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CARTESIAN SYMMETRY CLASSES ASSOCIATED WITH CERTAIN SUBGROUPS OF S_m

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Abstract. In this paper, the problem existing O-basis for Cartesian symmetry classes is discussed. The dimensions of Cartesian symmetry classes associated with a cyclic subgroup of the symmetric group S_m (generated by a product of disjoint cycles) and the product of cyclic subgroups of S_m are explicitly expressed in terms of the Ramanajun sum. Additionally, a necessary and sufficient condition for the existence of an O-basis for Cartesian symmetry classes associated with the irreducible characters of dihedral group is given. The dimensions of these classes are also computed.

1. **Introduction and Preliminaries**

The concept of Cartesian symmetry classes was first introduced by Tian-Gang Lei in [5]. In [9], the authors computed the dimension of Cartesian symmetry class $V^{\chi}(G)$ in terms of the fixed point character of S_m. They also obtained a formula for the dimension of $V^{\chi}(G)$ in terms of the rank of an idempotent matrix $M(\chi)$. Furthermore, they concluded some properties of generalized trace functions of matrices.

In this work, the problem existing O-basis for Cartesian symmetry classes is discussed. The dimensions of Cartesian symmetry classes associated with a cyclic subgroup of the symmetric group S_m (generated by a product of disjoint cycles) and the product of cyclic subgroups of S_m are explicitly expressed in terms of the Ramanajun sum. Additionally, a necessary and sufficient condition for the

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existence of an O-basis for Cartesian symmetry classes associated with the irreducible characters of dihedral group is given. The dimensions of these classes are also computed.

We commence with a review of Cartesian symmetry classes. For more details, we refer the reader to [5, 9].

Let V be a complex inner product space of dimension $n \geq 2$. Consider $\mathbb{E} = \{f_1, \ldots, f_n\}$ as an orthonormal basis of V. Let $\times^m V$ be the Cartesian product of m-copies of V. We have an induced inner product of $\times^m V$, which is defined by

$$< z^\times, w^\times >= \sum_{i=1}^{m} < z_i, w_i >,$$

where

$$z^\times = (z_1, \ldots, z_m),\ w^\times = (w_1, \ldots, w_m).$$

For every $1 \leq i \leq n,\ 1 \leq j \leq m$, we define

$$f_{ij} = (\delta_{1j} f_i, \delta_{2j} f_i, \ldots, \delta_{mj} f_i) \in \times^m V.$$

Then

$$\mathbb{E}^\times = \{f_{ij} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

is an orthonormal basis of $\times^m V$.

For each $\sigma \in S_m$, let

$$Fix(\sigma) = \{i \mid 1 \leq i \leq m,\ \sigma(i) = i\}$$

be the set of fixed points of σ and $l(\sigma) = m - |Fix(\sigma)|$ be the length of σ. In fact $l(\sigma)$ is the number of moved point of σ.

Let G be a subgroup of the symmetric group S_m. For any $\tau \in G$, we define the Cartesian permutation linear operator

$$Q(\tau) : \times^m V \longrightarrow \times^m V$$

by

$$Q(\tau)(u_1, \ldots, u_m) = (u_{\tau^{-1}(1)}, \ldots, u_{\tau^{-1}(m)}).$$

It is easy to see that $Q : \tau \mapsto Q(\tau)$ defines a faithful unitary representation of G over $\times^m V$.

Let λ be a complex irreducible character of G. Define the *Cartesian symmetrizer* C_λ as follows:

$$C_\lambda = \frac{\lambda(1)}{|G|} \sum_{\tau \in G} \lambda(\tau) Q(\tau).$$

Clearly $C_\lambda^* = C_\lambda$. As the symmetry classes of tensors [6, page 154], by using the orthogonality relations of characters, it can be easily verified that for any two irreducible characters λ and μ of G, we have

$$C_\lambda C_\mu = \delta_{\lambda,\mu} C_\lambda.$$

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Moreover,

$$\sum_{\lambda\in \mathrm{Irr}(G)} C_\lambda = I_{\times^m V},$$

the identity operator on Cartesian space $\times^m V$.

The range of the linear operator C_λ is called *the Cartesian symmetry class associated with G and λ*, denoted by $V^\lambda(G)$. Then

$$\times^m V = \sum_{\lambda\in \mathrm{Irr}(G)}^{\perp} V^\lambda(G) \text{ (orthogonal direct sum)}.$$

