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A STUDY ON THE STRUCTURE OF FINITE GROUPS WITH $c-$SUBNORMAL SUBGROUPS

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Abstract. In this paper, we use the definition of the concept "$c-$Subnormal Subgroup" to study the structure of a given finite group G which contains some $c-$subnormal subgroups. We prove two main theorems, which answer the question of what conditions must hold so that G is an element in a formation $\mathfrak{U}$ of supersoluble groups. Finally, we state many previous results which can be considered as special cases of these theorems.

1. Introduction

Throughout this paper, by G we mean any group with finite order. Recall that: A formation $\mathfrak{F}$ is a class of groups which is closed under homomorphic images and subdirect products, particularly, $\forall\, G \in \mathfrak{F}$, G has a smallest normal subgroup denoted by $G^{\mathfrak{F}}$ and whose quotient is still in $\mathfrak{F}$, a formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ then $G \in \mathfrak{F}$, and a formation $\mathfrak{F}$ is said to be hereditary if $H \leq G \in \mathfrak{F}$ then $H \in \mathfrak{F}$. In this paper, we use $\mathfrak{U}$ to denote the formation of the supersoluble groups. A chief factor H/K of a group G is called central if $G = C_G(H/K)$. The symbol $Z_\infty(G)$ denotes the hypercenter of a group G that is the product of all such normal subgroups H of G whose G-chief factors are central.

Keywords: Saturated formation, $c-$subnormal subgroup, sylow subgroup, supersoluble group.

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Following [1], we say that a subgroup H of a group G is c-normal in G if there exists a normal subgroup T in G such that $G = HT$ and $T \cap H \leq H_G$, where H_G is the largest normal subgroup of G contained in H.

Several authors have investigated the structure of a group G under the assumption that the maximal or the minimal subgroups of the Sylow subgroups of some subgroups of G are well situated in G. [2] proved that a group of odd order is supersoluble if its minimal subgroups are normal. Later on, [3] showed that a group G is supersoluble if it has a normal subgroup N with supersoluble quotient G/N such that all maximal subgroups of the Sylow subgroups of N are normal in G. [4] proved that: If G is a soluble group and all maximal subgroups of any Sylow subgroup of $F(G)$ are normal in G, then G is supersoluble. [1] generalized the results from [2] by replacing normality with c-normality. [5] obtained the same results by assuming the c-normality of the maximal or minimal subgroups of the fitting subgroup of a soluble group. By using the theory of formations, [6] extended further the results to a saturated formation containing the class of supersoluble groups $\mathcal{U}$. The most general results in this trend was the following theorem:

Theorem 1.1. [7] *Suppose that $\mathcal{F}$ is a saturated formation containing $\mathcal{U}$ and G is a group with a normal subgroup E such that $G/E \in \mathcal{F}$, then*

(1) *if all maximal subgroups of any Sylow subgroups of $F^*(E)$ are c-normal in G , then $G \in \mathcal{F}$.*
(2) *If all minimal subgroups and all cyclic subgroups of $F^*(E)$ are c-normal in G, then $G \in \mathcal{F}$.*

In this paper, we use the definition of the concept "c-subnormal subgroup", which is considered weaker than the concept of "c-normal subgroup", to prove some similar interesting results proved previously on finite groups which contain c-normal subgroups.

Definition 1.2. [8] *Suppose that $H \leq G$, if there exists a subnormal subgroup T of G such that $G = HT$, and $T \cap H \subseteq H_G$, then H is called a $c-$subnormal subgroup of G.*

It is obvious that if H is a c-normal subgroup in G, then H is a $c-$subnormal subgroup of G, while the converse is in general need not be true as shown in the following example:

Example 1.3. *For $m \geq 3$, let $P = M_m(2) = \langle x, y | x^{2^{m-1}} = y^2 = 1, x^y = x^{1+2^{m-2}} \rangle$, and $H = \langle y \rangle$. Clearly H is a c-subnormal subgroup in G, from [9], $\exists$ a 2-group G' such that $P \leq G'$, therefore, $B \leq \Phi(G)$, and the only supplement of H in G is G itself, because H is not a normal subgroup in G, therefore, H is not a c-normal subgroup in G.*

2. Materials and Methods

The used methodology in this paper depends on stating some previous definitions and Lemmas together with their references, and constructing some useful Lemmas with their proofs, then used them later to prove the main results.

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Lemma 2.1. [10] *Suppose that X is a subgroup of G, $\mathcal{F} \neq \Phi$ is a saturated formation and $Z = Z_\infty^{\mathcal{F}}(G)$ is the $\mathcal{F}-$hypercenter of the group G, then*

(1) *If $X \lhd G$, then $XZ/X \leq Z_\infty^{\mathcal{F}}(G/X)$.*
(2) *If $\mathcal{F}$ is hereditary, then $Z \cap X \leq Z_\infty^{\mathcal{U}}(X)$, where $Z_\infty^{\mathcal{U}}(X)$ is the $\mathcal{U}-$hypercenter of the group X.*

In Lemma 2.2 and Lemma 2.3, we state and prove some results on $c-$subnormal subgroups. These results are similar to the previous results proved in [10] but with $\mathcal{F}_c$-normal subgroups, where the saturated formation $\mathcal{F}$ contains the class of all supersoluble groups.

Lemma 2.2. *If $M \leq K \leq G$, then*

(1) *M is a $c-$subnormal subgroup in G if and only if G has a normal subgroup T such that $G = MT$, $M_G \leq T$, and $(M/M_G) \cap (T/M_G) \leq Z_\infty(G/M_G)$.*
(2) *If $M \lhd G$, then K/M is a $c-$subnormal subgroup of G/M if and only if K is a $c-$subnormal subgroup of G.*
(3) *If M is a normal subgroup of G, then for any $c-$subnormal subgroup E of G which satisfies the condition $(|M|, |E|) = 1$, the subgroup ME/M is a $c-$subnormal subgroup of G/M.*
(4) *If M is a c-subnormal subgroup in G, then M is a $c-$subnormal subgroup in K.*
(5) *If M is a $c-$subnormal $p-$subgroup in G, and $M/M_G \nsubseteq Z_\infty(G/M_G)$, then $\exists\, X \lhd G$ with $|G : X| = p$, and $G = MX$.*

