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A LATTICE THEORETIC CHARACTERIZATION FOR THE EXISTENCE OF A FAITHFUL IRREDUCIBLE REPRESENTATION

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Abstract. In a recent article Sébastien Palcoux formulated a sufficient condition on the subgroup lattice of a finite group G that guarantees the existence of a faithful irreducible complex representation of G, and asked whether his condition is also necessary. In this short note we give an affirmative answer using Kochendörffer's criterion for the existence of a faithful irreducible representation based on the structure of the socle of G.

1. Introduction

Sébastien Palcoux [4] gave a new sufficient condition for a finite group G that guarantees the existence of a faithful irreducible complex representation of G, and asked whether his condition is also necessary [4, Question 4.4]. We give an affirmative answer, so Palcoux's condition adds to the list of various properties equivalent to the existence of a faithful irreducible representation, i.e., describing abstract properties of irreducible linear groups. The search for such properties started with Burnside [1, p. 476–478] more than a hundred years ago. Later contributions were made by Shoda [7], Weisner [8], Kochendörffer [3], Gaschütz [2], and Zhmud' [9]. Our proof will be based on Kochendörffer's criterion (see Theorem 1.4 below).

Palcoux relates the problem to the subgroup lattice. At the first glance this approach does not look promising, because there exist pairs of groups with isomorphic subgroup lattice, such that one of them

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has, while the other does not have faithful irreducible representation. For example, the dihedral group of order 6 and the elementary abelian group of order 9 constitute such a pair of groups. It is also possible to find groups with isomorphic lattice of normal subgroups, but behaving differently with respect to the existence of faithful irreducible representations (see Proposition 3.1). However, as we will see, Palcoux's condition is not purely lattice theoretic. Namely, he assumes that a subgroup H in the assumptions of his theorem is core-free, and this property is not reflected by the subgroup lattice. Recall that a subgroup $H < G$ is called core-free if it does not contain any non-trivial normal subgroup of G, i.e., if $Core_G(H) = \bigcap_{g\in G} g^{-1}Hg = 1$.

First we need some definitions from lattice theory. Let L be a finite lattice, and denote by 1_L the largest element of L. For $x, y \in L$ with $x \le y$ define the interval $Int(x, y) = \{z \mid x \le z \le y\}$. We say that y covers x, if $Int(x, y)$ is a 2-element lattice. For any $x \ne 1_L$ we denote by x^* the join of all covers of x, and we will call $Int(x, x^*)$ the socle of the interval $Int(x, 1_L)$. A finite lattice is called Boolean iff it is a power of the 2-element lattice.

Theorem 1.1. [4, Theorem 3.13] *If G contains a core-free subgroup H such that the interval $Int(H, H^*)$ in the subgroup lattice of G is Boolean, then G has a faithful irreducible complex representation.*

The goal of this note is to prove the converse of this statement.

Theorem 1.2. *If a finite group G has a faithful irreducible complex representation, then there exists a core-free subgroup $H < G$ such that the interval $Int(H, H^*)$ in the subgroup lattice of G is Boolean.*

Our proof is based on Kochendörffer's criterion that uses the structure of the socle of G. Let us recall that $Soc(G)$, the socle of G is the normal subgroup generated by all minimal normal subgroups of G. Two minimal normal subgroups are considered G-isomorphic, if there is an isomorphism between them that commutes with the conjugation by any element of G. If M is a non-abelian minimal normal subgroup of G, then no other minimal normal subgroup is G-isomorphic to M. We will denote by $M_1, \dots, M_s$ the non-abelian minimal normal subgroups of G. Let $\mathcal{A}$ be the set of all abelian minimal normal subgroups of G, and let $\mathcal{A}_1, \dots, \mathcal{A}_r$ be the G-isomorphism classes in $\mathcal{A}$. Set $S_i = \langle M \mid M \in \mathcal{A}_i\rangle$ for $1 \le i \le r$. Finally, choose one abelian minimal normal subgroup A_i from $\mathcal{A}_i$ for each $1 \le i \le r$. Then A_i is an elementary abelian p_i-group for some prime number p_i. Moreover, G acts irreducibly on A_i, hence the endomorphism ring $F_i = \mathrm{End}_G(A_i)$ is a finite field and A_i is a vector space over F_i. Let $d_i = \dim_{F_i}(A_i)$. Now G acts completely reducibly on S_i, so S_i decomposes into the direct product of some minimal normal subgroups of G, each one G-isomorphic to A_i. Denote the number of factors in this direct decomposition by n_i. The invariants d_i and n_i defined here will play the crucial role in Kochendörffer's criterion.

