y_\mu|^2 = \mu_1\mu_2\cdots\mu_k \end{cases}$$

if m is odd. Then (ii) follows. Note that we have made an arbitrary choice for the split classes $\mathcal{C}_\mu^+$ and $\mathcal{C}_\mu^-$ so that, in fact, the values of x_μ and y_μ are interchangeable.

Step 5. We now show that if $\epsilon \in \mathrm{Split}_n$ and $x_\nu = y_\nu$ for all $\nu \in \mathrm{Split}_n$ such that $\nu \neq \epsilon$, then necessarily $\epsilon = \mu$. By contradiction, if $\epsilon \neq \mu$ we would have $x_\mu = y_\mu$. Since $x_\mu + y_\mu = (-1)^m$ (by virtue of (4.4)), we would have $x_\mu = (-1)^m/2$. But this is impossible since characters take only algebraic integer values (this holds for any finite group G: indeed if $\sigma \in \widehat{G}$ and $g \in G$, one has $\chi^\sigma(g)$ is the sum of the eigenvalues of $\sigma(g)$ which are $|G|$-roots of the unity).

Step 6. Given two partitions $\eta = (\eta_1, \eta_2, \ldots, \eta_\ell), \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_j) \vdash n$ we write $\eta \trianglelefteq \gamma$ if $j \leq \ell$ and $\sum_{p=1}^i \eta_p \leq \sum_{p=1}^i \gamma_p$ for all $i = 1, 2, \ldots, j$. It is well known (see, for instance, [3, Section 3.6.1]) that $\trianglelefteq$ is a partial order on the set of all partitions of n. Now, given $\eta \in \mathcal{S}I_n$, from the Murnaghan-Nakayama rule (Theorem 2.1) it follows that if $\chi^\eta(C_\nu) \neq 0$, one necessarily has $\nu \trianglelefteq (h_{1,1}^\eta, h_{2,2}^\eta, \ldots, h_{k,k}^\eta)$. Indeed, a rim hook tableau of shape η and content ν should have all the 1's in the hook $H_{1,1}^\eta$ all the 1's and the 2's in $H_{1,1}^\eta \coprod H_{2,2}^\eta$ and so on.

Step 7 With θ as in Step 4, we set $\phi = \theta\overline{\theta}$. Observe that ϕ is a so-called *generalized character* of $\mathfrak{A}_n$, i.e. a $\mathbb{Z}$-linear combination of characters of irreducible representations (this follows from the fact that the conjugate of a character is the character of the conjugate representation and the product of two characters is the character of the tensor product of the corresponding representations). By Step 1, $\phi(\mathcal{C}_\nu^+) = \phi(\mathcal{C}_\nu^-)$ if $\nu \in \mathrm{Split}_n$ and ϕ vanishes on all other $\mathfrak{A}_n$-conjugacy classes. We now define $\psi\colon \mathfrak{S}_n \to [0, +\infty)$ by setting

$$\psi(g) = \begin{cases} \phi(g) & \text{if } g \in \mathfrak{A}_n \\ 0 & \text{if } g \in \mathfrak{S}_n \setminus \mathfrak{A}_n \end{cases} \tag{4.5}$$

for all $g \in \mathfrak{S}_n$.

Let $\eta \in \mathcal{S}I_n$. Then, by Step 4 we have that $\langle \phi, \chi_+^\eta \rangle =_{(*)} \langle \phi, \chi_-^\eta \rangle$, moreover, since $\chi^\eta|_{\mathfrak{A}_n} =_{(**)} \chi_+^\eta + \chi_-^\eta$, we have

$$\begin{aligned} \langle \psi, \chi^\eta \rangle &= \frac{1}{|\mathfrak{S}_n|} \sum_{g \in \mathfrak{S}_n} \psi(g)\chi^\eta(g) \\ &= \frac{1}{2|\mathfrak{A}_n|} \sum_{g \in \mathfrak{A}_n} \phi(g)\chi^\eta(g) \\ (\text{by } =_{(**)}) \quad &= \frac{1}{2|\mathfrak{A}_n|} \sum_{g \in \mathfrak{A}_n} \phi(g)[\chi_+^\eta(g) + \chi_-^\eta(g)] \\ &= \frac{1}{2}[\langle \phi, \chi_+^\eta \rangle + \langle \phi, \chi_-^\eta \rangle] \\ (\text{by } =_{(*)}) \quad &= \langle \phi, \chi_+^\eta \rangle. \end{aligned}$$

