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# ON THE STRUCTURE OF SOME LEFT BRACES

ADOLFO BALLESTER-BOLINCHES, RAMÓN ESTEBAN-ROMERO*, LEONID A. KURDACHENKO AND VICENT PÉREZ-CALABUIG

Abstract. Given an element $a$ of a left brace $A$ satisfying some nilpotency conditions, we describe the smallest subbrace of $A$ containing $a$. We also present a description of the left braces satisfying the minimal condition for subbraces.

## 1. Introduction

A *left brace* $(A, +, \cdot)$ consists of a set $A$ with two binary operations $+$ and $\cdot$ such that $(A, +)$ is an abelian group, $(A, \cdot)$ is a group, and $a(b + c) = ab - a + ac$ for every $a$, $b$, $c \in A$. When the operations are clear from the context, we refer to the brace $(A, +, \cdot)$ simply as $A$. It is usual to omit the product symbol and to evaluate first the multiplications and then the additions; $-a$ denotes the additive opposite of $a \in A$ and $a - b$ denotes $a + (-b)$ for $a$, $b \in A$. As both group identities coincide, we shall denote it by 0.

This structure was introduced by Rump as a generalisation of Jacobson radical rings in [24,25] and is useful to obtain involutive non-degenerate set-theoretic solutions of the Yang-Baxter equation. This is a consistency equation in fundamental physics, first introduced in the field of statistical mechanics, that was obtained by the Nobel laureate Yang [30] and Baxter [8] and that represents the fact that,

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*Corresponding author.

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in some scattering situations, particles may preserve their momentum while changing their quantum internal states.

The theory of left braces and its generalisations, like the skew left braces defined by Guarnieri and Vendramin [16], in which the commutativity of the addition is not required, have attracted the attention of many mathematicians. However, even for some concrete types of braces, little is known about the internal structure of left braces. Therefore, every piece of information about the structure of concrete left braces is important. We can cite as the first results in this line the ones in the paper [26] of Rump, where he classifies the left braces with cyclic additive group. Other papers that present the description of concrete left braces include [1–4, 6, 7, 13, 14, 21–23].

We review in this paper some recent results obtained by the authors about one-generated left braces satisfying a nilpotency condition and left braces satisfying the minimal condition for subbraces.

## 2. **Preliminary results**

We recall here some basic results about left braces that can be found, for instance, in [12] or [17]. Given a left brace $A$ and two elements $a, b \in A$, we define $\lambda_a(b) = -a + ab$ and $a * b = -a + ab - b = \lambda_a(b) - b$. We see that $\lambda_a \colon A \longrightarrow A$ is an automorphism of $(A, +)$ and that if $\lambda \colon (A, \cdot) \longrightarrow \operatorname{Aut}(A, +)$ is given by $\lambda(a) = \lambda_a$ for $a \in A$, then $\lambda$ is a group homomorphism, that is, $\lambda$ is an action of $(A, \cdot)$ on $(A, +)$.

A subset $S$ of a left brace $A$ is called a *subbrace* of $A$ if $S$ is both a subgroup of $(A, +)$ and $(A, \cdot)$, that is, $S$, with the restrictions of the addition and the multiplication of $A$ to $S$ becomes a left brace.

A subbrace $L$ of $A$ is called a *left ideal* of $A$ if $a * b \in L$ for every element $a \in A$ and every element $b \in L$. This is equivalent to saying that $\lambda_a(b) \in L$ for every $a \in A$ and every $b \in L$. A subbrace $L$ of $A$ is said to be and *ideal* of $A$ if $a * z \in L$ and $z * a \in L$ for all elements $a \in A$ and $z \in L$. This is equivalent to saying that $L$ is a left ideal of $A$ and that $L$ is a normal subgroup of the multiplicative group $(A, \cdot)$. If $K$ and $L$ are subbraces of $A$, we denote by $K * L$ the subgroup of the additive group generated by the elements $x * y$ with $x \in K$, $y \in L$. It is well known that if $A$ is a left brace and $L$ is a left ideal of $A$, then $L * A$ and $A * L$ are left ideals of $A$. Moreover, if $L$ is an ideal of $A$, then $L * A$ is an ideal of $A$.

A left brace $A$ is called *trivial* or *abelian* if $a * b = 0$ for all $a, b \in A$. This condition is equivalent to $a * b = 0$ or $a + b = ab$ for all elements $a, b \in A$.