The following classical result of Remak [6] describes the structure of the socle.

Lemma 1.3. *Let $N \le Soc(G)$ be a normal subgroup of G.*

(i) *We have $N = (N \cap S_1) \times \cdots \times (N \cap S_r) \times (N \cap M_1) \times \cdots \times (N \cap M_s)$.*

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(ii) *If $N \leq S_i$ for some $1 \leq i \leq r$, then N is the direct product of some minimal normal subgroups, each G-isomorphic to A_i. Furthermore, $N \mapsto \mathrm{Hom}_G(A_i, N)$ is a lattice isomorphism between $\{N \leq S_i \mid N \lhd G\}$ and the subspace lattice of the n_i-dimensional vector space $\mathrm{Hom}_G(A_i, S_i)$ over F_i.*

(iii) *If $N \leq M_i$ for some $1 \leq i \leq s$, then either $N = M_i$ or $N = 1$.*

Concerning (i), (iii), and the first assertion in (ii) we refer to the paper of Remak [6], in particular Theorems 3a, 5, and 1, respectively. For the last assertion of (ii) see [5, Lemma 1(ii)]. Using the previously introduced notation, we can formulate the following necessary and sufficient condition for the existence of a faithful irreducible complex representation of G. (Notice, however, that the original result is more general, it is not restricted to representations over the field of complex numbers.)

Theorem 1.4. [3, Satz 5] *A finite group G possesses a faithful irreducible representation over the complex field if and only if the invariants of its socle satisfy $n_i \leq d_i$ for each $1 \leq i \leq r$, i.e., the multiplicity of a G-isomorphism type of an abelian minimal normal subgroup A_i in the direct decomposition of the socle of G does not exceed the dimension of A_i over its G-endomorphism ring F_i.*

Another proof of this theorem that utilizes the Möbius function of the lattice of normal subgroups and uses only very basic representation theory was given in [5].

2. Proof of the necessity of Palcoux's criterion

First we show three simple lemmas that we will use in our proof.

Lemma 2.1. *Let V be an irreducible $\mathbb{F}_pG$-module, $F = End_{\mathbb{F}_pG}(V)$, and let $W = V_1 \oplus \cdots \oplus V_n$ be the direct sum of n copies of V. If $n \leq \dim_F(V)$, then there exists a subgroup U of the additive group of the module W such that U does not contain any nontrivial submodule of W and $|W : U| = p$.*

Proof. Let $x \cdot y$ be a non-degenerate F-bilinear form on V, and $\mathrm{Tr} : F \to \mathbb{F}_p$ the trace function. Let us choose $v_1, \ldots, v_n \in V$ linearly independent vectors over F, and consider the map $\psi : W \to \mathbb{F}_p$ defined by $\psi(x_1, \ldots, x_n) = \mathrm{Tr}(\sum_{i=1}^n v_i \cdot x_i)$. Let $U = \mathrm{Ker}(\psi)$. Obviously, $|W : U| = p$. Any minimal submodule of W has the form $\{(\alpha_1 x, \ldots, \alpha_n x) \mid x \in V\}$ for some $\alpha_1, \ldots, \alpha_n \in F$, not all zero. Such a submodule cannot be contained in U, since $\mathrm{Tr}(\sum_{i=1}^n v_i \cdot \alpha_i x) = \mathrm{Tr}((\sum_{i=1}^n \alpha_i v_i) \cdot x)$ is not identically zero. □

Lemma 2.2. *Let W be a completely reducible $\mathbb{F}_pG$-module with homogeneous components $W_1, \ldots, W_m$. Assume that each W_j $(j = 1, \ldots, m)$ contains a subgroup U_j such that U_j does not contain any nontrivial submodule of W_j and $|W_j : U_j| = p$. Then there exists a subgroup U in W such that U does not contain any nontrivial submodule of W and $|W : U| = p$.*