Recalling that ϕ is a generalized character, we deduce that

$$\langle \psi, \chi^\eta \rangle \in \mathbb{Z}. \tag{4.6}$$

The class function ψ vanishes on all nonsplit conjugacy classes, but it is not identically zero (since $\chi_+^\lambda \neq \chi_-^\lambda$). Therefore, we can select a partition $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \mathrm{Split}_n$ such that $\psi(\mathcal{C}_\epsilon) \neq 0$ and it is minimal with respect to the partial order $\trianglelefteq$. Denote by $\eta = (\eta_1, \eta_2, \ldots, \eta_\ell)$ the selfconjugate partition corresponding to ϵ. Then (arguing as in Step 4)

$$\begin{aligned} \langle \psi, \chi^\eta \rangle &= \frac{1}{n!} \sum_{\mu \vdash n} |\mathcal{C}_\mu| \psi(\mathcal{C}_\mu)\chi^\eta(\mathcal{C}_\mu) \\ &= \frac{1}{n!} \sum_{\mu \in \mathrm{Split}_n} |\mathcal{C}_\mu| \psi(C_\mu)\chi^\eta(C_\mu) \\ (\text{by minimality of } \varepsilon \text{ and by Step 6}) \quad &= \frac{1}{n!} |\mathcal{C}_\epsilon| \psi(\mathcal{C}_\epsilon)\chi^\eta(\mathcal{C}_\epsilon) \\ (\text{by the Murnaghan-Nakayama rule}) \quad &= \frac{1}{n!} |\mathcal{C}_\epsilon| \psi(\mathcal{C}_\epsilon)(-1)^{n-k}. \end{aligned}$$

From (4.6) we then deduce that

$$\frac{1}{n!} |\mathcal{C}_\epsilon| \psi(\mathcal{C}_\epsilon) \in \{1, 2, \ldots\}. \tag{4.7}$$

On the other hand, reasoning again as in Step 4, we have

$$\begin{aligned} 1 &= \frac{1}{2}\langle \theta, \theta \rangle \\ &= \frac{1}{n!} \sum_{\substack{\nu \vdash n \\ \mathcal{C}_\nu \subseteq \mathfrak{A}_n}} |\mathcal{C}_\nu| \phi(\mathcal{C}_\nu) \\ (\text{by } (4.5)) \quad &= \frac{1}{n!} \sum_{\nu \vdash n} |\mathcal{C}_\nu| \psi(\mathcal{C}_\nu). \end{aligned} \tag{4.8}$$

Finally, since ψ takes only nonnegative values (ψ can be expressed as the squared absolute value of a function), from (4.7) and (4.8), we deduce that

$$\frac{1}{n!} |C_\mu| \psi(C_\mu) = \delta_{\mu, \epsilon}$$

This shows that the condition in Step 4 (taking into account Step 5) is not restrictive and therefore the proof is complete. □

Corollary 4.6. *Let $\lambda \vdash n$.*

(i) *If $\lambda \in \mathcal{N}_n$, then the corresponding irreducible representation $\tilde{S}^\lambda$ of $\mathfrak{A}_n$ is real;*
(ii) *if $\lambda \in \mathcal{S}I_n$ and $m = m(\mu(\lambda))$ is even, then $S^\lambda_\pm$ are real;*
(iii) *f $\lambda \in \mathcal{S}I_n$ and $m = m(\mu(\lambda))$ is odd, then $S^\lambda_\pm$ are complex.*

Proof. We know that the representations of $\mathfrak{S}_n$ are real (see [3, Corollary 3.4.3]). In [2, Theorem 3.3] it is shown that every irreducible representation of $\mathfrak{A}_n$ is either real or complex, and then Frobenius theorem (Theorem 4.5) exactly determines when these two possibilities occur. □

Example 4.7. The restriction of the $\mathfrak{S}_4$ representation $S^{2,2}$ to $\mathfrak{A}_4$ splits into the two irreducible representations $S^{2,2}_+$ and $S^{2,2}_-$. These are both complex since $\lambda = (2,2)$ corresponds to $\mu = (3,1)$ and therefore $m = \frac{1}{2}(2+0) = 1$ is odd.