Let $A$ be a left brace. Let $A^{(1)} = A$ and define $A^{(\alpha+1)} = A^{(\alpha)} * A$ for all ordinals $\alpha$, and $A^{(\alpha)} = \bigcup_{\mu<\alpha} A^{(\mu)}$ for all limit ordinals $\alpha$. In a similar way, let $A^1 = A$ and define $A^{\alpha+1} = A * A^{\alpha}$ for all ordinals $\alpha$ and $A^{\alpha} = \bigcup_{\mu<\alpha} A^{\mu}$ for all limit ordinals $\alpha$. We have that $A^{\alpha}$ is a left ideal for each ordinal $\alpha$ and $A^{(\alpha)}$ is an ideal for each ordinal $\alpha$.

If $A$ is a left brace and there exists a natural number $k$ such that $A^k = \{0\}$, we say that $A$ is *left nilpotent*. If there exists a natural number $m$ such that $A^{(m)} = \{0\}$, then we say that $A$ is *right nilpotent*.

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Let $A$ be a left brace. The *socle* of $A$ is defined as

$$\begin{aligned}\operatorname{Soc}(A) &= \{a \in A \mid ax = a + x \text{ for every } x \in A\}\\ &= \{a \in A \mid a * x = 0 \text{ for every } x \in A\}.\end{aligned}$$

We have that $\operatorname{Soc}(A) = \operatorname{Ker}\lambda$ and $\operatorname{Soc}(A)$ is an ideal of $A$. Let $M$ be a subset of $A$. We call

$$\begin{aligned}\operatorname{Ann}_A(M) &= \{a \in A \mid ax = x + a = xa \text{ for every } x \in M\}\\ &= \{a \in A \mid a * x = x * a = 0 \text{ for every } x \in M\},\end{aligned}$$

the *annihilator* of $M$ in $A$. It can be shown that $\operatorname{Ann}_A(M)$ is a subgroup of the centraliser of $M$ in the multiplicative group $(A, \cdot)$. In the case that $M$ is an ideal of $A$, we have that $\operatorname{Ann}_A(M)$ is a normal subgroup of $(A, \cdot)$. In particular, if $M = A$, we call $\zeta(A) = \operatorname{Ann}_A(A) = \operatorname{Soc}(A) \cap \operatorname{Z}(A, \cdot)$ the *centre* of $A$. The centre of a (skew) left brace (also known as the *annihilator ideal* of a (skew) left brace) was first introduced in [11] in the context of ideal extensions of (skew) left braces and also appears in [9] to introduce central nilpotency of (skew) left braces.

## 3. The subbrace generated by an element of a left brace $A$ with $A^{(3)} = \{0\}$: a theorem of Rump revisited

One of the first natural problems we can consider when analysing left braces is the description of the left braces generated by a single element (one-generator left braces). It is clear that the intersection of a family of subbraces of a left brace $A$ is again a subbrace of $A$. Given a subset $M$ of $A$, we can consider the intersection $\operatorname{br}(M)$ of all subbraces of $A$ containing $M$, called the *subbrace of $A$ generated by $M$*. In the case that $M = \{a\}$ is a singleton, we write $\operatorname{br}(\{a\}) = \operatorname{br}(a)$ and we call it the *subbrace of $A$ generated by $a$*. If $A = \operatorname{br}(a)$ for an element $a \in A$, we say that $A$ is *one-generated*.

The general problem of describing the subbrace generated by an element $a$ of a brace $A$ seems to be complicated and, as a first step, we begin by describing one-generator left braces $A$ satisfying the condition $A^{(3)} = \{0\}$. Rump [27] gave a description of these braces by means of other algebraic structures like $q$-braces and cycle sets. We will recover his description with a different approach based on abelian groups that act on itself, so that they become modules for themselves, and a result of Stefanello and Trappeniers [29, Theorem 3.13], that states that the condition $A^{(3)} = \{0\}$ is equivalent to the fact that the map $\lambda\colon (A, +) \longrightarrow \operatorname{Aut}(A, +)$ is a group homomorphism and $A^2$ is contained in $\operatorname{Ker}(\lambda)$ by [29, Theorem 3.13].

**Proposition 3.1.** *Let us suppose that $(B, +)$ is an abelian group acting on itself by means of an action $\lambda\colon (B, +) \longrightarrow \operatorname{Aut}(B, +)$ given by $\lambda(x) = \lambda_x$ for $x \in B$. Let us suppose also that if $x$, $y \in B$, then $\lambda_{\lambda_y(x)} = \lambda_x$. We can define a product $\cdot$ on $B$ by means of $xy = x + \lambda_x(y)$ for $x$, $y \in B$. Then $(B, +, \cdot)$ is a left brace with $B^{(3)} = \{0\}$.*

The following construction, based on actions of an abelian group on itself, can be used to construct a left brace $A$ with $A^{(3)} = 0$. Again, its proof is based on [29, Theorem 3.13].