The elements

$$u^\lambda = C_\lambda(u_1,\ldots,u_m)$$

of $V^\lambda(G)$ are called *λ-symmetrized vectors*. Note that $V^\lambda(G)$ is spanned by the *standard λ-symmetrized vectors*

$$f_{ij}^\lambda = C_\lambda(f_{ij}) \ (i=1,2,\ldots,n, j=1,2,\ldots,m).$$

Theorem 1.1. *Let G be a subgroup of the symmetric group S_m and $\lambda \in Irr(G)$. Then for each $1 \le i, r \le n, 1 \le j, s \le m$, we have*

$$\langle f_{ij}^\lambda, f_{rs}^\lambda\rangle = \begin{cases} \frac{\lambda(1)}{|G|}\sum_{\sigma\in G_j}\lambda(\sigma\tau) & \text{if } i=r \text{ and } s=\tau(j) \text{ for some } \tau\in G \\ 0 & \text{otherwise} \end{cases}.$$

In particular

$$\| f_{ij}^\lambda \|^2 = \frac{\lambda(1)[\lambda, 1_{G_j}]}{[G:G_j]}, \tag{1.1}$$

where G_j is the stabilizer of j in G and $[\ ,\ ]$ the inner product of characters.

Proof. Since C_λ is an orthogonal projection on Cartesian space $\times^m V$, so we have

$$\begin{aligned} \langle f_{ij}^\lambda, f_{rs}^\lambda\rangle &= \langle C_\lambda(f_{ij}), C_\lambda(f_{rs})\rangle \\ &= \langle C_\lambda(f_{ij}), f_{rs}\rangle \\ &= \frac{\lambda(1)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)\langle Q(\sigma)f_{ij}, f_{rs}\rangle \\ &= \frac{\lambda(1)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)\langle f_{i\sigma(j)}, f_{rs}\rangle \\ &= \delta_{ir}\frac{\lambda(1)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)\delta_{\sigma(j),s}. \end{aligned}$$

If $i \ne r$ or if j and s are not in the same orbit, then $\langle f_{ij}^\lambda, f_{rs}^\lambda\rangle = 0$. Suppose $i = r$ and $s = \tau(j)$ for some $\tau \in G$. Then $\delta_{\sigma(j),s} = 1$ if and only if $\tau^{-1}\sigma \in G_j$. This implies that

$$\langle f_{ij}^\lambda, f_{rs}^\lambda\rangle = \frac{\lambda(1)}{|G|}\sum_{\sigma\in G_j}\lambda(\sigma\tau),$$

and the assertion holds. □

Let $\mathfrak{D}$ be a set of representatives of orbits of the action of G on the set $\mathbf{I}_m = \{1, 2, \ldots, m\}$ and $\mathcal{O} = \{j \mid 1 \leq j \leq m,\ [\lambda, 1_{G_j}] \neq 0\}$. We put $\bar{\mathfrak{D}} = \mathfrak{D} \cap \mathcal{O}$. It is easy to see that the set

$$\{f^{\lambda}_{ij} \ \mid\ 1 \leq i \leq n, j \in \bar{D}\}$$

is an orthogonal set of non-zero vectors in $V^{\lambda}(G)$. Notice that the above set may not be a basis for $V^{\lambda}(G)$.

For any $1 \leq i \leq n$ and $j \in \bar{\mathfrak{D}}$, define the cyclic subspace

$$V^{\lambda}_{ij} = \langle f^{\lambda}_{i\sigma(j)} \mid \sigma \in G\rangle.$$

Clearly

$$V^{\lambda}(G) = \sum^{\perp}_{1\leq i\leq n,\ j\in\bar{\mathfrak{D}}} V^{\lambda}_{ij}, \tag{1.2}$$

the orthogonal direct sum of the cyclic subspaces V^{λ}_{ij} $(1 \leq i \leq n,\ j \in \bar{\mathfrak{D}})$. If λ is a linear character of G and $j \in \bar{\mathfrak{D}}$, then it is easy to see that $f^{\lambda}_{i\sigma(j)} = \lambda(\sigma^{-1})f^{\lambda}_{ij}$, so $\dim V^{\lambda}_{ij} = 1$ and the set

$$\{f^{\lambda}_{ij} \mid 1 \leq i \leq n, j \in \bar{\mathfrak{D}}\}$$

is an orthogonal basis of $V^{\lambda}(G)$. Therefore, we have $\dim V^{\lambda}(G) = (\dim V)|\bar{\mathfrak{D}}|$. For example we have

$$\dim V^{1}(S_m) = \dim V, \quad \dim V^{\varepsilon}(S_m) = \begin{cases} \dim V & \text{if } m = 2 \\ 0 & \text{if } m \geq 3 \end{cases},$$

where 1 and ε are the principal and the alternating character of S_m, respectively.