4.3. **Ambivalence of the groups $\mathfrak{A}_n$.**

Theorem 4.8. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathrm{Split}_n$. As above, we denote by $\mathcal{C}^+_\mu$, $\mathcal{C}^-_\mu$ the corresponding split classes in $\mathfrak{A}_n$ and set $m = \frac{1}{2}\sum_{j=1}^k (\mu_j - 1) \equiv \frac{1}{2}(n-k)$. If m is even, then $\mathcal{C}^+_\mu$ and $\mathcal{C}^-_\mu$ are both ambivalent. If m is odd then $\mathcal{C}^+_\mu = \{\pi^{-1} : \pi \in \mathcal{C}^-_\mu\}$. In particular, $\mathcal{C}^+_\mu$ and $\mathcal{C}^-_\mu$ are both non-ambivalent.*

Proof. Let $\pi \in \mathcal{C}^-_\mu$ and $\xi \in \mathfrak{S}_n$ as in Step 2 of the proof of Theorem 4.5. Then $\xi \in \mathfrak{A}_n$ if and only if m is even. Therefore, if m is even then $\mathcal{C}^-_\mu$ (and similarly $\mathcal{C}^+_\mu$) is ambivalent. On the other hand, if m is odd, $\mathcal{C}^-_\mu$ cannot be ambivalent: if $\sigma\pi\sigma^{-1} = \pi^{-1}$ with $\sigma \in \mathfrak{A}_n$, then $(\xi\sigma)^{-1}\pi(\xi\sigma) = \pi$ and $\xi\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$; but this is impossible since $\mathcal{C}_\mu$ is split (cf. the beginning of the proof of Lemma 4.1). Finally, since $\pi^{-1} \in \mathcal{C}^+_\mu$, the last two assertions for $\mathcal{C}^+_\mu$ follow as well. □

Corollary 4.9. *The group $\mathfrak{A}_n$ is ambivalent if and only if $n = 1, 2, 5, 6, 10, 14$.*

Proof. For $n \in \{1, 2, 5, 6, 10, 14\}$, all conjugacy classes are non-split (and therefore ambivalent) except for the following ones:

- $(5) \vdash 5 \quad (m = 2)$
- $(5,1) \vdash 6 \quad (m = 2)$
- $(9,1), (7,3) \vdash 10 \quad (m = 4)$
- $(13,1), (11,3), (9,5) \vdash 14 \quad (m = 6)$.

Since m is even, from Theorem 4.8 it follows that these split classes are also ambivalent.

Conversely, for the other values of n, we consider the following split classes:

- $(4k-1, 1) \vdash n = 4k \quad (m = 2k-1), k \geq 1$
- $(4k-3, 3, 1) \vdash n = 4k+1 \quad (m = 2k-1), k \geq 2$
- $(4(k-1)-3, 5, 3, 1) \vdash n = 4k+2 \quad (m = 2k-1), k \geq 4$
- $(4k+3) \vdash n = 4k+3 \quad (m = 2k+1), k \geq 0$.

Since now m is odd, we deduce, again from Theorem 4.8, that these split classes are non-ambivalent. □

References

[1] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, *Harmonic analysis on finite groups: representation theory, Gelfand pairs and Markov chains.* Cambridge Studies in Advanced Mathematics **108**, Cambridge University Press, Cambridge, 2008.

[2] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, Clifford theory and applications. Functional analysis, *J. Math. Sci. (N. Y.)*, **156** no. 1 (2009) 29–43.

[3] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, *Representation theory of the symmetric groups: the Okounkov-Vershik approach, character formulas, and partition algebras.* Cambridge Studies in Advanced Mathematics, **121**, Cambridge University Press, Cambridge, 2010.

[4] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, *Representation Theory and Harmonic Analysis of wreath products of finite groups.* London Mathematical Society Lecture Note Series, **410**, Cambridge University Press, Cambridge, 2013.

[5] Ch. W. Curtis and I. Reiner, *Methods of Representation Theory. With Applications to Finite Groups and Orders*, **1**, Pure Appl. Math., John Wiley & Sons, New York, 1981.

[6] W. Fulton and J. Harris, *Representation theory. A first course. Graduate Texts in Mathematics*, **129**, Springer-Verlag, New York, 1991.

[7] G. D. James, and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, **16**, Addison-Wesley, Reading, MA, 1981.

Tullio Ceccherini-Silberstein

Dipartimento di Ingegneria, Università del Sannio, C.so Garibaldi 107, 82100 Benevento, Italy

Email: `tceccher@mat.uniroma3.it`

Fabio Scarabotti

Dipartimento SBAI, Sapienza Università di Roma, via A. Scarpa 8, 00161 Roma, Italy

Email: `fabio.scarabotti@sbai.uniroma1.it`

Filippo Tolli

Dipartimento di Matematica e Fisica, Università Roma TRE, L. San Leonardo Murialdo 1, 00146 Roma, Italy

Email: `tolli@mat.uniroma3.it`