Proof. **(1)** Suppose that M is a $c-$subnormal subgroup of G, $T \lhd G$ with $MT = G$, and $(T \cap M)M_G/M_G \leq Z_\infty(G/M_G)$. If $T_0 = TM_G$, then $T_0 \lhd G$, and by Lemma 2.1, $T_0/M_G \cap M/M_G = (T_0 \cap M)/M_G = (T \cap M)M_G/M_G \leq Z_\infty(G/M_G)$.
(2) If we suppose that K/M is a $c-$subnormal subgroup in G/M, by (1), G/M must have a normal subgroup T/M where $(K/M)(T/M) = G/M$, $(K/M)_{G/M} \leq T/M$, and $(T/M)/(K/M)_{G/M} \cap (K/M)/(K/M)_{G/M} \leq Z_\infty((G/M)/(K/M)_{G/M})$, then T is a normal subgroup of G, and $G = KT$, because $(T/M)/(K/M)_{G/M} \cap (K/M)/(K/M)_{G/M} = ((T/M)/(K_G/M)) \cap ((K/M)/(K_G/M)) = ((T \cap K)/M)/(K_G/M)$, and $Z_\infty((G/M)/(K/M)_{G/M}) = Z_\infty((G/M)/(K_G/M))$, we get $(T \cap K)/K_G = (T/K_G) \cap (K/K_G) \leq Z_\infty(G/K_G)$. therefore, K is a $c-$subnormal subgroup in G. Similarly if K is a $c-$subnormal subgroup of G, then K/M is a $c-$subnormal subgroup in G/M.
(3) Suppose that E is a $c-$subnormal subgroup of G, by (1), $\exists\, T \lhd G$ so that $G = ET$, $E_G \leq T$, and $(E/E_G) \cap (T/E_G) \leq Z_\infty(G/E_G)$. Now, we will show that ME/M is a $c-$subnormal subgroup in G/M, by (2) it is enough to prove that ME is a $c-$subnormal subgroup of G. Because $(|M|, |E|) = 1$, $M \leq T$, we have, $T \cap ME = M(T \cap E) \leq MZ$ where $Z/E_G = Z_\infty(G/E_G)$, from the G-isomorphism $MZ/ME_G = ME_GZ/ME_G \simeq Z/Z \cap ME_G = Z/E_G(Z \cap M)$ we have $MZ/ME_G \leq X/ME_G = Z_\infty(G/ME_G)$, therefore, $(ME \cap T)/ME_G \leq X/ME_G$. Let $D = (ME)_G$, by Lemma 2.2, $(X/ME_G)(D/ME_G)/(D/ME_G) \leq Z_\infty((G/ME_G)/(D/ME_G))$, therefore, $(TD/D) \cap (ME/D) = D(T \cap ME)/D \leq Z_\infty(G/D)$, and ME is

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a $c-$subnormal subgroup in G.

(4) Assume that $T \lhd G$, $MT = G$, $M_G \leq T$, $(T/M_G) \cap (M/M_G) \leq Z_\infty(G/M_G)$, and $T_1 = M_G(K \cap T)$. Since $K = K \cap MT = M(K \cap T)$, we get $K = MT_1$. It is clear that $T_1 \lhd K$, Moreover, $(M/M_G) \cap (T_1/M_G) = M_G(M \cap T \cap K)/M_G \leq Z/M_G = Z_\infty(G/M_G) \cap K/M_G$, by Lemma 2.1, $Z/M_G \leq Z_\infty(K/M_G)$, and $(Z/M_G)(M_K/M_G)/(M_K/M_G) \leq Z_\infty(K/M_G)/$
$(M_K/M_G))$, therefore, $(T_1/M_K) \cap (M/M_K) \leq Z_\infty(K/M_K)$, and M is a $c-$subnormal subgroup in K.

(5) Since M is a $c-$subnormal subgroup of G, $\exists T \lhd G$ such that $MT = G$, and $(M \cap T)M_G/M_G = (M \cap TM_G)/M_G \leq Z_\infty(G/M_G)$, but since $M/M_G \nsubseteq Z_\infty(G/M_G)$, $\exists\ X \lhd G$ where X is a maximal in $G = MX$, and $|G : X| = p$. ☐

Lemma 2.3. *Suppose that the saturated formation $\mathfrak{F}$ contains every nilpotent group, and G is with a soluble $\mathfrak{F}$-residual $G^{\mathfrak{F}} = P$. If $\mathfrak{F}$ contains all maximal subgroups of G which do not contain P, then P is a $p-$group for some prime number p, Moreover, if all cyclic subgroups of P with prime order or with order equal to 4 (in case $p = 2$, and P is a non-abelian) which do not have a supersoluble supplement in G are $c-$subnormal subgroups of G then $|P/\Phi(P)| = p$.*

Proof. By [11] $P = G^{\mathfrak{F}}$ is a $p-$group for some prime p, and the following hold: **(1)** A G-chief factor of the subgroup P is $P/\Phi(P)$, **(2)** The exponent of P is equal to either 4 or p (in case P is non-abelian and $p = 2$). Suppose that all cyclic subgroups of P with prime order or order equal to 4, which do not have a supersoluble supplement in G are $c-$subnormal subgroups of G, $\Phi = \Phi(P)$, $X/\Phi \leq P/\Phi$ is with a prime order, $x \in X/\Phi$, and $L = \langle x \rangle$, then $|L| = p$ or 4, which means that either L is a $c-$subnormal subgroup of G, or L has a supersoluble supplement subgroup T of G, since $\Phi \leq \Phi(G)$ we have $T\Phi \neq G$. Moreover, since $LT = G$, we have $(T\Phi/\Phi)(L\Phi/\Phi) = (T\Phi/\Phi)(X/\Phi) = G/\Phi$. therefore, $|G/\Phi : T\Phi/\Phi| = p$, $|P/\Phi(P)| = p$, because $G/\Phi = (P/\Phi)(T\Phi/\Phi)$. If L is a $c-$subnormal subgroup of G, and $L/L_G \nsubseteq Z_\infty(G/L_G)$, by Lemma 2.2, $\exists\ M \lhd G$ with $|G : M| = p$, Moreover, $G = LM$, here $L \nsubseteq T$, since G/T is a group with prime order, $L \leq P \leq T$, by the definition of P. therefore, by this contradiction we have: $L/L_G \leq Z_\infty(G/L_G)$, and $L\Phi/\Phi = X/\Phi \leq Z_\infty(G/\Phi) \cap (P/\Phi)$. Now from (1) we get $|P/\Phi(P)| = p$. ☐