Proof. Take surjective additive maps $\psi_j : W_j \to \mathbb{F}_p$ with $\mathrm{Ker}(\psi_j) = U_j$, let $\psi : W \to \mathbb{F}_p$ be defined by $\psi(\sum_{j=1}^m w_j) = \sum_{j=1}^m \psi_j(w_j)$ and let $U = \mathrm{Ker}(\psi)$. Since any minimal submodule of W is contained in one of the homogeneous components W_j, the result follows. □

Lemma 2.3. *Let T be a nonabelian simple group, $R < T$ a maximal subgroup, and L an arbitrary group. Then every subgroup of $T \times L$ that contains R has the form either $R \times X$ or $T \times X$ for some subgroup $X \leq L$.*

Proof. Let $H \leq T \times L$ be a subgroup containing R. By the maximality of R, the image of H at the first projection is either R or T. Denote by X the image of H at the second projection. If $H \leq R \times X$, then obviously $H = R \times X$. In the other case H is a subdirect product of the simple group T and X. Since the kernel of the second projection is non-trivial, as it contains R, this kernel must be T, hence $H = T \times X$. □

Proof of Theorem 1.2. Assume that G has a faithful irreducible complex representation. Then Theorem 1.4 yields $n_i \leq d_i$ for each $i = 1, \ldots, r$. This property of the invariants of $Soc(G)$ is what we will use in the proof.

We now take another form of the decomposition of the socle of G. Remember, that every abelian component S_i of the socle is an elementary abelian p_i-group for some prime p_i ($i = 1, \ldots, r$). Some of these prime numbers can be equal, so we introduce the notation $\pi = \{p_1, \ldots, p_r\}$, where this set may have less than r elements. For $p \in \pi$ let $W^{(p)} = O_p(Soc(G))$ be the product of those S_i's that are p-groups (that is, $p_i = p$). Then

$$Soc(G) = \prod_{p \in \pi} W^{(p)} \times \prod_{i=1}^{s} M_i.$$

Since $n_i \leq d_i$ for each $i = 1, \ldots, r$, Lemmas 2.1 and 2.2 yield a subgroup $U^{(p)} < W^{(p)}$ such that $U^{(p)}$ is core-free in G and $|W^{(p)} : U^{(p)}| = p$. Every non-abelian minimal normal subgroup M_i ($1 \leq i \leq s$) is a direct product of (pairwise isomorphic) simple groups: $M_i = T_{i1} \times \cdots \times T_{it_i}$ ($t_i \geq 1$). For each $i = 1, \ldots, s$ let us choose a maximal subgroup R_i in the simple group T_{i1} and let $U_i = R_i \times T_{i2} \times \cdots \times T_{it_i}$. Then U_i is a maximal subgroup in M_i and it is core-free in G. Let

$$U = \prod_{p \in \pi} U^{(p)} \times \prod_{i=1}^{s} U_i.$$

By its construction U is a core-free subgroup in G. Let H be a maximal core-free subgroup in G containing U. Our goal is to show that the interval $Int(H, H^*)$ is Boolean.

Before proving that, we consider the interval $Int(U, Soc(G))$. Let $U \leq X \leq Soc(G)$. Define $X^{(p)} = X \cap W^{(p)}$ (for $p \in \pi$) and $X_i = X \cap M_i$ (for $i = 1, \ldots, s$). Clearly, $X^{(p)} \geq U \cap W^{(p)} = U^{(p)}$, and since $|W^{(p)} : U^{(p)}| = p$, we have either $X^{(p)} = U^{(p)}$ or $X^{(p)} = W^{(p)}$. Similarly, by the maximality of U_i in M_i, either $X_i = U_i$ or $X_i = M_i$ holds. Repeated application of Lemma 2.3 yields that $X = (X \cap \prod_{p \in \pi} W^{(p)}) \times \prod_{i=1}^{s} X_i$. Since $\prod_{p \in \pi} W^{(p)} / \prod_{p \in \pi} U^{(p)}$ is a cyclic group of square-free order $\prod_{p \in \pi} p$, it follows that