**Proposition 3.2.** *Let $(A, +, \cdot)$ be a left brace with $A^{(3)} = \{0\}$. Let $a \in A$. For $i \in \mathbb{Z}$, we define $a_i = \lambda_{a^i}(a) = -a^i + a^{i+1}$. In particular, $a_0 = -a^0 + a^1 = -0 + a = a$. We observe that $\lambda_{a^j}(a_i) = \lambda_{a^j}(\lambda_{a^i}(a)) = \lambda_{a^j a^i}(a) = \lambda_{a^{i+j}}(a) = a_{i+j}$. The set*

$$B = \Big\{\sum_{i\in\mathbb{Z}} x_i a_i \mid x_i \in \mathbb{Z}, i \in \mathbb{Z},\ \text{and } \{i \in \mathbb{Z} \mid x_i \neq 0\} \text{ is finite}\Big\}.$$

*coincides with the subbrace of $A$ generated by $a \in A$. Furthermore, if*

$$D = \Big\{\sum_{i\in\mathbb{Z}} x_i a_i \in B \mid \sum_{i\in\mathbb{Z}} x_i = 0\Big\},$$

*then $B * B = D$.*

The construction of a free one-generated left brace in the category of all left braces $A$ with $A^{(3)} = \{0\}$ follows from an application of Propositions 3.1 and 3.2.

**Proposition 3.3.** *Let $C$ be a free abelian group with basis $\{c_i \mid i \in \mathbb{Z}\}$. Given $x \in C$, then $x = \sum_{i\in\mathbb{Z}} x_i c_i$ with $x_i \in \mathbb{Z}$ for all $i \in \mathbb{Z}$ and $x_i \neq 0$ for a finite number of $i \in \mathbb{Z}$. We define an action of $x \in C$ on $C$ by means of $\lambda_x(y) = \sum_{i\in\mathbb{Z}} y_i c_{i+\sum_{j\in\mathbb{Z}} x_j} = \sum_{i\in\mathbb{Z}} y_{i-\sum_{j\in\mathbb{Z}} x_j} c_i$ for $y = \sum_{i\in\mathbb{Z}} y_i c_i$. Then $(C, +, \cdot)$ with the multiplication given by $xy = x + \lambda_x(y)$ for $x$, $y \in C$ is a left brace generated by $a = c_0$ with $C^{(3)} = \{0\}$ and $C^j \neq \{0\}$ for all $j \in \mathbb{N}$. Moreover, if $A$ is a left brace with $A^{(3)} = \{0\}$ and $b \in A$, then there exists a brace epimorphism $\alpha$ from $C$ and the subbrace of $A$ generated by $b$ such that $\alpha(c_0) = b$.*

Variations in the construction of Proposition 3.3 also give interesting left braces.

**Example 3.4.** *Let $C = \bigoplus_{n\in\mathbb{Z}} \langle c_n\rangle$, with $c_n$ of order $p$ for $n \in \mathbb{Z}$, be an elementary abelian $p$-group. Let $D = \mathbb{Z} \times C$. Given $x \in D$, then $x = (n, \sum_{i\in\mathbb{Z}} x_i c_i)$ with $n \in \mathbb{Z}$, $x_i \in \mathbb{F}_p$ for all $i \in \mathbb{Z}$ and $x_i \neq 0$ for a finite number of $i \in \mathbb{Z}$. We define an action of $D$ on $D$ by means of $\lambda_x(y) = \sum_{i\in\mathbb{Z}} y_i c_{i+\sum_{j\in\mathbb{Z}} x_j} = \sum_{i\in\mathbb{Z}} y_{i-\sum_{j\in\mathbb{Z}} x_j} c_i$ for $y = \sum_{i\in\mathbb{Z}} y_i c_i$. Then $(D, +, \cdot)$ with the multiplication given by $xy = x + \lambda_x(y)$ for $x$, $y \in D$ is a left brace generated by $a = c_0$ with $D^{(3)} = \{0\}$ by Proposition 3.1 and $D^j \neq \{0\}$ for all $j \in \mathbb{N}$. Moreover, if $A$ is a left brace with $A^{(3)} = \{0\}$ and $b \in A$, then there exists a brace epimorphism $\alpha$ from $D$ and the subbrace of $A$ generated by $b$ such that $\alpha(c_0) = b$.*