Suppose λ is a non-linear irreducible character of G. We will now proceed to construct a basis for $V^{\lambda}(G)$. For each $j \in \bar{\mathfrak{D}}$, we choose the set $\{j_1, \ldots, j_{s_j}\}$ from the orbit of j such that $\{f^{\lambda}_{ij_1}, \ldots, f^{\lambda}_{ij_{s_j}}\}$ forms a basis for the cyclic subspace V^{λ}_{ij}. This procedure is executed for each $k \in \bar{\mathfrak{D}}$. If $\bar{\mathfrak{D}} = \{j, k, l, \ldots\}$ with $j < k < l < \cdots$, then we define

$$\hat{\mathfrak{D}} = \{j_1, \ldots, j_{s_j}; k_1, \ldots, k_{s_k}; \ldots\}$$

with an indicated order. Consequently,

$$\mathbb{E}^{\lambda} = \{f^{\lambda}_{ij} \mid 1 \leq i \leq n,\ j \in \hat{\mathfrak{D}}\}$$

forms a basis for $V^{\lambda}(G)$. Note that this basis may not be orthogonal. Therefore,

$$\dim V^{\lambda}(G) = (\dim V)|\hat{\mathfrak{D}}|.$$

If the subspace W of $V^{\lambda}(G)$ possesses a basis consisting of orthogonal standard λ-symmetrized vectors, we will refer to W as having an *O-basis*.

In section 2 we give a formula for the dimension of the cyclic subspace V_{ij}^{λ}. Then we discuss the problem existing O-basis for $V^{\lambda}(G)$. In section 3, we determine the dimensions of Cartesian symmetry classes associated with a cyclic subgroup of the symmetric group S_m (generated by a product of disjoint cycles) and the product of cyclic subgroups of S_m. These dimensions are expressed in terms of the Ramanujan sum. In Section 4, we establish a necessary and sufficient condition for the existence of an O-basis for Cartesian symmetry classes associated with the irreducible characters of the dihedral group. Additionally, we compute the dimensions of these classes.

2. Orthogonal bases of Cartesian symmetry classes

In this section we discuss the problem existing O-basis for $V^{\lambda}(G)$ consisting of standard λ-symmetrized vectors. We first prove the following theorem.

Theorem 2.1. *For every $1 \leq i \leq n$ and $j \in \bar{\mathfrak{D}}$, we have*

$$\dim V_{ij}^{\lambda} = \lambda(1)[\lambda, 1_{G_j}].$$

Proof. Let $[G : G_j] = k$, $G = \bigcup_{i=1}^{k} \tau_i G_j$ be the left coset decomposition of G_j in G. Hence

$$Orb\ (j) = \{\tau_1(j), \ldots, \tau_k(j)\}.$$

We put

$$V_{ij} = \langle f_{i\sigma(j)} \mid \sigma \in G \rangle.$$

Then

$$\mathbb{E}_{ij} = \{f_{i\tau_1(j)}, \ldots, f_{i\tau_k(j)}\}$$

is a basis for V_{ij} but the set

$$\{f_{i\tau_1(j)}^{\lambda}, \ldots, f_{i\tau_k(j)}^{\lambda}\}$$

may not be a basis for V_{ij}^{λ}. Notice that $V_{ij}^{\lambda} = C_{\lambda}(V_{ij})$. Since V_{ij} is invariant under C_{λ}, so we put

$$C_{ij}(G, \lambda) = C_{\lambda} \mid_{V_{ij}} .$$

This restriction is a linear operator on V_{ij}. Since C_{λ} is an orthogonal projection on $\times^m V$, so $C_{ij}(G, \lambda)$ is also an orthogonal projection on V_{ij}.

Let $M = (m_{ij}) = [C_{ij}(G, \lambda)]_{\mathbb{E}_{ij}}$. Then

$$\dim V_{ij}^{\lambda} = rank\ C_{ij}(G, \lambda) = tr\ C_{ij}(G, \lambda) = tr\ M.$$

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Now we compute the entries of the matrix M. We have

$$\begin{aligned}
C_{ij}(G,\lambda)f_{i\tau_q(j)} =& C_\lambda(f_{i\tau_q(j)})\\
=&\frac{\lambda(1)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)Q(\sigma)(f_{i\tau_q(j)})\\
=&\frac{\lambda(1)}{|G|}\sum_{\sigma\in G}\lambda(\sigma)f_{i\sigma\tau_q(j)}\\
=&\frac{\lambda(1)}{|G|}\sum_{\pi\in G}\lambda(\pi\tau_q^{-1})f_{i\pi(j)} \qquad (\sigma\tau_q=\pi)\\
=&\frac{\lambda(1)}{|G|}\sum_{p=1}^{k}\sum_{\pi\in G_j}\lambda(\tau_p\pi\tau_q^{-1})f_{i\tau_p\pi(j)}\\
=&\frac{\lambda(1)}{|G|}\sum_{p=1}^{k}\sum_{\pi\in G_j}\lambda(\tau_p\pi\tau_q^{-1})f_{i\tau_p(j)}\\
=&\sum_{p=1}^{k}\left(\frac{\lambda(1)}{|G|}\sum_{\pi\in G_j}\lambda(\tau_p\pi\tau_q^{-1})\right)f_{i\tau_p(j)}.
\end{aligned}$$

