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ON THE THEORY AND GENERALIZATION OF Σ-GROUPS

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ABSTRACT. In this work we present a systematic study of n-layered modules which are closely related to Σ-modules. For each integer $n \geq 1$ we prove some results for n-layered modules concerning when Σ-modules are direct sum of countably generated modules. Moreover, we discover additional restriction which leads to coinciding of n-layered modules and m-layered modules for $n > m$.

1. Introduction and background material

The theory of abelian groups is fundamental with respect to the investigation of TAG-modules. The concept of TAG (torsion abelian group like) module was introduced and initially investigated by Singh [26]. The notion of TAG-modules is one of the most important tools in module theory and many different forms of generalizations of torsion abelian groups have been introduced and investigated for these modules. This notion has many more pleasing properties which have been the focus of attention of many authors since 1976 (see, for instance, [1, 28]).

Over an arbitrary (associative, unitary) ring R, a module M is called a TAG-module if it satisfies the following two conditions relating to uniserial modules.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$.

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In 1987 Singh followed this up in his another work [27], and introduced the notion of $QTAG$-modules by applying condition (i) only. The study was then followed by numerous developments on the topic. In particular, a lot of variations of the concept have been introduced and studied (see, for example, [10, 11, 23, 24, 25] and the references cited therein). It is worthwhile noticing that many of the developments in this direction are analogous to the earlier development of torsion abelian groups.

We begin by reviewing some terminology. Let all rings into consideration be associative with unity $(1 \neq 0)$, and let modules be unital $QTAG$-modules. In addition, throughout, the letter n will denote non-negative integers. By the term "uniserial module" we will mean a module M over a ring R, whose submodules are totally ordered by inclusion, i.e., for any two submodules N and L of M, either $N \subseteq L$ or $L \subseteq N$. Likewise, we shall say M is uniform if intersection of any two of its non-zero submodules is non-zero. In particular, an element u in a module M is called the uniform element if uR is non-zero uniform (hence uniserial) module. Concerning decomposition series, we suppose that all decomposition series are unique. For any module M, the symbol $d(M)$ will denote its decomposition length. In addition, if u is an uniform element of M (i.e., $u \in M$), then $e(u)$ is called the exponent of u, and $e(u) = d(uR)$. As usual, for such a module M, we set the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$. For every non-negative integer n, $H_n(M) = \{x \in M \mid H_M(x) \geq n\}$ denotes the submodule of M generated by the elements of height at least n and for some submodule N of M, $H^n(M) = \{x \in M \mid d(xR/(xR \cap N)) \leq n\}$ is the submodule of M generated by the elements of exponents at most n. For a module M, the letter M^1 will always denote in the sequel the submodule of M, containing elements of infinite height. Moreover, we denote by $Soc(M)$, the socle of M, i.e., the sum of all simple submodules of M.

Next, we review the following concepts. The module M is defined to be bounded [26] if $\exists\ n \geq 0$ such that $H_M(u) \leq n$ for some $u \in M$. The module M is termed $(\omega + n)$-projective [17] if there exists $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules; thus $M^1 \subseteq N$. A submodule N of a module M is said to be an h-pure [13] in M if for every non-negative integer n the equality $N \cap H_n(M) = H_n(N)$ holds.

Moreover, we recall another critical terminology from [19]. If α is an ordinal, and M is a $QTAG$-module, then the infinite height $H_\alpha(M)$ is defined inductively as follows: $H_0(M) = M$, and if $\alpha > 0$, then $H_\alpha(M) = \cap_{\gamma<\alpha} H_\gamma(M)$. Clearly, $H_\alpha(M)$ is a submodule of M, consisting of elements of height at least α. This submodule is also known as α-th Ulm submodule.

We add some familiarity as well in terms of infinite height of M. A submodule N of M is said to be α-pure [20] if, for all ordinal γ, there exists an ordinal α (depending on N) such that $H_\gamma(M) \cap N = H_\gamma(N)$. Besides, a submodule N of M is named isotype, if it is α-pure for every ordinal α. A submodule N of M is said to be α-high if it is maximal with respect to the property that $N \cap H_\alpha(M) = \{0\}$. A ω-high submodule is usually referred to simply as a high submodule [14]. A submodule N of M is said to be nice [16] in M, if the equality $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ holds for all ordinals α, i.e. every coset of M modulo N may be represented by an element of the same height.