**Example 3.5.** *Consider the infinite cyclic group $\mathbb{Z}$, written aditively, and the additive group $C$ of the $p$-adic fractions. We can define on $D = \mathbb{Z} \times C$ and addition elementwise and an action $\lambda$ of $(D, +)$ on $(D, +)$ by means of $\lambda_{(n,u)}(m, v) = (m, p^n v)$. By Proposition 3.1, we construct a left brace $(D, +, \cdot)$ with $D^{(3)} = \{0\}$. By cardinality considerations, it cannot be a one-generated one-brace. Since the additive group of $D$ is a direct product of an infinite cyclic group and the additive group of $p$-adic fractions, it is minimax. The multiplicative group of $D$ turns out to be a semidirect product of the normal subgroup $C$ and the infinite cyclic group $\langle(1, 0)\rangle$. In particular, it is metabelian, minimax, and torsion free.*

## 4. Smoktunowicz-nilpotent one-generated left braces

We say that a left brace is *Smoktunowicz-nilpotent* or *nilpotent in the sense of Smoktunowicz* if there exist natural numbers $m$ and $n$ such that $A^{(m)} = A^n = \{0\}$, that is, when $A$ is simultaneously left and right nilpotent. These braces have been studied for the first time by Agata Smoktunowicz in [28]. For left braces, Smoktunowicz-nilpotency coincides with central nilpotency introduced in [9] (see also [17] for more details).

The same techniques used in the previous section to describe the subbrace generated by an element $a$ of a left brace $A$ can be used here. However, the left nilpotency of the brace imposes restrictions in the structure of the subbrace $S$ generated by $a$ that are not apparent from the description in Proposition 3.2. We present a variation of the description in Proposition 3.2 that takes into account the left nilpotency and gives more information about the subbraces $S^k$ for $k$ a natural number.

Some of the results will be stated in terms of generalised binominal coefficients of the form $\binom{n}{k}$, where $n$ is an integer that can be eventually negative and $k$ is a non-negative integer. They are defined as

$$\binom{n}{0} = 1, \qquad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{if } k > 0.$$

These generalised binomial coefficients satisfy some of the well-known properties of binomial coefficients, like the following ones:

**Lemma 4.1.** (1) *Let $n \in \mathbb{Z}$ and $k$ a non-negative integer. Then:*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

(2) *(Theorem of Chu-Vandermonde) Let $n$ and $t$ be integer numbers and $r$ a non-negative integer. Then*

$$\binom{n+t}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{t}{r-k}.$$

We review some of the properties of the star operation. In the absence of parentheses, the star operations will be computed before the additions.

**Lemma 4.2.** *(see, for instance,* [17, Lemma 2.1]*) Let $(A, +, \cdot)$ be a left brace. Then, for every $a$, $b$, $c \in A$, we have:*

(1) $a * (b + c) = a * b + a * c$.

(2) $(ab) * c = a * (b * c) + b * c + a * c$.

In the following statements, we will consider only left braces $A$ satisfying $A^{(3)} = \{0\}$.

**Proposition 4.3.** *Let $A$ be a left brace such that $A^{(3)} = \{0\}$. Then:*

(1) *For every $x \in A^2$, $y \in A$, we have that $x * y = 0$, that is, $xy = x + y = yx$. In particular, the multiplicative group of $A^2$ is abelian.*

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(2) *For every $x$, $y$, $z \in A$, we have that $(x+z)*y = x*(z*y) + x*y + z*y = (xz)*y$.*

**Corollary 4.4.** *Let $A$ be a left brace such that $A^{(3)} = \{0\}$. Then, for every $x$, $y \in A$, $b$, $c \in A^2$, we have that*

$$(x+b)*(y+c) = x*y + x*c.$$

*In particular, $(x+b)*y = x*y$.*

**Proposition 4.5.** *Let $A$ be a left brace and suppose that $A^{(3)} = \{0\}$. Let $a$ be an element of $A$. Define $a_1 = a$, $a_{j+1} = a*a_j$, $j \geq 1$. Then, for every positive integer $n$,*

$$a^n = \sum_{k=1}^{n} \binom{n}{k} a_k = \binom{n}{1} a_1 + \binom{n}{2} a_2 + \binom{n}{3} a_3 + \cdots + \binom{n}{n-1} a_{n-1} + \binom{n}{n} a_n.$$

**Proposition 4.6.** *Let $A$ be a left brace such that $A^{(3)} = A^{m+1} = \{0\}$. Define $a_1 = a$, $a_{j+1} = a*a_j$ for $1 \leq j \leq m-1$. Let $n$ be an integer. Then*

$$(na)*a_j = \sum_{k=1}^{m-j} \binom{n}{k} a_{k+j}.$$

**Theorem 4.7.** *Let $A$ be a left brace such that $A^{(3)} = A^{m+1} = \{0\}$ and let $a \in A$. Define $a_1 = a$, $a_{j+1} = a*a_j$ for $j \geq 1$. Then the set*

$$S = \left\{ \sum_{k=1}^{m} t_k a_k \mid t_k \in \mathbb{Z}, 1 \leq k \leq m \right\}$$