$$X = \prod_{p \in \pi} X^{(p)} \times \prod_{i=1}^{s} X_i,$$

where $X^{(p)}$ is either $U^{(p)}$ or $W^{(p)}$ and X_i is either U_i or M_i. Thus $Int(U, Soc(G))$ is Boolean. Furthermore, we see that the core of X is the direct product of those $X^{(p)}$'s and X_i's that are equal to $W^{(p)}$ or M_i, respectively. Hence we have

$$X = Core_G(X)U.$$

Now we show that H^*, the join of the covers of H, is equal to $Soc(G)H$. On the one hand, if $K > H$ is a cover of H, then by the maximal choice of H, the core of K is non-trivial, so K contains a minimal normal subgroup N. Now $H < NH \leq K$, hence $K = NH \leq Soc(G)H$ holds for any cover K of H, hence $H^* \leq Soc(G)H$. On the other hand, for any minimal normal subgroup $N \lhd G$ we see that NH covers H, since $N \cap H$ is a maximal subgroup of N. (Namely, if $N \leq W^{(p)}$, then it has index p in N, and if $N = M_i$, then it is U_i.) Therefore $Soc(G)H \leq H^*$ also holds.

Finally, we prove that the intervals $Int(H, H^*)$ and $Int(U, Soc(G))$ are isomorphic, so $Int(H, H^*)$ is indeed Boolean. We show that the two obvious maps between these two intervals are inverses of each other, and since they are trivially order-preserving, they are isomorphisms of the intervals. Every subgroup $X \in Int(U, Soc(G))$ will be mapped to $XH = Core_G(X)UH = Core_G(X)H \leq Soc(G)H = H^*$. Conversely, every subgroup $Y \in Int(H, H^*)$ is mapped to $Y \cap Soc(G) \geq H \cap Soc(G) = U$. On the one hand, for $X \in Int(U, Soc(G))$ we get

$$XH \cap Soc(G) = Core_G(X)H \cap Soc(G) = Core_G(X)(H \cap Soc(G)) = Core_G(X)U = X.$$

On the other hand, for $Y \in Int(H, H^*)$ we obtain $(Y \cap Soc(G))H = Y \cap Soc(G)H = Y$. So the two maps are inverses of each other, as we have claimed. □

3. Some groups with isomorphic lattice of normal subgroups

In the Introduction we mentioned that the existence of a faithful irreducible representation cannot be decided by looking at the lattice of normal subgroups. Now we give a construction that provides examples showing this.

Proposition 3.1. *There exist pairs of groups with isomorphic lattice of normal subgroups such that one of them has a faithful irreducible representation while the other one does not have such a representation.*

Proof. Let S be a finite nonabelian simple group with a nontrivial absolutely irreducible representation on the vector space V over a field $\mathbb{F}_p$ (p a prime). Let $V_1, \dots, V_k$ be isomorphic copies of the S-module V, and let $A = V_1 \oplus \cdots \oplus V_k$ be a homogeneous S-module. Let us form the semidirect product $G = A \rtimes S$. Clearly, a subgroup of A is normal in G iff it is an S-submodule. Since the representation of S is absolutely irreducible, the lattice of S-submodules of A is isomorphic to the lattice of subspaces of $\mathbb{F}_p^k$ (see Lemma 1.3(ii)). Furthermore, if N is a normal subgroup of G not contained in A, then $A < NA \leq G$, so by the simplicity of $G/A \cong S$, we have $NA = G$, thus $N \geq N[N, A][A, A] = N[G, A] = NA = G$. So the lattice of normal subgroups of G is isomorphic to the lattice of subspaces of $\mathbb{F}_p^k$ with a new largest element added. Notice that this lattice does not depend on the dimension of V. However, for the existence of a faithful irreducible representation we need $k \leq \dim V$. Hence, if we have two absolutely irreducible representations of S over $\mathbb{F}_p$ that have different dimensions, than choosing k to be equal to the larger dimension, if we start with one of the representations of S, then this construction yields a group having a faithful irreducible representation, while the other representation of S will lead to a group without a faithful irreducible representation. However, the lattice of normal subgroups is the same (up to isomorphism) for both groups. □

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