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It is well to note that various results for TAG-modules are also valid for $QTAG$-modules [20]. Our present work is motivated by the many significant results from the papers [2, 3, 5]. In what follows, all notations and notions are standard and will be in agreement with those used in [9].

2. Chief results

The study of Σ-modules, which was introduced by Khan [14], is very large and possesses a key role in the theory of $QTAG$-modules. A $QTAG$ module M is said to be a Σ-module if its high submodules are the direct sum of uniserial modules. It is well known that if M is a Σ-module, then all its high submodules are the direct sum of uniserial modules. On the other hand, referring to a classical criterion from [21], a $QTAG$ module M is said to be a Σ-module if $Soc(M) = \cup_{k<\omega} M_k$, where for all $k < \omega$, $M_k \subseteq M_{k+1} \subseteq Soc(M)$ and $M_k \cap H_k(M) = Soc(H_\omega(M))$. Generalizing this concept, in [25] was introduced the following notion: The $QTAG$-module M is called an n-layered module if $1 \leq n < \omega$, $H^n(M) = \cup_{k<\omega} M_k$, where for all $k < \omega$, $M_k \subseteq M_{k+1} \subseteq H^n(M)$ and $M_k \cap H_k(M) = H^n(H_\omega(M))$. With this terminology, the 1-layered module (or for simpleness just a layered module) is precisely the Σ-module. Moreover, we observe that n-layered modules are m-layered modules provided that $n \geq m$. However, the converse implication fails; specifically, there exists an example of a Σ-module which is not a 2-Σ-module (see, [24]).

Before stating and proving our main attainments, we shall list some important but helpful properties of the so-defined class of n-layered modules. These properties lead to the generalizations of some known facts discussed in [2, 4, 6, 7, 12].

Proposition 2.1. [22]. *Let N be a countably generated (nice) submodule of the $QTAG$-module M. If M is an n-layered module, then M/N is an n-layered module.*

Proposition 2.2. [18]. *Let M be a $QTAG$-module of length less than or equal to $(\omega + n - 1)$. Then M is an n-layered module if and only if M is a direct sum of countably generated modules.*

Proposition 2.3. [18]. *Let M be a $QTAG$-module. Then M is an n-layered module if and only if every $(\omega + n - 1)$-high submodule of M is a direct sum of countably generated modules.*

Proposition 2.4. [25]. *Any n-layered module is $(\omega + n - 1)$-projective if and only if it is a direct sum of countably generated modules of length at most $(\omega + n - 1)$. In particular, $(\omega + n)$-projective n-layered modules are direct sum of countably generated modules of length not exceeding $(\omega + n)$.*

And so, we will verify the validity of the following theorem.

Theorem 2.5. *Let $n \geq 1$ be an integer, and M a $QTAG$-module such that $H_{\omega+n-1}(M)$ is countably generated. Then M is an n-layered module if and only if M is the direct sum of a countably generated module and a direct sum of countably generated module of length at most $(\omega + n - 1)$.*

In particular, n-layered modules of length less than or equal to $\omega + n - 1$ are direct sum of countably generated modules.

Proof. Since $H_{\omega+n-1}(M)$ is countably generated, and hence nice in M, it therefore follows from Proposition 2.1 that $M/H_{\omega+n-1}(M)$ is an n-layered module. Furthermore, with Proposition 2.2 at hand, we subsequently deduce that $M/H_{\omega+n-1}(M)$ is a direct sum of countably generated modules. Again, since $H_{\omega+n-1}(M)$ is countably generated, we see that M is also countably generated.

The last part is now immediate. □

The following definition appeared in [10]: A $QTAG$-module M is Σ-uniserial if it is isomorphic to a direct sum of uniserial modules. Notice that Σ-uniserial modules are separable (i.e., ω-bounded). More generally, for any integer $n \geq 0$, the $QTAG$-module M is $(\omega+n)$-projective precisely when there is a submodule $N \subseteq H^n(M)$ with the property that M/N is Σ-uniserial; notice that such a module is necessarily $(\omega+n)$ bounded.

Recall that, in [23], a $QTAG$-module M is said to be an $(\omega+n)$-totally $(\omega+n)$-projective module if each of its $(\omega+n)$-bounded submodule is $(\omega+n)$-projective.

Analogous to proper group in [8], we will say an $\omega+n$-totally $(\omega+n)$-projective module M is proper if it does not belong to either of these two classes; i.e., iff it is not $(\omega+n)$-projective and not ω-totally Σ-uniserial.