*is the subbrace of $A$ generated by $a$.*

*Moreover, for $1 \leq i \leq m$, we have that*

$$S^i = \left\{ \sum_{k=i}^{m} t_k a_k \mid t_k \in \mathbb{Z}, i \leq k \leq m \right\}$$

*and $S^i = 0$ for $i \geq m+1$.*

Our next step is the construction of a free one-generated left brace $B_m$ with $B_m^{(3)} = B_m^{m+1} = \{0\}$ for a natural number $m$. We will do it by means of the construction in Proposition 3.1.

We define in $B_m = \overbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}^{(m)}$ the addition in the usual form $(n_1, \dots, n_m) + (t_1, \dots, t_m) = (n_1 + t_1, \dots, n_m + t_m)$. Given $n = (n_1, \dots, n_m) \in B_m$ and $t = (t_1, \dots, t_m) \in B_m$, we define $\lambda_n(t) = (z_1, \dots, z_m)$ where, for $1 \leq j \leq m$,

$$z_i = \sum_{k=0}^{i-1} \binom{n_1}{k} t_{i-k}.$$

Clearly, $\lambda_n$ is a automorphism of $(B,+)$ for all $n \in B_m$ and the Chu-Vandermonde equality implies that $\lambda$ is a homomorphism from $(B,+)$ to $\mathrm{Aut}(B,+)$. As a consequence of Proposition 3.1, $B_m$ acquires a left brace structure with the multiplication given by $nt = n + \lambda_n(t)$. Furthermore, $B_m^{(3)} = B_m^{m+1} = \{\mathbf{0}\}$, where $\mathbf{0} = (0, \dots, 0)$, and for $1 \le i \le m$, we have that

$$B_m^i = \{(x_1, \dots, x_m) \in B_m \mid x_r = 0 \text{ for } 1 \le r < i\}.$$

Furthermore, this left brace is generated by $\mathbf{b} = (1, 0, \dots, 0)$. We have that this left brace satisfies the following universal property.

**Theorem 4.8.** *Suppose that $A$ is a left brace with $A^{(3)} = A^{m+1} = \{0\}$. Let $a \in A$ and consider the left brace $B_m$. Let us call $a_1 = a$, $a_{j+1} = a * a_j$ for $1 \le j \le m-1$, and $\mathbf{b} = (1, 0, \dots, 0) \in B$. The map $\alpha\colon B_m \longrightarrow A$ defined by*

$$\alpha(n_1, \dots, n_m) = \sum_{i=1}^{m} n_i a_i, \quad (n_1, \dots, n_m) \in B_m$$

*is a left brace homomorphism such that $\alpha(\mathbf{b}) = a$.*

## 5. Artinian left braces

Another interesting problem in the study of an algebraic structure is to describe when this structure satisfies the minimal or the maximal condition for a relevant subset of the lattice of its substructures. Let us denote by $\mathcal{L}(A)$ the family of all subbraces of a left brace $A$. Then $\mathcal{L}(A)$ is an ordered set with the inclusion that becomes a lattice where the meet operation is the intersection and the join operation between two subbraces corresponds to the subbrace generated by the union of both subbraces. Recall that an ordered set $M$ satisfies the *minimal condition* if every non-empty subset of $M$ has a minimal element. Moreover, an ordered set $M$ satisfies the *descending chain condition* if for every chain $a_1 \ge a_2 \ge \cdots \ge a_j \ge a_{j+1} \ge$ of elements of $M$ there is a natural number $k$ such that $a_k = a_{k+n}$ for all $n \in \mathbb{N}$. It is well known that the minimal condition and the descending chain condition are equivalent. By replacing $\ge$ by $\le$ and "minimal" by "maximal" we can dually define the *maximal condition* and the *ascending chain condition*.

In the case of a left brace $A$, if $\mathcal{S}$ is a subfamily of $\mathcal{L}(A)$, we say that $A$ satisfies the minimal condition for $\mathcal{S}$-subbraces (min-$\mathcal{S}$) if the family $\mathcal{S}$, ordered by inclusion, satisfies the minimal condition. If $\mathcal{S} = \mathcal{L}(A)$, we obtain the braces that satisfy the minimal condition for subbraces. We will call them *Artinian left braces*. We must warn the reader that the word "Artinian" has been used in a different way in [18] to denote (skew) left braces satisfying the minimal condition for ideals, instead of subbraces. We have considered that our use is more consistent with the use in other algebraic structures like groups and rings. We will see in Example 5.4 below that both properties are different.