Thus

$$m_{pq}=\frac{\lambda(1)}{|G|}\sum_{\pi\in G_j}\lambda(\tau_p\pi\tau_q^{-1}).$$

Hence

$$\begin{aligned}
\dim V_{ij}^{\lambda}=trM=\sum_{p=1}^{k}m_{pp}=\sum_{p=1}^{k}\frac{\lambda(1)}{|G|}\sum_{\pi\in G_j}\lambda(\tau_p\pi\tau_p^{-1})=\sum_{p=1}^{k}\frac{\lambda(1)}{|G|}\sum_{\pi\in G_j}\lambda(\pi)\\
=\frac{k}{|G|}\lambda(1)\sum_{\pi\in G_j}\lambda(\pi)=\lambda(1)\frac{1}{|G_j|}\sum_{\pi\in G_j}\lambda(\pi)=\lambda(1)[\lambda,1_{G_j}],
\end{aligned}$$

so the result follows. □

By using Theorem 2.1 and Equation (1.2), we have the following corollary.

Corollary 2.2. *Let G be a subgroup of S_m. Then for any $\lambda\in Irr(G)$, we have*

$$\dim V^{\lambda}(G)=(\dim V)\lambda(1)\sum_{j\in\bar{\mathfrak{D}}}[\lambda,1_{G_j}].$$

Similar to the proof of Theorem 1 in [8], we can prove the following important theorem.

Theorem 2.3. *Let λ be a non-linear irreducible character of G and suppose there exist $1\le i\le n$ and $j\in\bar{\mathfrak{D}}$ such that*

$$\frac{\sqrt{2}}{2}<\parallel f_{ij}^{\lambda}\parallel<1.$$

Then $V^{\lambda}(G)$ has no O-basis.

We will show that $\frac{\sqrt{2}}{2}$ is the best lower bound for $\parallel f^{\lambda}_{ij} \parallel$ in Theorem 2.3. Also, we will give a counterexample that the converse of this theorem is not true.

Recall that G is 2-transitive if $m \geq 2$ and G_i is transitive on $\mathbf{I}_m \setminus \{i\}$ for all $1 \leq i \leq m$. By [3, Corollary 5.17], G is 2-transitive if and only if $\lambda = Fix - 1 \in \mathrm{Irr}(G)$. Now we have the following corollary.

Corollary 2.4. *Let $G \leq S_m$ $(m \geq 3)$ be 2-transitive and $\lambda = Fix - 1$. Then $V^{\lambda}(G)$ has no O-basis. In particular, if $G = S_m$ $(m \geq 3)$ or $G = A_m$ $(m \geq 4)$, then $V^{\lambda}(G)$ does not have an O-basis.*

Proof. Clearly $G_m = G \cap S_{m-1}$, so $[\lambda, 1_{G_m}] = 1$ by Burnside's counting theorem (see [4, Proposition 29.4]). Also $[G : G_m] = [S_m : S_{m-1}] = m$. By Equation (1.1), for all $1 \leq i \leq n$ we have

$$\parallel f^{\lambda}_{im} \parallel^2 = \frac{m-1}{m},$$

so the result holds by the previous theorem. □

We will say that $\times^m V$ has O-basis if for each irreducible character λ of G, the Cartesian symmetry class $V^{\lambda}(G)$ has an O-basis. We know that $\times^m V = \sum^{\perp}_{\lambda \in \mathrm{Irr}(G)} V^{\lambda}(G)$ (orthogonal direct sum), so the following result follows immediately from the above corollary.

Corollary 2.5. *Assume $m \geq 3$. If $G \leq S_m$ is 2-transitive, then $\times^m V$ does not have an O-basis.*

3. Cyclic Cartesian symmetry classes

The well-known Ramanujan sum is given by

$$c_m(q) = \sum_{\substack{s=0, \\ (s,m)=1}}^{m-1} \exp(\frac{2\pi i q s}{m}),$$

where m is a positive integer, and q is a non-negative integer (see [1]).

Theorem 3.1. *Let G be the cyclic subgroup of S_m, generated by the m-cycle $\sigma = (1\ 2\ \cdots\ m)$. Then for any $\lambda \in Irr(G)$, we have*

$$\dim V^{\lambda}(G) = \dim V.$$

Proof. By [3, Corollary 2.6], a finite group G is abelian if and only if every irreducible character of it is linear. Let $\lambda \in \mathrm{Irr}(G)$. Then $\lambda(1) = 1$. It is easy to see that the action of G on $\mathbf{I}_m$ is transitive, so $|\mathfrak{D}| = 1$. Assume that $\mathfrak{D} = \{1\}$. Clearly $G_1 = \{1\}$ and $[\lambda, 1_{G_1}] = 1$. Therefore $\dim V^{\lambda}(G) = \dim V$, by Corollary 2.2. □

Now, let's consider the cyclic group $G = \langle \sigma_1 \sigma_2 \cdots \sigma_p \rangle$, where σ_i $(1 \leq i \leq p)$ are disjoint cycles in S_m. Assume that the order of σ_i $(1 \leq i \leq p)$ is m_i. The irreducible characters of G are given by

$$\lambda_q\left((\sigma_1 \cdots \sigma_p)^s\right) = \exp\left(\frac{2\pi i q s}{[m_1, \ldots, m_p]}\right), \quad s = 0, \ldots, [m_1, \ldots, m_p] - 1,$$

where $[m_1, \ldots, m_p]$ denotes the least common multiple of the integers $m_1, \ldots, m_p$.