Remark 2.6. *There are no proper ω-totally ω-projective modules.*

Corollary 2.7. *If M is a proper $(\omega+n)$-totally $(\omega+n)$-projective module, then M is not a Σ-module.*

Proof. Indeed, suppose the contrary. Hence, in view of [23, Corollary 2.1], $H_\omega(M)$ has to be countably generated. Consequently, by Theorem 2.5, M is ω-totally ω-projective module. This contradicts by the previous Remark 2.6. □

Next, we concentrate on the following theorem.

Theorem 2.8. *Suppose M is an n-layered module. Then M is an $(\omega+n)$-totally $(\omega+n)$-projective module if and only if M is a direct sum of countably generated modules.*

Proof. In accordance with Corollary 2.7, one may see that M is not a proper $(\omega+n)$-totally $(\omega+n)$-projective module. By the definition of proper module, it is easily observed that M is either $(\omega+n)$-projective or M is a direct sum of countably generated modules. We next employ Proposition 2.4 to get that M is a direct sum of countably generated modules, as wanted. □

We come now to a significant characterization of n-layered modules.

Proposition 2.9. *Suppose M is a $QTAG$-module. Then the following are equivalent.*

(i) *M is an n-layered module;*
(ii) *each $(\omega+n-1)$-high submodule of M is a direct sum of countably generated modules;*
(iii) *each $(\omega+n-1)$-high submodule of M is an n-layered module;*
(iv) *each $(\omega+n)$-high submodule of M is an n-layered module;*

(v) *each α-high submodule of M is an n-layered module for arbitrary ordinal α.*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows immediately from Proposition 2.3.

The implication $(ii) \Rightarrow (iii)$ is obvious and, besides, the implication $(iii) \Rightarrow (ii)$ follows directly from Theorem 2.5.

As for the fourth and fifth parts, we shall demonstrate that they are both equivalent to the first part. In fact, it is elementary to see that α-high submodules are themselves isotype in the $QTAG$-module M, for any ordinal α. This insures at once with the aid of [25] that h-pure submodules of n-layered modules are again n-layered modules. So (iv) and (v) hold provided that (i) is valid. In order to verify the converse, we let (v) hold. Suppose N is an arbitrary α-high submodule of M. Then N must be n-layered. Suppose next that L is some $(\omega+n-1)$-high submodule of M; in fact, assume $\alpha \geq (\omega+n)$ by utilizing (iii). Since $L \cap H_\alpha(M) \subseteq L \cap H_{\omega+n-1}(M) = 0$, it follows that L embeds in N. But L is isotype in M and hence in N. Consequently, L is an n-layered module, so that (iii) follows and, by what we have just shown above, (i) follows as well. The implication $(iv) \Rightarrow (i)$ follows via identical arguments as above. □

The following points represent some major properties of a direct sum of countably generated modules that will be intensively used in the sequel.

(a) If a $QTAG$-module M has one $(\omega+n-1)$-high submodule which is a direct sum of countably generated modules, then all its $(\omega+n-1)$-high submodules are direct sum of countably generated modules.

(b) If $\alpha < \omega+n-1$ and a $QTAG$-module M has an $(\omega+n-1)$-high submodule that is a direct sum of countably generated modules, then it has an α-high submodule that is a direct sum of countably generated modules.

(c) If N is an $(\omega+n)$-high submodule of a $QTAG$-module M, then under canonical map $M \to M/H_{\omega+n}(M)$, N maps to an $(\omega+n)$-high submodule of $M/H_{\omega+n}(M)$.

And so, we prepare to prove the following.

Theorem 2.10. *Suppose M is a $QTAG$-module and $\omega \leq \alpha \leq \omega_1$ is an ordinal. Then*

(i) *M is an n-layered module if and only if $M/H_{\alpha+n}(M)$ is an n-layered module.*

(ii) *If $M/H_\alpha(M)$ is an n-layered module, then M is an n-layered module.*

Proof. (i) By the usage of the above points (a), (b) and (c), we detect that M is an n-layered module if and only if $M/H_{\alpha+n}(M)$ is an n-layered module.