In the case of groups, the typical example of group satisfying the minimal condition for subgroups is the *Prüfer $p$-group* for a prime $p$. It is generated by elements $a_n$, $n \in \mathbb{N}$, with the relations $a_1^p = 1$, $a_{n+1}^p = a_n$ for $n \in \mathbb{N}$ in multiplicative notation ($pa_1 = 0$, $pa_{n+1} = a_n$ for $n \in \mathbb{N}$ in additive notation). A group $G$ is a *Chernikov group* if it has a normal subgroup $D$ that is a direct product of finitely

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many Prüfer groups, such that $G/D$ is finite. They are the groups satisfying the minimal conditions on subgroups and having a normal abelian subgroup of finite index.

We call a left brace $A$ *weakly soluble* (see [5]) if $A$ has a finite series

$$\{0\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$$

of subbraces such that $A_j$ is an ideal of $A_{j+1}$ for all $j$ and every section $A_{j+1}/A_j$ is an abelian brace for $0 \leq j \leq n-1$.

**Theorem 5.1.** *Let $A$ be a weakly soluble left brace. Then $A$ is Artinian if, and only if, the additive and the multiplicative groups of $A$ are Chernikov groups. Conversely, if the additive group of $A$ is Chernikov, then $A$ satisfies the minimal condition for subbraces.*

Let $L$ be an ideal of a left brace $A$. We say that $L$ is $A$-*quasifinite* if every ideal $K$ of $A$ contained in $L$ and that does not coincide with $L$ is finite. The next natural step is to describe the left braces whose additive group is Chernikov.

**Theorem 5.2.** *Let $A$ be an infinite left brace whose additive group is Chernikov and let $D$ be the divisible part of the additive group of $A$. Then the following assertions hold:*

1. *$D$ is contained in the socle of $A$.*
2. *$D$ is an ideal of $A$. Moreover, $D = \bigoplus_{p\in\pi(D)} D_p$, where $D_p$ is the Sylow $p$-subgroup of the additive group of $D$ and $D_p$ is an ideal of $A$ for each prime $p \in \pi(D)$.*
3. *$D_p = Z_p + K_p$, where $Z_p$ and $K_p$ are ideals of $A$ such that $Z_p \subseteq \zeta(A)$, $A * K_p = K_p$, and the intersection $Z_p \cap K_p$ is finite. Moreover, if $p$ does not divide $|A/\mathrm{Ann}_A(D_p)|$, then $Z_p \cap K_p = \{0\}$. In particular, $D = Z + K$, where $Z$ and $K$ are ideals of $A$ such that the centre of $A$ contains $Z$, $A * K = K$, the intersection $Z \cap K$ is finite, and the additive groups of $Z$ and $K$ are divisible.*
4. *$D_p = L_{1,p} + \cdots + L_{t(p),p}$, where $L_{j,p}$ are $A$-quasifinite ideals of $A$ whose additive group is divisible and the intersections $L_{j,p} \cap (L_{1,p} + \cdots + L_{j-1,p} + L_{j+1,p} + \cdots + L_{t(p),p})$ are finite for $1 \leq j \leq t(p)$. Moreover, if $p$ does not divide $|A/\mathrm{Ann}_A(D_p)|$, then $D_p = L_{1,p} \oplus \cdots \oplus L_{t(p),p}$. In this case, $A/\mathrm{Ann}_A(D_p)$ can be embedded in a direct product $G_1 \times \cdots \times G_{t(p)}$, where $G_j$ is an irreducible subgroup of $\mathrm{GL}_{s(j)}(p)$, with $p^{s(j)} = |\Omega_1(L_{j,p})|$, $1 \leq j \leq t(p)$.*

In the case of commutative rings, there is a strong relation between the minimal and the maximal conditions: if a commutive ring $R$ satisfies the minimal condition for ideals, then it satisfies the maximal condition for ideals (see, for instance, [20, Section 4.5]). The situation in left braces is different.