Lemma 2.4. [12] *Suppose that $P \leq G$ is a noncyclic Sylow p-subgroup, and Q is a Sylow q-subgroup of G, where p, q are prime divisors of the order of G, if all maximal subgroups of P have q-closed supplements in G, then $Q \lhd G$.*

Lemma 2.5. [12] *Suppose that $\mathfrak{F}$ denotes a saturated formation, $\mathfrak{U}$ denotes a formation of supersoluble groups such that $\mathfrak{U} \subseteq \mathfrak{F}$ and $\exists\ E \lhd G$ such that G/E belongs to $\mathfrak{F}$. If E is cyclic, then G belongs to $\mathfrak{F}$.*

Lemma 2.6. [13] *Let $P = P_1 \times \cdots \times P_r, r > 1$ be a $p-$subgroup of G, and P_k are all minimal normal subgroups of G, $\forall\ k = 1, 2, \ldots, r$. If $D \leq P$ with $|D| > 1$ and $\forall\ H \leq P$ with $|H| = |D|$, $H \lhd G$, then $|P_k|$ is a prime number $\forall\ k$.*

Lemma 2.7. *Suppose that $N \lhd G$ is an elementary abelian subgroup, if $\exists\ D \leq N$ with $|D| > 1$, and for every subgroup H of N with $|H| = |D|$ is a $c-$subnormal subgroup of G, then a maximal subgroup of N is a normal subgroup of G.*

Proof. By contradiction, suppose G is a counterexample with minimal order. First assume $H \leq N$ satisfying $|H| = |D|$ we have $H/H_G \nsubseteq (G/H_G)$, therefore, by Lemma 2.2, $\exists\ T \lhd G$ with $|G : T| = p$, and $G = HT$. therefore, $G = NT$, $T \cap N \leq N$ is a maximal, and $T \cap N \lhd G$, which is a contradiction, and $\forall H \leq N$ which satisfy $|H| = |D|$, $H/H_G \leq Z_\infty(G/H_G)$. If $H \neq H_G$, then by Lemma 2.3, the assumption for G/H_G is still true, therefore, N/H_G contains a maximal subgroup M/H_G which is a normal in G/H_G, but then a maximal subgroup M of N is a normal subgroup of G, and this is a contradiction to our choice of G, so $\forall H \leq N$ with $|H| = |D|$, we have $H \lhd G$, by Lemma 2.6, the order of all minimal normal subgroups in G which are contained in N is a prime number, so the maximal subgroup of N is a normal subgroup of G, which is a contradiction. □

Lemma 2.8. [14] *Assume $P \lhd G$ and P is nilpotent, if $\Phi(G) \cap P = 1$, then P is the direct product of some minimal normal subgroups of G.*

Lemma 2.9. [14] *Assume $A \lhd G$, $B \lhd G$, and A is a subgroup of $\Phi(G)$. If the quotient B/A is a nilpotent and A is a subgroup of B, then B is a nilpotent.*

Definition 2.10. *If p is a prime number, G is a group that contains a normal Sylow p-subgroup, then G is called p-closed group.*

Lemma 2.11. [11] *If p is a prime number, and the class $\mathfrak{U}$ contains all $p-$closed groups, then $\mathfrak{U}$ must be a saturated formation class.*

Definition 2.12. *The product of all normal quasi-nilpotent subgroups of G is called the generalized fitting subgroup and denoted by $F^*(G)$.*

Lemma 2.13. [15]

(1) *If $N \lhd G$, then $F^*(N) \leq F^*(G)$.*
(2) *If $N \lhd G$, and $N \leq F^*(G)$, then $F^*(G)/N$ is a subgroup of $F^*(G/N)$.*
(3) *If $F(G)$ is a subgroup of $F^*(G) = F^*(F^*(G))$, and $F^*(G)$ is a soluble, then $F^*(G) = F(G)$.*
(4) *If $E(G)$ is the layer of G, then $E(G) \cap F(G) = Z(E(G))$ and $F^*(G) = F(G)E(G)$.*
(5) *$C_G(F^*(G)) \leq F(G)$.*

Lemma 2.14. [5] *If P is a normal $p-$subgroup of G for some prime number p, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

Lemma 2.15. *From [15], if P is a normal $p-$subgroup of G for some prime number p, and $P \subseteq Z(G)$, then $F^*(G/P) = F^*(G)/P$.*

Lemma 2.16. [12] *Let $G = AB$, where $A \lhd G$, $B \lhd G$, if $|A|$ is a prime number, $O_{p'}(G) = 1$ and $B \in \mathfrak{F}$, then $G \in \mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation.*

Lemma 2.17. [16] *Assume that P is a normal $p-$subgroup of G for some prime number p, if every cyclic subgroup of P is of order equal to 4 or p, and belongs to $Z_\infty(G)$, then $P \leq Z_\infty(G)$.*

3. Results and Discussion

In this section, we state the main two results of this paper and discuss their proofs.

Theorem 3.1. *Suppose that E is a normal subgroup of G, where the quotient G/E belongs to a saturated formation $\mathfrak{F}$ that contains all supersoluble groups, if $\exists\ D \leq P \leq E$, where P is a noncyclic Sylow subgroup, $|D| > 1$, and all subgroups H of P such that H does not have a supersoluble supplement in G, with order $|H| = 2|D|$ and with order $|H| = |D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are $c-$subnormal subgroups in G, then G belongs to $\mathfrak{F}$.*

Proof. By contradiction. Consider a counterexample (G, E) for which $|G||E|$ is minimal, we will prove the following claims:

(i) If X is a Hall subgroup of E, then the assumption for (X, X) is true, Moreover, if $X \lhd G$, then the assumption for $(G/X, E/X)$ is also true.