(ii) Assume that N is an $(\omega+n)$-high submodule of M. Then N is isotype in M, and it is apparent that $N/H_\alpha(N) \cong (N + H_\alpha(M))/H_\alpha(M)$ is isotype in $M/H_\alpha(M)$. Henceforth, according to [25], $N/H_\alpha(N)$ is an n-layered module. Now, we have two cases to consider. First if $\alpha \geq \omega+n$, then $H_\alpha(N) = 0$ and we are done. For the remaining case $\alpha < \omega+n$ we write $\alpha \leq \omega+n-1$ and so by virtue of Theorem 2.5, it is easily observed that $N/H_\alpha(N)$ is a direct sum of countably generated modules. Since $H_\alpha(N)$

is bounded, it is fairly to see that N is a direct sum of countably generated modules. Thus N is an n-layered module. Consequently, we apply Proposition 2.9 to obtain the wanted claim, i.e., M is an n-layered module. □

We continue the study with the following proposition.

Proposition 2.11. *Let M be a QTAG-module such that $M/H_{\omega+n}(M)$ is $(\omega+n)$-projective. If M is an n-layered module, then $M/H_{\omega}(M)$ is Σ-uniserial.*

Proof. Appealing to Theorem 2.10(i), $M/H_{\omega+n}(M)$ is an n-layered module. Next, in view of Proposition 2.4, $M/H_{\omega+n}(M)$ is a direct sum of countably generated modules. This, in tern, implies that $M/H_{\omega}(M)$ is Σ-uniserial. Since the isomorphism sequence

$$M/H_{\omega}(M) \cong M/H_{\omega+n}(M)/H_{\omega}(M)/H_{\omega+n}(M) \cong M/H_{\omega+n}(M)/H_{\omega}\left(M/H_{\omega+n}(M)\right)$$

holds, where $H_{\omega}\left(M/H_{\omega+n}(M)\right)$ is bounded. We are done. □

The cardinality of the minimal generating set of a *QTAG*-module M is denoted by the symbol $g(M)$ that plays a significant role in our further investigation. By analogy, for all ordinals α, one can define $f_M(\alpha)$, the α-th *Ulm* invariant of M (see [15]) as follows:

$$f_M(\alpha) = g\left(Soc\left(H_{\alpha}(M)\right)/Soc\left(H_{\alpha+1}(M)\right)\right).$$

In [24], it was proved that any $(\omega+1)$-projective Σ-module is a direct sum of countably generated modules of length at most $(\omega+1)$. Furthermore, it was demonstrated that, for any $n \geq 2$, there exists an $(\omega+n)$-projective Σ-module which is not a direct sum of countably generated modules. However, under some extra circumstances, this claim holds true.

So, we are now ready to prove below the following statement.

Proposition 2.12. *Let $n \geq 2$, and M a QTAG-module such that the $(\omega+k)$-th Ulm invariant of M is zero for each $0 \leq k < n-1$. If M is a Σ-module, then M is an n-layered module.*

Proof. Let N be high submodule of M, and if we let $\phi : M \to M/H_{\omega}(M)$ be a natural map. Then $H^n(M) = H^n(N) \oplus H^n\left(H_{\omega}(M)\right)$. Since $f_M(\omega+k) = 0$ for some $k \geq 0$, it easily follows that $H^k\left(H_{\omega}(M)\right) \subseteq H_{n-k}\left(H_{\omega}(M)\right)$ for $k \leq n$. Indeed, since N is a Σ-uniserial module, we get that $H^n(N) = \cup_{t<\omega} N_t$, $N_t \subseteq N_{t+1} \subseteq H^n(N)$, and $N_t \cap H_t(N) = 0$. Besides, it is simple to checked that $H^n(M) = \cup_{t<\omega} L_t$ where $L_t = N_t \oplus H^n\left(H_{\omega}(M)\right)$. Thereby, because of the h-pureness of N, it follows at once that

$$L_t \cap H_t(M) \subseteq H_{\omega}(M) + N_t \cap H_t(M) = H_{\omega}(M) + N_t \cap H_t(N) = H_{\omega}(M),$$

as expected. We are finished. □

A valuable consequence is the following

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Corollary 2.13. *Let M be an $(\omega+n)$-totally projective Σ-module, and let $f_M(\omega+k)=0$ for every $0\le k<n-1$ if $n>1$. Then M is a direct sum of countably generated modules of length not exceeding $(\omega+n)$.*

Proof. It follows by the usage of the same idea as in Proposition 2.12 along with [25, Theorem 1]. □

Now Proposition 2.12 can be somewhat refined to the new framework.