**Example 5.3.** *Let $\langle a\rangle$ be a cyclic group of order 2 and let*

$$C = C_{2^\infty} = \langle \{c_n : n \in \mathbb{N} \mid 2c_1 = 0,\, 2c_{n+1} = c_n,\, n \in \mathbb{N}\}\rangle$$

*be a Prüfer 2-group, both written additively. We define and action of $D = \langle a\rangle \times C$ on itself by setting $\lambda_{(0,u)}$ the identity of $D$ for all $u \in C$, $\lambda_{(a,u)}(0,v) = (0,-v)$ for $u$, $v \in C$, and $\lambda_{(a,u)}(a,v) = (a, c_1 - v)$ for $u$, $v \in C$. Then $\lambda_x\colon D \longrightarrow D$ is an automorphism of $D$ for all $x \in D$. It is straightforward to check that $\lambda\colon (D,+) \longrightarrow \mathrm{Aut}(D,+)$ is a group homomorphism. Furthermore, consider $x$, $y \in D$. The automorphism $\lambda_x$ depends only on the first component of $x$. Since the first component of $\lambda_y(x)$ coincides with the first component of $x$, we conclude that $\lambda_x = \lambda_{\lambda_y(x)}$. By Proposition 3.1, there is a product $\cdot$ in $D$ for which $(D,+,\cdot)$ becomes a left brace for which the lambda map is $\lambda$. By Theorem 5.1, since the additive group $(D,+)$ is Chernikov, the brace $(D,+,\cdot)$ satisfies the minimal condition for subbraces.*

*Recall that $x * y = \lambda_x(y) - y$. Since*

$$(0,u)*(0,v) = (0,v)-(0,v) = (0,0), \qquad (a,u)*(0,v) = (0,-v)-(0,v) = (0,-2v)$$

$$(0,v)*(a,v) = (a,v)-(a,v) = (0,0), \qquad (a,u)*(a,v) = (a,c_1-v)-(a,v) = (0,c_1-2v),$$

*and $C$ is a divisible 2-group, we have that $D*D = C*D = C$ and $C$ is an abelian ideal of $D$.*

*If $k$ is a natural number and $0 \le s \le 2k$, then*

$$(a,u)*(0,sc_k) = (0,-2sc_k) = (0,sc_k)*(a,u).$$

*We conclude that $\Omega_k(C)$ is an ideal of $D$ for each natural number $k$. Thus we have an ascending series*

$$\Omega_1(C) < \Omega_2(C) < \Omega_3(C) < \cdots < \Omega_k(C) < \Omega_{k+1}(C) <$$

*This series shows that the left brace $D$ does not satisfy the maximal condition for ideals. In particular, it does not satisfy the maximal condition for subbraces.*

Finally, we show that there are left braces that satisfy the minimal condition for ideals, as in [18], but do not satisfy the minimal condition for subbraces.

**Example 5.4.** *Let $\langle a\rangle$ be a finite cyclic group of order $n$ and $C$ a $\mathbb{Z}\langle a\rangle$-module. Denote the result of the action of $g \in \langle a\rangle$ on $u \in C$ by $a \bullet u$. Suppose also that $\mathrm{C}_C(g) = \{0\}$ for every element $g \in \langle a\rangle \setminus \{1\}$.*

*Choose an element $c_0 \in C \setminus \{0\}$. Let $D = D(a, C, c_0) = \langle a\rangle \times C$, where the addition on $D$ is defined as*

$$(a^k,u) + (a^t,v) = (a^{k+t}, u+v), \qquad ,0 \le k,t \le n-1.$$

*It is clear that $(D,+)$ is an abelian group. We define an action of $(D,+)$ on itself by means of*

$$\lambda_{(a^k,u)}(a^t,v) = (a^t, t(c_0 + c_1 + \cdots + c_{k-1}) + a^k \bullet v),$$

*where $c_j = a^j \bullet c_0$ for $0 \le j \le n-1$. We note that*

$$\begin{aligned} c_0 + c_1 + \cdots + c_{n-1} &= c_0 + a \bullet c_0 + a^2 \bullet c_0 + \cdots + a^{n-1} \bullet c_0 \\ &= (1 + a + a^2 + \cdots + a^{n-1}) \bullet c_0 \end{aligned}$$

*belongs to* $\mathrm{C}_C(a)$*, which implies that* $c_0 + c_1 + \cdots + c_{n-1} = 0$*. With this fact, we see that* $\lambda_{(a^k,u)}$ *is an automorphism of* $(D,+)$*, that* $\lambda\colon (D,+) \longrightarrow \mathrm{Aut}(D,+)$ *is a group homomorphism, and that* $\lambda_{(a^k,u)} = \lambda_{\lambda_{(a^r,w)}(a^k,u)}$ *for all* $0 \le k, r \le n-1$*,* $u$*,* $v \in C$*. We conclude by Proposition 3.1 that* $D$ *acquires a structure of left brace with the product given by* $(a^k,u)(a^t,v) = (a^k,u) + \lambda_{(a^k,u)}(a^t,v)$ *satisfying* $D^{(3)} = \{0\}$*.*