Proof(i). Assume $X \leq E$ is a Hall subgroup, and the Sylow subgroup $P \leq X$ is not cyclic, by assumption $\exists\ D \leq P$ with $|D| > 1$, and for all $H \leq P$ with $|D| = |H|$ and with $2|D| = |H|$, (in case P is a non-abelian $2-$ group, and with index $|P : D| > 2$), we have either a supersoluble supplement subgroup $T \leq G$ or $H \leq G$ is a $c-$subnormal subgroup, in the first case we have $X = HT \cap X = H(T \cap X)$, which means that $T \cap X$ which belongs to X is a supersoluble supplement of H, in the second case, H is a $c-$subnormal subgroup of X, so by Lemma 2.2, the assumption for (X, X) is true. Now, if $X \leq E$ is a Hall subgroup, $X \lhd G$, then $(G/X)/(E/X) \simeq G/X$ belongs to $\mathfrak{F}$. Assume P^*/X is a Sylow p-subgroup of E/X and not cyclic, where p is a prime number which divides the order of E/X, P is a Sylow p-subgroup of P^* such that $P^* = PX$, therefore, P is not a cyclic Sylow subgroup of E, and by the assumption, $\exists\ D \leq P$ with $|D| > 1$, and for all $H \leq P$ with $|D| = |H|$ and with $2|D| = |H|$, (in case P is a non-abelian 2 group, and with index $|P : D| > 2$) either has a supersoluble supplement, T belongs to G, or H is a $c-$subnormal subgroup of G. Assume $H^*/X \leq P^*/X$ with $|H^*/X| = |D|$, so $H^* = [X]H$ where $H \leq H^*$ is a Sylow p subgroup. Its clear that $|H| = |D|$, so $H^*/X = HX/X$ has a supersoluble supplement $TX/X \simeq T/X \cap T$ belonging to G/X or $H^*/X = HX/X$ is a $c-$subnormal subgroup of G/X, by Lemma 2.3, the assumption for $(G/X, E/X)$ is also true.

(ii) Assume $X \lhd E$ is not the identity, then $X = E$.

Proof(ii). Because $X \leq E$ is a characteristic, $X \lhd G$, so by (1), the assumption for $(G/X, E/X)$ is true, and by the choice of G, $G/X \in \mathfrak{F}$, therefore, the assumption for (G, X) is also true, and $E = X$ by our choice of (G, E).

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(3) If $P \leq E$ is a Sylow subgroup with order p, where p is the smallest prime number which divides the order of E, then P is not cyclic.

proof (3). Suppose P is cyclic, from [15], the group E is nilpotent with order p unless its cyclic, so G belongs to $\mathfrak{F}$ and $E = P$ by (2). Moreover, because G/E belongs to $\mathfrak{F}$, by Lemma 2.5, G belongs to $\mathfrak{F}$ too, this contradicts our choice of G.

Note that since P is not cyclic, and since $\exists\ D \leq P$ with $|D| > 1$, any subgroup $H \leq P$ with $|D| = |H|$ and with $2|D| = |H|$ (in case P is a non-abelian 2 group, and with index $|P : D| > 2$) which does not have a supersoluble supplement in G is a $c-$subnormal subgroup of G.

(4) If $E = P$ or $E = G$, then $|D| > p$.

proof (4). In case $E = G$, by (2): G must have a p-closed Schmidt subgroup because G is not a p-nilpotent, from [15] and by Lemma 2.3, the order of D is greater than p, while if $E = P$, take the maximal subgroup $B \leq G$ which does not contain E, we have $G/E \simeq B/B \cap E \in \mathfrak{F}$. If $|D| = p$, let $M = G^{\mathfrak{F}}$, and $\Phi = \Phi(M)$, so $M \leq E$, and by Lemma 2.3, $|M/\Phi| = p$, we have G/Φ belongs to $\mathfrak{F}$, and by Lemma 2.5, $M \leq \Phi$, so $M = \Phi$, which is a contradiction, therefore, $|D| > p$.

(5) (i) If the index $|P : D|$ is greater than p, $\forall\ H \leq P$ with $|H| = |D|$ which does not have a supersoluble supplement in G, then $H/H_G \leq Z_\infty(G/H_G)$.

(5) (ii) If P is a non-abelian 2-group and $|P : D| > 2$, then $\forall\ H \leq P$, with $|H| = 2|D|$ which does not have a supersoluble supplement in G, then $H/H_G \leq Z_\infty(G/H_G)$.

Proof (5)(i). If $\exists\ H \leq P$ with $|H| = |D|$, and H has neither a supersoluble supplement in G nor $H/H_G \leq Z_\infty(G/H_G)$, by Lemma 2.2, $\exists\ M \lhd G$ such that $|G : M| = p$ and $HM = G$, then $G/E \cap M$ belongs to $\mathfrak{F}$, because $\mathfrak{F}$ is a closed class under sub direct products, by Lemma 2.2, the assumption for $(G, E \cap M)$ is still true, because $p < |P : D|$ and $|G||M \cap E| < |G||E|$, this contradicts our choice of (G, E), so $\forall\ H \leq P$ with $|H| = |D|$ which does not have a supersoluble supplement in G, we have $H/H_G \leq Z_\infty(G/H_G)$.

Proof (5)(ii). The proof is similar to the proof of the case (5)(i).

(6) If $N \lhd G$ and N is a minimal in P, then $|N| \leq |D|$.

proof (6). Assume $|D| < |N|$ then $p < |N|$. If $H \leq N$, $|H| = |D|$, H has a supersoluble supplement S in G, then $SN = G$, $S \neq G$, by our choice of G, so $N \cap S \leq N$ is not an identity and a proper, since $N = N \cap HS = H(N \cap S)$, so $N \cap S \lhd G$, this is a contradiction to the minimality of N, so $\forall\ H \leq N$ with $|H| = |D|$, H is a $c-$subnormal subgroup of G. Let $S \lhd G$ with $HS = G$, and $(H \cap S)H_G/H_G = H \cap S/1 \leq Z_\infty(G/1)$, so $N \leq Z_\infty(G)$, $|N| = p$, which is a contradiction.