In the following theorem we obtain $\dim V^{\lambda_q}(G)$ in terms of Ramanujan sum.

Theorem 3.2. *Let $G = \langle \sigma \rangle$ be a finite cyclic group, where $\sigma \in S_m$ and $\sigma = \sigma_1 \cdots \sigma_p$ is the decomposition of σ into disjoint cycles. If $l(\sigma) = m,\ l(\sigma_i) = m_i,\ 1 \leq i \leq p,$ and $\lambda_q,\ 0 \leq q < [m_1, \ldots, m_p]$, be the irreducible character of G. Then*

$$\dim V^{\lambda_q}(G) = (\dim V) \sum_{i=1}^{p} \frac{1}{m'_i} \sum_{d \mid m'_i} c_{\frac{m'_i}{d}}(q),$$

where $m'_i = [m_1, \ldots, m_p]/m_i$. In particular, if G is generated by an m-cycle in S_m, then $\dim V^{\lambda_q}(G) = \dim V$.

Proof. To compute the dimension of $V^{\lambda_q}(G)$ we use Corollary 2.2. For every $1 \leq i \leq p$, suppose

$$\mathcal{O}_i = \{1 \leq j \leq m \mid \sigma_i(j) \neq j\}$$

be the set of moved points of σ_i. Then $\mathcal{O}_1, \ldots, \mathcal{O}_p$ are distinct orbits of G. If $j \in \mathcal{O}_i$ then

$$\begin{aligned} G_j =& \{(\sigma_1 \cdots \sigma_p)^s \mid 0 \leq s \leq [m_1, \ldots, m_p] - 1, \sigma_i^s(j) = j\} \\ =& \{\sigma_1^s \cdots \sigma_p^s \mid 0 \leq s \leq [m_1, \ldots, m_p] - 1, \sigma_i^s \in \langle \sigma_i \rangle_j = 1\} \\ =& \{\sigma_1^s \cdots \sigma_{i-1}^s \sigma_{i+1}^s \cdots \sigma_p^s \mid 0 \leq s \leq [m_1, \ldots, m_p] - 1, m_i | s\}. \end{aligned}$$

In this case, it is easy to see that $|G_j| = [m_1, \ldots, m_p]/m_i$. Now for $j \in \mathcal{O}_i$ we have

$$\begin{aligned} [\lambda_q, 1_{G_j}] =& \frac{1}{|G_j|} \sum_{g \in G_j} \lambda_q(g) \\ =& \frac{1}{|G_j|} \sum_{\substack{s=0, \\ m_i | s}}^{[m_1, \ldots, m_p]-1} \lambda_q((\sigma_1 \cdots \sigma_p)^s) \\ =& \frac{m_i}{[m_1, \ldots, m_p]} \sum_{\substack{s=0, \\ m_i | s}}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi i q s}{[m_1, \ldots, m_p]}\right). \end{aligned}$$

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For any $1 \leq i \leq p$ we choose $j_i \in \mathcal{O}_i$. Then $\mathfrak{D} = \{j_1, \ldots, j_p\}$. Therefore

$$\begin{aligned}
\dim V^{\lambda_q}(G) &= (\dim V)\lambda_q(1) \sum_{j \in \bar{\mathfrak{D}}} [\lambda_q, 1_{G_j}] \\
&= (\dim V) \sum_{j \in \mathfrak{D}} [\lambda_q, 1_{G_j}] \\
&= (\dim V) \sum_{i=1}^{p} [\lambda_q, 1_{G_{j_i}}] \\
&= (\dim V) \sum_{i=1}^{p} \frac{m_i}{[m_1, \ldots, m_p]} \sum_{\substack{s=0, \\ m_i | s}}^{[m_1,\ldots,m_p]-1} \exp\left(\frac{2\pi i q s}{[m_1, \ldots, m_p]}\right).
\end{aligned}$$