*Let us particularise this construction to a simple* $\mathbb{F}_pG$*-module for* $G = C_{3^\infty}$*, the Prüfer quasicyclic* 3*-group and* $p > 3$ *a prime. By* [19, Corollary 2.4] *or* [15, Theorem 2.10]*, there exists a simple* $\mathbb{F}_pG$*-module* $C$*. Note that the additive group of* $C$ *is an infinite elementary abelian* $p$*-group. Consider a non-zero element* $c_{01}$ *of* $C$*. Let* $D_1 = D(a_1, C, c_{01}) = \langle a_1\rangle \times C$ *be the left brace constructed above. Given an element* $u \in C$ *we construct the left brace* $D(a_2, C, u)$ *on the direct product* $\langle a_2\rangle \times C$*. Now we choose an element* $u$ *in the following way. Consider* $J = \mathbb{F}_p\langle a_2\rangle$*. Since* $\langle a_2\rangle$ *is a finite cyclic subgroup, the* $J$*-submodule of* $C$ *generated by* $c_{01}$ *is finite. Then it contains a simple* $J$*-module* $W$*. We have that* $C = \bigoplus_{x\in S} Wx$ *for some subset* $S$ *of* $G$ *(see, for example,* [19, Lemma 5.4]*). Since* $G$ *is abelian,* $\mathrm{Ann}_J(Wx) = \mathrm{Ann}_J(W)$ *for every* $x \in S$*. Then the* $J$*-submodule of* $C$ *generated by* $c_{01}$ *is isomorphic to* $W$ *as a* $J$*-module by* [10, Lemma 4]*. In particular, we obtain that the* $J$*-submodule of* $C$ *generated by* $c_{01}$ *is simple, in other words, it coincides with* $W$*. We have that* $W$ *is* $J$*-isomorphic to* $J/\mathrm{Ann}_J(c_{01})$*, where* $\mathrm{Ann}_J(c_{01}) = Jf(x)$ *for a polynomial* $f$ *irreducible over* $\mathbb{F}_p$*. There is a polynomial* $h$ *such that* $(x^2 + x + 1)h(x) \in 1 + J$*. Set* $u = (1/3)h(a_2)c_{01} = c_{02}$*. We have that* $(a_2, 0) * (a_2, 0) = (1, c_{02})$*. We can check that* $(a_1, 0) * (a_1, 0) = (1, c_{01})$*. We conclude that the left brace* $D(a_1, C, c_{01})$ *is a subbrace of* $D(a_2, C, c_{02})$*.*

*With similar arguments, we can construct an ascending chain of left braces*

$$D(a_1, C, c_{01}) \subseteq D(a_2, C, c_{02}) \subseteq \cdots \subseteq D(a_n, C, a_{0n}) \subseteq D(a_{n+1}, C, c_{0,n+1}) \subseteq$$

*Let* $D$ *be the union of all these left braces. We have that* $C * D = \{0\}$*,* $D^{(3)} = \{0\}$*,* $D^{(2)} \subseteq C \subseteq \mathrm{Soc}(D)$*. We note that every* $\mathbb{F}_pG$*-submodule of* $C$ *is an ideal of* $D$*. Since* $C$ *is a simple* $\mathbb{F}_p(G)$*-module,* $D^{(2)} = C$ *is a minimal ideal of* $D$*. The additive group of* $D/C$ *is a quasicyclic* 3*-group, in particular, it satisfies the minimal condition for subbraces. Hence* $D$ *satisfies the minimal condition on ideals. However, the condition* $D^{(3)} = \{0\}$ *implies that* $D^{(2)}$ *is an abelian ideal of* $D$*, in particular, every subgroup of* $D^{(2)}$ *is a subbrace. Since* $D^{(2)}$ *is an infinite elementary abelian* $p$*-group, it does not satisfy the minimal condition for subbraces.*

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**Adolfo Ballester-Bolinches**

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100, Burjassot, València, Spain

Email: Adolfo.Ballester@uv.es

**Ramón Esteban-Romero**

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100, Burjassot, València, Spain

Email: Ramon.Esteban@uv.es

**Leonid A. Kurdachenko**

Department of Algebra and Geometry, Oles Honchar Dnipro National University, Dnipro 49010, Ukraine

Part of this research has been done during some visits of this author to the Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100, Burjassot, València, Spain

Email: lkurdachenko@gmail.com

**Vicent Pérez-Calabuig**

Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, 46100, Burjassot, València, Spain

Email: Vicent.Perez-Calabuig@uv.es

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