(7) If $E = P$, or $E = G$, $N \lhd G$ is a minimal abelian in E, then the assumption for $(G/N, E/N)$ is still true, Moreover, G/N is in $\mathfrak{F}$.

proof (7). The case $|P : D| = p$ or $|N| < |D|$ is obvious, assume $|P : D| > p$, and $|N| = |D|$, by (5), $\forall\ H \leq P$ with $|H| = |D|$ which does not have a supersoluble supplement in G we get $H/H_G \leq Z_\infty(G/H_G)$. Moreover, if P is a non-abelian 2-group with $|P : D| > 2$, then $\forall\ H \leq P$ with $|H| = 2|D|$

not having a supersoluble supplement in G, Moreover, $H/H_G \leq Z_\infty(G/H_G)$, by (4), N is not cyclic, so all subgroups of G which contain N are not cyclic too. Assume $N \leq K \leq P$ with $|K : N| = p$. Because K is not cyclic, K has a maximal subgroup $L \neq N$. If L has a supersoluble supplement in G, then we deduce the same for K. Otherwise, by (5) we have $L/L_G \leq Z_\infty(G/L_G)$, so $(L_GN)L/L_GN = K/L_GN \leq Z_\infty(G/L_GN)$, and $K/K_G \leq Z_\infty(G/K_G)$ by Lemma 2.1, therefore, the subgroup K is a c-subnormal subgroup in G, so K/N is a c-subnormal subgroup in G/N by Lemma 2.2, so if P/N is abelian, then by Lemma 2.2, the assumption for G/N is also true. If P/N is a non-abelian 2-group, then P is a non-abelian 2-group, so all subgroups of P with $|P| = 2|D|$ which do not have a supersoluble supplement in G are $c-$subnormal subgroups in G, here, similarly one can prove that all $X \leq P$ which contain N and with order $|X : N| = 4$ hase either a supersoluble supplement in G, or $c-$subnormal subgroups in G, so the assumption for $(G/N, E/N)$ is still true.

(8) If $E = G$, then at least one of the maximal subgroups of P does not have a supersoluble supplement in G .

proof (8). Use (2) and Lemma 2.4.

(9) E is a soluble.

proof (9). By our choice of G and by (1), we consider only the case when $G = E$, Moreover, by (7) we must prove that $P_G \neq 1$. If $P_G = 1$, and p divides $|Z_\infty(G)|$ and R is the Sylow p-subgroup of $Z_\infty(G)$, then $R \leq P$, so $R \leq P_G = 1$. This contradiction shows that $Z_\infty(G) \cap P = 1$. Assume $|P : D| = p$, by Lemma 2.4 and by (2), there exists more than one maximal subgroup of P, calling it M, which does not have any supersoluble supplement in G. Because $P_G = 1$, $M_G = 1$, so by assumption $\exists\, T \lhd G$ such that $T \cap M \leq Z_\infty(G)$. But $P \cap Z_\infty(G) = 1$, so $T \cap M = 1$, and any Sylow p-subgroup of T has the order p, by Lemma 2.2(4), the assumption for (T, T) is still true, so T is a supersoluble, this contradicts our choice of M, so if $p < |P : D|$ then by (5) for every subgroup H of P with $|H| = |D|$ we have $H \leq Z_\infty(G)$, therefore, we have the contradiction $1 \neq H \leq Z_\infty(G) \cap P = 1$.

(10) If q is the largest prime divisor of $|E|$, then E is a q-closed.

proof (10). By (1), consider only the case where $G = E$, now E is soluble by (8), but also, by (1) the assumption for (H, H) is true, where $H \leq G$ is a Hall subgroup, if $|G| = p^a q^b$ where $a, b \in bbN$. If G is not q-closed, then by (7) and our choice of G, $\forall\, N \lhd G$ which are minimal in P, we have G/N is supersoluble, by Lemma 2.11, $N \nsubseteq \Phi(G)$, and N is unique. Moreover, by Lemma 2.5, $|N| > p$. Hence $Z_\infty(G) \cap P = 1$. Now, We claim that $N = O_p(G)$, if $X \leq G$ is a maximal with $G = [N]X$, then $O_p(G) = O_p(G) \cap NX = N(O_p(G) \cap X)$. Since $O_p(G) \leq F(G) \leq C_G(N)$, we have $O_p(G) \cap X \lhd G$, therefore, $O_p(G) \cap X = 1$, so, $N = O_p(G)$, and $N \nsubseteq \Phi(G)$.

If $p = |P : D|$, then $\forall\, A \leq P$ which are maximal and contains N, we get $AX = G$, therefore, $X \simeq G/N$ is a supersoluble supplement of A which belongs to G, so if we use Lemma 2.4, a maximal subgroup V of P does not contain N and does not contain a supersoluble supplement in G. therefore, by the assumption V is a $c-$subnormal subgroup of G. Let $L = V_G$, then $L \leq N = O_p(G)$. But

$N \nsubseteq V$, $N \lhd G$ is a minimal, therefore, $L = 1$, by the hypothesis G has a normal subgroup T such that $VT = G$, and $T \cap V \leq Z_\infty(G)$. But $Z_\infty(G) \cap P = 1$, so $T \cap V = 1$, then $|T| = pq^b$, therefore, by Lemma 2.2(4), the assumption for (T, T) is true, and T is supersoluble, which contradicts our choice of V. If $p < |P : D|$, assume H is a subgroup of P satisfying $|H| = |D|$, and $N \nsubseteq H$, so $H_G = 1$, by (5) either $H \leq Z_\infty(G)$ or H contains a supersoluble supplement in G, but $Z_\infty(G) \cap P = 1$, therefore, the second case is impossible. If $N \leq H$, then from $G = [N]X$ we conclude that H contains a supersoluble supplement X in G, therefore, all maximal subgroups of P have a supersoluble supplement in G, but this is a contradiction to Lemma 2.2.

(11) $E = P$.

proof (11). Assume that Q is a Sylow q-subgroup of the group E, where q is the largest prime number which divides the order of E, by (9) we have $Q \lhd E$, and by (2) we have $Q = E = P$.

Finally, we have the following contradiction: If $N \lhd G$ is a minimal and belongs to P, by (7) N is unique, $G/N \in \mathfrak{F}$, and $O_p(G) = N = P$. Assume $H \leq N$ with $|H| = |D|$, $T \lhd G$ with $G = TH$, and $T \cap H \leq Z_\infty(G)$, by Lemma 2.2(5), we have either $N \cap T = 1$ or $N \leq T$, from the former case $G = NT = HT$, we have $|N| = |H|$, and this is a contradiction to our choice of H, so $N \leq T$, and $N \leq Z_\infty(G)$, with $|N| = p$, but then by Lemma 2.5, $G \in \mathfrak{F}$. □

Theorem 3.2. *Assume that $E \lhd G$, so that the quotient G/E belongs to a saturated formation $\mathfrak{F}$ that contains all supersoluble groups. If $\exists\ D \leq P \leq F^*(E)$, where P is any noncyclic Sylow subgroup, such that $|D| > 1$ and $\forall\ H \leq P$ with $|D| = |H|$ and with $2|D| = |H|$ (if P is a non-abelian 2-group with index $|P : D| > 2$) are c−subnormal subgroups in G, then G belongs to $\mathfrak{F}$.*

Proof. Assume (G, E) is a counterexample where $|G||E|$ is a minimal, $F^*(E) = F^*$, and $F(E) = F$, note that in case E is a soluble group, p is considered the smallest prime number which divides the order of F, while in case E is not a soluble group, p is considered the largest prime number which divides the order of F. Let P be the Sylow p-subgroup of F, we will prove the following claims:

(1) $F = F^* \neq E$, $C_G(F) = C_G(F^*) \leq F$.

proof (1). By Lemma 2.2, the assumption for (F^*, F^*) is still true, by Theorem 3.1, F^* is supersoluble, and $F^* = F$ by Lemma 2.13, hence if $F = E$, then by Theorem 3.1, G belongs to $\mathfrak{F}$, this is a contradiction to our choice of G, so $F = F^* \neq E$. Now by Lemma 2.13, $C_G(F) = C_G(F^*) \leq F$.