Now letting $s'_i = s/m_i$ and $m'_i = [m_1, \ldots, m_p]/m_i$, we obtain

$$\begin{aligned}
\dim V^{\lambda_q}(G) &= (\dim V) \sum_{i=1}^{p} \frac{1}{m'_i} \sum_{s'_i=0}^{m'_i-1} \exp\left(\frac{2\pi i q s'_i}{m'_i}\right) \\
&= (\dim V) \sum_{i=1}^{p} \frac{1}{m'_i} \sum_{d | m'_i} \sum_{\substack{s'_i=0 \\ (s'_i, m'_i)=d}}^{m'_i-1} \exp\left(\frac{2\pi i q s'_i}{m'_i}\right) \\
&= (\dim V) \sum_{i=1}^{p} \frac{1}{m'_i} \sum_{d | m'_i} c_{\frac{m'_i}{d}}(q).
\end{aligned}$$

□

Our the other goal is to obtain the dimension of $V^{\lambda}(G)$, when $G \leq S_m$ has the structure

$$G =< \sigma_1 > \cdots < \sigma_k >,$$

with σ_l $(1 \leq l \leq k)$ representing disjoint cycles in S_m of specific orders, denoted as $m_1, \ldots, m_k$, respectively. The irreducible characters of G are all linear and can be expressed as products of the irreducible characters of the cyclic groups $\langle \sigma_l \rangle$, where $l = 1, \ldots, k$:

$$\lambda_{(q_1,\ldots,q_k)} = \lambda_{q_1} \cdots \lambda_{q_k}, \ 0 \leq q_i \leq m_i - 1, \ 1 \leq i \leq k$$

and the character λ_{q_l} is defined as $\lambda_{q_l}(\sigma_l^{j_l}) = \exp\left(\dfrac{2\pi i q_l j_l}{m_l}\right)$.

Theorem 3.3. *Suppose $G =< \sigma_1 > \cdots < \sigma_k >$, where $\sigma = \sigma_1 \cdots \sigma_k$ is the decomposition of $\sigma \in S_m$ into disjoint cycles with $2 \leq k$. If $l(\sigma_l) = m_l$, $1 \leq l \leq k$, and $\lambda = \lambda_{(q_1,\ldots,q_k)}$, where $0 \leq q_l \leq m_l - 1$, $1 \leq l \leq k$. Then*

$$\dim V^{\lambda}(G) = (\dim V) \left(l(\sigma)[\lambda, 1_G] + \sum_{j=1}^{k} \prod_{l \neq j} \frac{1}{m_l} \sum_{d | m_l} c_{\frac{m_l}{d}}(q_l) \right).$$

In the case $k = 1$, $\dim V^{\lambda}(G) = (\dim V)\,(l(\sigma)[\lambda, 1_G] + 1)$.

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Proof. Let $j \in \mathfrak{D}$. If $\sigma_i(j) = j$ for all $1 \leq i \leq k$, then $G_j = G$ and so $[\lambda, 1_{G_j}] = [\lambda, 1_G]$. Otherwise there exists $1 \leq i \leq k$, such that $\sigma_i(j) \neq j$. Then $G_j = \prod_{i \neq j} < \sigma_i >$. Thus we have

$$\begin{aligned}
[\lambda, 1_{G_j}] =& \frac{1}{|G_j|} \sum_{g \in G_j} \lambda(g) \\
=& \frac{1}{m_1 \cdots m_{j-1} m_{j+1} \cdots m_k} \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_k=0}^{m_k-1} \lambda_{q_1}(\sigma_1^{j_1}) \cdots \lambda_{q_k}(\sigma_k^{j_k}) \\
=& \prod_{l \neq j} \left(\frac{1}{m_l} \sum_{j_l=0}^{m_l-1} \lambda_{q_l}(\sigma_l^{j_l}) \right) \\
=& \prod_{l \neq j} \left(\frac{1}{m_l} \sum_{j_l=0}^{m_l-1} \exp(\frac{2\pi i q_l j_l}{m_l}) \right) \\
=& \prod_{l \neq j} \left(\frac{1}{m_l} \sum_{d | m_l} \sum_{\substack{j_l=0, \\ (j_l, m_l)=d}}^{m_l-1} \exp(\frac{2\pi i q_l j_l}{m_l}) \right) \\
=& \prod_{l \neq j} \left(\frac{1}{m_l} \sum_{d | m_l} c_{\frac{m_l}{d}}(q_l) \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\dim V^\lambda(G) &= (\dim V)(\lambda(1) \sum_{j \in \mathfrak{D}} [\lambda, 1_{G_j}]) \\
&= (\dim V) \left(\sum_{j \in \mathrm{Fix}(\sigma)} [\lambda, 1_G] + \sum_{j=1}^{k} [\lambda, 1_{G_j}] \right) \\
&= (\dim V) \left((m - l(\sigma))[\lambda, 1_G] + \sum_{j=1}^{k} \prod_{l \neq j} \frac{1}{m_l} \sum_{d | m_l} c_{\frac{m_l}{d}}(q_l) \right).
\end{aligned}$$

so the result holds. □

4. Cartesian symmetry classes associated with dihedral group

In this section, we first obtain the dimensions of Cartesian symmetry classes associated with the irreducible characters of the dihedral group D_{2m}.