Proposition 2.14. *Let $n\ge 2$, and M a QTAG-module such that the $(\omega+k)$-th Ulm invariant of M is zero for each $0\le k<n-1$. Then M is a Σ-module if and only if M is an n-layered module.*

Proof. In virtue of Proposition 2.12, the sufficiency is true.

Concerning the necessity, we shall prove at first by induction on the integer r. First, if $1\le r\le n$, we have $H^r(H_\omega(M))\subseteq H_{n-r}(H_\omega(M))$. If now $r=n$, we obtain the desired inclusion. To that goal, given u is an uniform element in $Soc(H_\omega(M))$. Since $Soc(H_{\omega+k}(M))=Soc(H_{\omega+k+1}(M))$ for each non-negative $k<n-1$, one may write $Soc(H_\omega(M))=\cdots=Soc(H_{\omega+n-1}(M))\subseteq H_{n-1}(H_\omega(M))$ which gives the required inclusion for $r=1$.

Next, with this in hand, we ascertain the same induction hypothesis that $H^{n-2}(H_\omega(M))\subseteq H_2(H_\omega(M))$ and letting $u\in H^{n-1}(H_\omega(M))$. So, there exists $v\in Soc(H_{\omega+n-2}(M))=Soc(H_{\omega+n-1}(M))$ such that $d(uR/vR)=n-2$, and hence $v\in H_{n-2}(Soc(H_{\omega+1}(M)))$ where $d(uR/vR)=n-2$. Consequently, $u\in H^{n-1}(H_{\omega+1}(M))+H^{n-2}(H_\omega(M))\subseteq H_1(H_\omega(M))$.

On the other hand, since $Soc(M)=Soc(N)\oplus Soc(H_\omega(M))$ for some high submodule N of M. This, in tern implies that $H^n(M)=H^n(N)\oplus H^n(H_\omega(M))$. Furthermore, supposing $u\in H^n(M)$, there exists $v\in Soc(M)$ and that $v=x+y$ where $x\in Soc(N)$, $y\in Soc(H_\omega(M))$ and $d(uR/vR)=n-1$. Notice that N is h-pure in M, we obtain that $v=b+y$ where $d(uR/vR)=d(aR/bR)=n-1$ for some $a\in H^n(N)$. Since $Soc(H_\omega(M))=H_{n-1}(Soc(H_\omega(M)))$, one may see that $y\in H_{n-1}(H_\omega(M))$. Consequently, $u\in H^n(N)+H^n(H_\omega(M))+H^{n-1}(M)=H^n(N)+H^n(H_\omega(M))$. This allows us to obtain that $H^{n-1}(M)=H^{n-1}(N)+H^{n-1}(H_\omega(M))$, as claimed.

Finally, since N is Σ-uniserial, it is self-evident that $H^n(N)=\cup_{t<\omega}N_t$, $N_t\subseteq N_{t+1}\subseteq H^n(N)$ such that $N_t\cap H_t(N)=0$, and so it is also clear to see that

$$H^n(M)=\cup_{t<\omega}(N_t\oplus H^n(H_\omega(M)).$$

From the h-purity of N in M, we get that

$$\begin{aligned}(N_t\oplus H_\omega(M))\cap H_t(M)&=N_t\cap H_t(M)+H_\omega(M),\\&=N_t\cap H_t(N)+H_\omega(M),\\&=H_\omega(M).\end{aligned}$$

Henceforth, all the conditions are satisfied for n-layered module and the proof of the proposition is completed. □

With the last statement in hand, one may derive the following.

Corollary 2.15. *Let $n > m \geq 1$, and M a QTAG-module such that $f_M(\omega + k) = 0$ for every $0 \leq k < n - 1$. Then M is an m-layered module if and only if M is an n-layered module.*

Proof. It was seen in [25] that each m-layered module is n-layered module and hence Proposition 2.14 works to get the claim. □

3. Open problems

We close with the following intriguing problems.

Problem 3.1. *Find the intersection between the classes of $(\omega + n)$-totally $(\omega + n)$-projective modules and n-layered modules.*

Problem 3.2. *What is the structure of $(\omega + n)$-totally $(\omega + n)$-projective n-layered modules for $n \geq 2$?*

Problem 3.3. *What are the conditions under which an m-layered module to be an n-layered module, provided that $m < n$?*

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