(2) If H is a proper normal subgroup of G, such that $F \leq H \leq E$, then H is supersoluble.

proof (2). Use Lemma 2.13(1), $F^*(H) \leq F^* = F \leq H$, this means that $F^*(H) = F^*$, therefore, the assumption for (H, H) is true, and by the choice of G, H is a supersoluble.

(3) Any subgroup E which does not equal to G is supersoluble.

proof (3). This is a direct result of (2).

(4) If E is a soluble group, $F(E/P) = V/P$, and $Q \leq V$ is a Sylow q-subgroup for a prime number q which divides the order of V/P, then $p \neq q$, and either Q is a subgroup of F or $q < p$, and $C_Q(P) = 1$.

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proof (4). Because V/P is nilpotent, Moreover, $QP/P \leq V/P$ is a Sylow q-subgroup, we have QP/P is a characteristic belonging to V/P, $QP \lhd E$, so $p \neq q$, and by Theorem 3.1, QP is supersoluble. The case $p < q$, we get Q is a normal subgroup of QP, therefore, Q is a subgroup of $F = F(E)$. While in the case $q < p$, then $p > 2$ because p which divides the order of the q'-subgroup F is taken to be the smallest. Assume that $C = C_{PQ}(P)$, Since $P \lhd PQ$, $C \lhd PQ$, and C is clearly a nilpotent. Hence $C_Q(P) = C \cap Q$ is a normal in PQ. Hence $C_Q(P) \leq O_q(PQ)$. But $O_q(PQ) \leq F$, since F is a q'-subgroup, so $C_Q(P) = 1$.

(5) $p > 2$.

proof (5). If E is soluble, and $p = 2$, then by (4) : $F/P = F(E/P)$, moreover, by Lemma 2.13 and by (1) of this theorem, we have $F^*(E/P) = F(E/P) = F^*/P$, therefore, by Lemma 2.2, the assumption for $(G/P, E/P)$ is true, now because $G/E \simeq (G/P)/(E/P) \in \mathfrak{F}$, G/P must be in $\mathfrak{F}$, so we have the contradiction G belongs to $\mathfrak{F}$ by Theorem 3.1, and E is not a soluble, since here the prime number p is taken to be the largest prime which divides $|F|$, we have by (1): $F^* = F$ is a 2-group. Assume $Q \leq E$ with $|Q| = q \neq 2$, and $X = FQ$, by Theorem 3.1, X is supersoluble, therefore Q is a normal subgroup of X, and Q is a subgroup of $C_E(F)$, but by (1), $C_E(F) = C_E(F^*) \leq F$, which is a contradiction.

(6) $P \nsubseteq Z_\infty(E)$.

proof (6). Suppose that $P \subseteq Z_\infty(E)$, then by Lemma 2.15, $F^*(E/P) = F^*(E)/P$, by Lemma 2.2, the assumption for $(G/P, E/P)$ is true. Hence G/P belongs to $\mathfrak{F}$, by Lemma 2.5, $G \in \mathfrak{F}$, which is a contradiction.

(7) $\Omega_1(P) \nsubseteq Z_\infty(E)$.

proof (7). Suppose $\Omega_1(P) \subseteq Z_\infty(E)$, then by (5) and Lemma 2.16, we have $P \leq Z_\infty(E)$, which contradicts (6).

(8) P is not a cyclic.

proof (8). This is a direct result from (7).

Note that, since P is not a cyclic, $\exists\ D \leq P$, with $|D| > 1$ and $\forall\, H \leq P$ with $|H| = |D|$ are $c-$subnormal subgroups of G.

(9) $p < |D|$.

proof (9). Assume $p = |D|$, then by (7), $\exists\ H \leq P$, where H has an order equal to p, and H does not belong to $Z_\infty(E)$. If $H \lhd G$, then consider $C = C_G(L)$, therefore, $F^* = F \leq C \cap E$, and $F^*(C \cap E) \leq F^*(E)$, so $F^* = F^*(C \cap E)$. Moreover, since $|H| = p$, G/C is an abelian, we have $G/C \cap E \in \mathfrak{F}$, therefore, by Lemma 2.2, the assumption for $(G, C \cap E)$ is true. Since $L \nsubseteq Z_\infty(E)$, $E \nsubseteq C$, so $|E| > |C \cap E|$, a contradiction to our choice of (G, E), therefore, $H_G = 1$, and by Lemma 2.2, H has a normal complement T in G. One can show that the assumption for (G, V) is still true, where $V = E \cap T$, because $T \leq G$ is a proper subgroup, Moreover, $G = ET$, we get $|E| > |V|$, this is a contradiction to our choice of (G, E).

(10) If $L \lhd G$ is a minimal, and L is a subgroup of P, then $|L| < p$.

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proof (10). If $|L| = p$, and $C = C_E(L)$, the assumption for $(G/L, C/L)$ is true. clearly $G/C = G/E \cap C_G(L)$ belongs to $\mathfrak{F}$, Moreover, because L is a subgroup of $Z(C)$, we have $F = F^* \leq C$ and L is a subgroup of $Z(F)$, so $F^*/L = F^*(C/L)$. Moreover, if $H/L \leq G/L$ such that $|D| = |H|$, by (9) we get $1 < |H/L| < |P/L|$, Moreover, H/L is a $c-$subnormal subgroup of G/L, by (5) and Lemma 2.2, the assumption for G/L is still true, therefore G/L belongs to $\mathfrak{F}$, and by Lemma 2.5, G belongs to $\mathfrak{F}$ which is a contradiction.