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The subgroup D_{2m} of S_m $(m \geq 3)$ generated by the elements $r = (1\ 2\ \cdots\ m)$ and

$$\begin{aligned} s &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & i & \cdots & m-1 & m \\ 1 & m & m-1 & m-2 & \cdots & m+2-i & \cdots & 3 & 2 \end{pmatrix} \\ &= \prod_{2 \leq i < m+2-i} (i\ m+2-i) \end{aligned}$$

is the *dihedral group of degree* m (see [2, page 50]). The generators r and s satisfy $r^m = s^2 = (1)$; if $0 < k < m$, $r^k \neq (1)$ and $s^{-1}rs = r^{-1}$. In particular,

$$D_{2m} = \{r^k,\ sr^k |\ 0 \leq k < m\}.$$

If m is even, i.e., $m = 2k$ $(k \geq 2)$, then D_{2m} has $k+3$ conjugacy classes. If m is odd, i.e., $m = 2k+1$ $(k \geq 1)$, then D_{2m} has $k+2$ conjugacy classes.

For each integer h with $0 < h < m/2$, D_{2m} has an irreducible character ψ_h of degree 2 given by

$$\psi_h(r^k) = 2\cos\frac{2kh\pi}{m}, \quad \psi_h(sr^k) = 0, \quad 0 \leq k < m.$$

The other characters of D_{2m} are of degree 1, namely λ_j. The character table of D_{2m} is shown in Table 1 (see [4, page 182]).

TABLE 1. Character table of D_{2m}

m is odd	r^k	sr^k	m is even	r^k	sr^k
λ_1	1	1	λ_1	1	1
λ_2	1	-1	λ_2	1	-1
ψ_h	$2\cos\frac{2k\pi h}{m}$	0	λ_3	$(-1)^k$	$(-1)^k$
-	-	-	λ_4	$(-1)^k$	$(-1)^{k+1}$
-	-	-	ψ_h	$2\cos\frac{2k\pi h}{m}$	0

Clearly, the action of $G = D_{2m}$ on $\mathbf{I}_m = \{1, 2, \ldots, m\}$ is transitive, so $\mathfrak{D} = \{1\}$. On the other hand, $G_1 = \{1, s\}$. Thus for any $\lambda \in \mathrm{Irr}(G)$, we have

$$\dim V^{\lambda}(G) = (\dim V)\ \lambda(1)\ [\lambda, 1_{G_1}] = (\dim V)\ \frac{\lambda(1)}{2}\ [\lambda(1) + \lambda(s)]. \tag{4.1}$$

Now, by using Table 1 and Equation (4.1), we have the following result.

Theorem 4.1. *Let $G = D_{2m}, m \geq 3$, and $n = \dim V \geq 2$. Then*

Case (i) *If m is even, then*

(a) $\dim V^{\lambda_1}(G) = \dim V^{\lambda_3}(G) = n$,

(b) $\dim V^{\lambda_2}(G) = \dim V^{\lambda_4}(G) = 0$,

(c) $\dim V^{\psi_h}(G) = 2n$ $(0 < h < \frac{m}{2})$.

Case (ii) *If m is odd, then*

(a) $\dim V^{\lambda_1}(G) = n$,

(b) $\dim V^{\lambda_2}(G) = 0$,

(c) $\dim V^{\psi_h}(G) = 2n$ $(0 < h < \frac{m}{2})$.

In the following theorem we give a necessary and sufficient condition for the existence of an O -basis for Cartesian symmetry classes $V^{\psi_h}(G)$ $(0 < h < \frac{m}{2})$.

Theorem 4.2. *Let $G = D_{2m}$ $(m \geq 3)$ and $\psi = \psi_h$ $(0 < h < \frac{m}{2})$. Then $V^{\psi}(G)$ has O-basis if and only if $m \equiv 0$ (mod $4h_2$), where $h = h_2 h_{2'}$ with h_2 a power of 2 and $h_{2'}$ odd.*

Proof. Since G acts on $\mathbf{I}_m$ transitively, $\mathfrak{D} = \{1\}$. Clearly, $G_1 = \{1, s\}$. So $[\psi, 1_{G_1}] = \frac{1}{2}[\psi(1) + \psi(s)] = 1 \neq 0$. This implies that $1 \in \bar{\mathfrak{D}}$ and so $V^{\psi}(G) = \sum_{1 \leq i \leq n}^{\perp} V_{i1}^{\psi}$. Notice that $\dim V_{i1}^{\psi} = \psi(1)[\psi, 1_{G_1}] = 2$ and the set $\{f_{ir(1)}^{\psi}, f_{is(1)}^{\psi}\}$ is a basis for it. By using Theorem 1.1, we have

$$\langle f_{ir(1)}^{\psi}, f_{is(1)}^{\psi} \rangle = \frac{\psi(1)}{|G|} \sum_{g \in G_1} \psi(gsr^{-1}) = \frac{1}{m}[\psi(sr^{-1}) + \psi(r^{-1})] = \frac{2}{m} \cos \frac{2\pi h}{m}.$$

It is observed that $\langle f_{ir(1)}^{\psi}, f_{is(1)}^{\psi} \rangle = 0$ if and only if $\cos \frac{2\pi h}{m} = 0$, or equivalently $m \equiv 0 \pmod{4h_2}$. This completes the proof. □

In the following, for every positive integer h, we will refer to h_2 as the 2-part of h.

Corollary 4.3. *Let $G = D_{2m}, m \geq 3$, and $\dim V \geq 2$. Then $\times^m V$ has an O-basis if and only if m is a power of 2.*

Proof. Suppose m_2 is the 2-part of m. Assume that $m_2 < m$. Then $0 < m_2 < \frac{m}{2}$ and $m \not\equiv 0 \pmod{4m_2}$. Hence, if $\psi = \psi_{m_2}$, then Theorem 4.2 implies that $V^{\psi}(G)$ has no O-basis, and so $\times^m V$.
Conversely, assume m is a power of 2. If $0 < h < \frac{m}{2}$, then $h_2 \leq \frac{m}{4}$ and $m \equiv 0 \pmod{4h_2}$, where h_2 is the 2-part of h. Now Theorem 4.2 implies that $V^{\psi_h}(G)$ has an O-basis, and so $\times^m V$. □

Remark 4.4. *Let $G = D_8 \leq S_4$. Then G has only one non-linear irreducible character $\psi(r^k) = 2\cos\frac{k\pi}{2}$, $\psi(sr^k) = 0$, $0 \leq k < 4$. By Theorem 4.2, $V^{\psi}(G)$ has an O-basis. Again, by the proof of Theorem 4.2, $\mathfrak{D} = \bar{\mathfrak{D}} = \{1\}$ and $G_1 = \{1, s\}$. So*

$$\parallel f_{i1}^{\psi} \parallel^2 = \frac{\psi(1)}{[G : G_1]}[\psi, 1_{G_1}] = \frac{1}{4}[\psi(1) + \psi(s)] = \frac{1}{2}.$$

Therefore $\parallel f_{i1}^{\psi} \parallel = \frac{\sqrt{2}}{2}$. This example shows that $\frac{\sqrt{2}}{2}$ is the best lower bound for $\parallel f_{ij}^{\lambda} \parallel$ in Theorem 2.3.

Now we give a counterexample that the converse of Theorem 2.3 is not true.

Example 4.5. *Let $q > 3$ be a prime and*

$$G = D_{2q} =< r, s \mid r^q = 1 = s^2 \text{ and } s^{-1}rs = r^{-1} >$$

be the subgroup of S_q. Then the non-linear irreducible characters of D_{2q} are ψ_h, where $0 < h < q/2$ and $\psi_h(r^k) = 2\cos\frac{2k\pi h}{q}$, $\psi_h(sr^k) = 0$, $0 \leq k < q$. Let $\psi = \psi_h$, $0 < h < \frac{q}{2}$. By Theorem 4.2, $V^\psi(G)$ has no O-basis. Again, by the proof of Theorem 4.2, $\mathfrak{D} = \bar{\mathfrak{D}} = \{1\}$ and $G_1 = \{1, s\}$. Therefore

$$\parallel f_{i1}^{\psi} \parallel^2 = \frac{\psi(1)}{[G : G_1]}[\psi, 1_{G_1}] = \frac{1}{q}[\psi(1) + \psi(s)] = \frac{2}{q} < \frac{1}{2}.$$

Remark 4.6. *Let H be a subgroup of S_6 generated by the elements $a = (1\ 2\ 3)(4\ 5)$ and $b = (1\ 2)$. The generators a and b satisfy $a^6 = b^2 = (1)$; if $0 \leq k < 6$, $a^k \neq (1)$ and $b^{-1}ab = a^{-1}$. Hence, $H \cong D_{12}$. It is evident that $\mathfrak{D} = \{1, 4, 6\}$. Consider the principal character λ_1 of H. Then $\bar{\mathfrak{D}} = \mathfrak{D}$ and*

$$\dim V^{\lambda_1}(H) = (\dim V)|\mathfrak{D}| = 3\ (\dim V).$$

By Theorem 4.1, $\dim V^{\lambda_1}(D_{12}) = \dim V$. This example shows that the dimension of Cartesian symmetry class associated with a group G depends on the permutation structure of the group.

As the symmetry classes of tensors [7, Section 2]*, one may find that the proof of Theorem 4.2 is independent of a certain permutation structure of D_{2m}.*

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