(11) If $P \cap \Phi(G) \neq 1$, $L \lhd G$ is a minimal in $P \cap \Phi(G)$, then $F^*(E/L) \neq F^*/L$.

proof (11). If $\Phi(G) \cap P = 1$, then by Lemma 2.8, there are some normal subgroups in G which are minimal and whose direct product equals P, therefore, by Lemma 2.7, $\exists\ M \lhd G$, such that $M \leq P$ and M is a maximal, by [17], if $L \lhd G$ is a minimal and in P, then the order of L equals to p, and this is a contradiction to (9), so $P \cap \Phi(G) \neq 1$. Now, if $L \lhd G$ is a minimal such that $L \leq \Phi(G) \cap P$, and $F^*(E/L) = F^*/L$, then we need to prove that the assumption is still true for $(G/L, E/L)$. It is clear that $|L| \leq |D|$, so in case $|P : D| = p$, the assumption is still true for G/L, Moreover, by (5), in case $|L| < |D|$, the assumption for G/L is still true. Hence, if $|P : D| > p$, and $|D| = |L|$, by (8), L is not a cyclic, so all subgroups of G which contain L are not cyclic. Assume that L and M are different maximal subgroups in K, $L \leq K \leq P$, and $M \leq K \leq P$, we need to prove that K is a $c-$subnormal subgroup of G. If $K \lhd G$, it is evident. So let $K_G = L$. First suppose that $L/L_G \nsubseteq Z_\infty(G/L_G)$, by Lemma 2.2, $\exists\ S \lhd G$ so that $G = KS = MS$, and $p = |G : S|$, now, because L is a subgroup of $\Phi(G)$, we get $L \leq S$, therefore, $S \cap K = L$, and K is a $c-$subnormal subgroup of G. Now suppose that $L/L_G \leq Z_\infty(G/L_G)$, then $(L_GN)L/L_GN = K/L_GN \leq Z_\infty(G/L_GN)$, so $K/K_G \leq Z_\infty(G/K_G)$, by Lemma 2.1(1), K is a $c-$subnormal subgroup, therefore, the assumption for G/L is still true, and G/L belongs to $\mathfrak{F}$, but then G belongs to $\mathfrak{F}$, because $L \leq \Phi(G)$, and $\mathfrak{F}$ is a saturated formation, which is a contradiction, so $F^*(E/L) \neq F^*/L$.

(12) $G = E$ is not a soluble.

proof (12). Assume $L \lhd G$ is a minimal belonging to $P \cap \Phi(G)$, by Lemma 2.9, $F(E/L) = F/L$. Moreover, by Lemma 2.13, $F(E/L) = F^*(E/L)$ and so by (1), $F(E/L) = F^*(E/L) = F^*/L$, but this is a contradiction to (11).

(13): G has the only maximal normal supersoluble subgroup M containing F, and G/M is a non-abelian simple group.

proof (13). Use (2), and (12) of this theorem.

(14) Suppose that G/F is a non-abelian simple group, X is a minimal normal subgroup of G belongs to $P \cap \Phi(G)$, then G/X is a quasi nilpotent group.

proof (14). By (11), $F^*(E/X) \neq F^*/X$, so the subgroup $F/X = F^*/X$ is a proper in $F^*(G/X)$, therefore, by Lemma 2.13(2),(4), we have $F^*(G/X) = F(G/X)E(G/X)$, where $E(G/X)$ is the layer of G/X, by (13) a chief series of G has a unique non-abelian factor, since $E(G/X)/Z(E(G/X))$ is a direct product of non-abelian simple groups, $F^*(G/X) = G/X$ is a quasi nilpotent, by Lemma 2.13,

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$F(G/X) \cap E(G/X) = Z(E(G/X))$, $G/F \simeq (G/X)/(F/X)$ is a non-abelian simple group.

(15) $P = F^*$.

proof (15). If $q \neq p$, Q is a Sylow q-subgroup of F^*, then by (14), Q is a subgroup of $Z_\infty(G)$, by Lemma 2.14, $F^*/Q = F^*(G/Q)$, therefore, by Lemma 2.2, the assumption for $(G/Q, G/Q)$ is still true, and by our choice of G, we have G/Q is a supersoluble, so G is a soluble which contradict (12).

(16) $\Phi(P) = 1$.

proof (16). Assume $L \lhd G$ is a minimal belonging to $\Phi(P)$, then G/X is quasi nilpotent by (14), by Lemma 2.14, G is quasi nilpotent, so $F = F^* = G$, which is a contradiction.

(17) $|P : D| > p$.

proof (17). If $p = |P : D|$, by (11), $\Phi(G) \cap P \neq 1$. Assume $N \lhd G$ is a minimal and contained in $\Phi(G) \cap P$, by (16) if $V \leq P$ is a maximal, then $P = NV$, assume $T \lhd G$ so that $G = VT$, and $(V \cap T)V_G/V_G \leq Z_\infty(G/V_G)$. If either $T \cap V = 1$ or $V_G = 1$, and $T \cap V \neq 1$, G has a normal subgroup L such that $L \leq P$, and $|L| = p$, which contradicts (10), so $V_G \neq 1$. But $N \nsubseteq V$, so $N \cap H_G = 1$, therefore, G contains a normal subgroup $L \neq N$ belonging to P, and the minimal, by (14) the order of N, or X or both is a prime number, a contradiction to (10).

(18) $O_{p'}(G) = 1$.

proof (18). If $O_{p'}(G) \neq 1$, by (14) and (15), $G = O_{p'}(G) \times P = O_{p'}(G)P = F^* = F$, which is a contradiction.

(19) $G = O^p(G)$.

proof (19). If $G \neq O^p(G)$, then $\exists\ T \lhd G$ with $|G : T| = p$. To show that T satisfies the assumption: Note that $F \cap T = F^*(T)$, clearly $F \cap T$ is a subgroup of $F^*(T)$, by (13), $T/F \cap T$ is a non-abelian simple group, so if $T \cap F \neq F^*(T)$, we have $T = F^*(T)$, therefore, $TF = G = F = F^*$ is a nilpotent, which is a contradiction, so $T \cap F = F^*(T)$, and by (17), the assumption is still true for T, so T belongs to $\mathfrak{F}$, by Lemma 2.17 and (18), we get the contradiction: G belongs to $\mathfrak{F}$.

(20) $\forall\ \ H \leq P$ with $|H| = |D|$, we have $H_G \neq 1$.

proof (20). If $H \leq P$ satisfying $|H| = |D|$ then $H_G = 1$, and $\exists T \lhd G$, such that $T \cap H \leq Z_\infty(G)$, then either $P \cap Z_\infty(G) \neq 1$ or $\exists\ D \lhd G$ such that $G : D = p$, by (10) the former case is impossible, while in the second case we have $O^p(G) \neq G$, which contradicts (19).

A final contradiction to this theorem: Assume that $N \lhd G$ is a minimal belonging to P, by (16): $P = NM$ where $M \leq P$ is some maximal subgroup. Assume $H \leq P$ so that H is a subgroup of M with $|H| = |D|$, by (20), $H \neq H_G$, therefore, $N \nsubseteq H_G$, $N \cap H_G = 1$, and $\exists\ L \lhd G$ which is a minimal, $L \neq N$, and L belongs to P. By (14), either L or N has a prime order, a contradiction to (10). □

4. Some applications on the main results

To show the generality of the theorems in section 3, we state some previous corollaries with their references, which can be considered as special cases of Theorem 3.1 and Theorem 3.2.

Corollary 4.1. [2] *Assume that $|G|$ is an odd, and $H \leq G$, with $|H|$ is a prime, if $H \lhd G$, then G is a supersoluble group.*

Corollary 4.2. [3] *Assume that $P \leq G$ is a Sylow subgroup, and $H \leq P$ is a maximal subgroup, if $H \lhd G$, then G is a supersoluble group.*

Corollary 4.3. [1] *Assume that $H \leq G$, with $|H|$ equals 4 or a prime number, if H is a c-normal in G, then G is a supersoluble group.*

Corollary 4.4. [1] *Assume that $P \leq G$ is a Sylow subgroup, and $H \leq P$ is a maximal subgroup, if we suppose that H is a c-normal subgroup of G, then G is a supersoluble group.*

Corollary 4.5. [18] *Assume that $P \leq G$ is a Sylow subgroup, and $H \leq P$ is a maximal subgroup, so that H has no supersoluble supplement in G, if $H \lhd G$, then G is a supersoluble group.*

Corollary 4.6. [19] *Assume that $P \leq G$ is a Sylow subgroup, and $H \leq P$ is a maximal subgroup which does not have a supersoluble supplement in G, if H is a c-normal subgroup of G, then G is a supersoluble group.*

Corollary 4.7. [20] *Assume that $\mathfrak{F}$ is a saturated formation contains all supersoluble groups, If all cyclic and minimal subgroups of $G^{\mathfrak{F}}$ with order 4 are c-normal subgroups of G, then G belongs to $\mathfrak{F}$.*

Corollary 4.8. [21] *Assume that $E \lhd G$, $|E|$ is an odd number, and G/E is a supersoluble, if P is not a cyclic Sylow subgroup and $\exists\ D \leq P \leq E$ with $|D| > 1$, and if any $H \leq P$ with $|H| = |D|$ is a c-normal subgroup of G, then G is a supersoluble group.*

Corollary 4.9. [4] *Assume that each maximal subgroup of the Sylow subgroups of $F(E)$ is normal subgroup belonging to a soluble group G, then G is a supersoluble group.*

Corollary 4.10. [5] *Assume that $E \lhd G$ is a soluble, with supersoluble quotient G/E, if any maximal subgroup of the Sylow subgroups of $F(E)$ is c-normal belonging to G, then G is a supersoluble group.*

Corollary 4.11. [5] *Assume that E is a soluble normal subgroup of G with supersoluble quotient G/E. If any subgroup of $F(E)$ of order 4 or of prime order is a c-normal belonging to G, then G is a supersoluble group.*

Corollary 4.12. [6] *Let $E \lhd G$ be a soluble subgroup such that G/E belongs to a saturated formation $\mathfrak{F}$ which contains all supersoluble groups. If any maximal subgroup of the Sylow subgroups of $F(E)$ is a c-normal belongs to G, then G belongs to $\mathfrak{F}$.*

Corollary 4.13. [6] *Let $E \lhd G$ be a soluble so that G/E belongs to a saturated formation $\mathfrak{F}$ that contains any supersoluble group. If any minimal subgroup and any cyclic subgroup with order 4 of $F(E)$ are c-normal in G, then G belongs to $\mathfrak{F}$.*

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Corollary 4.14. [7] *Let $E \lhd G$ be a soluble so that G/E belongs to a saturated formation $\mathfrak{F}$ that contains any supersoluble group. If any maximal subgroup of the Sylow subgroup of $F^*(E)$ is a $c-$normal belonging to G, then G belongs to $\mathfrak{F}$.*

Corollary 4.15. [7] *Let $E \lhd G$ be a soluble so that G/E belongs to a saturated formation $\mathfrak{F}$ that contains any supersoluble group. If any minimal subgroup and any cyclic subgroup with order 4 of $F^*(E)$ are c-normal belonging to G, then G belongs to $\mathfrak{F}$.*

Definition 4.16. [22] *A subgroup $H \leq G$ is said to be $\mathfrak{U}$-S-supplemented in G if $\exists\ T \leq G$, such that $G = HT$, and $T/T \cap H_G$ is supersoluble.*

Note that, if H is the $\mathfrak{U}$-S-supplemented subgroup of G, then H must have a supersoluble supplement belonging to G.

Corollary 4.17. [22] *Assume that $E \lhd G$ is soluble with supersoluble quotient G/E. If each maximal subgroup of the Sylow subgroup of E is the $\mathfrak{U}$-S-supplemented belonging to G, then G is a supersoluble group.*

5. Conclusion

We used the definition of the concept "$c-$subnormal subgroup", which is considered weaker than the previous concepts such as $c-$normal subgroup and $\mathcal{U}_c$-normal subgroup, and studied the structure of a given finite group G which contains "$c-$subnormal subgroup". We proved some Lemmas and theorems which focused on answering the question: If a given finite group G contains a class of $c-$subnormal subgroups, what conditions must hold so that G belongs to a saturated formation of supersoluble groups. We stated many previous corollaries together with their references, which are considered as special cases of the proved theorems. We believe that there will be more results which could be concluded in the future from these proved theorems.

Data Availability The data used to support the findings of this study have not been made available because the study is theoretical and does not depend on any collected data.

Conflicts of Interest The authors declare that there are no conflicts of interest.

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