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# SOME RESULTS ON NON-PROGRESSIVE SPREAD OF INFLUENCE IN GRAPHS

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Abstract. This paper studies the non-progressive spread of influence with threshold model in social networks. Consider a graph $G$ with a threshold function $\tau$ on its vertex set. Spread of influence is a discrete dynamic process as follows. For a given set of initially infected vertices at time step 0 each vertex $v$ gets infected at time step $i$, $i \geq 1$, if and only if the number of infected neighbors are at least $\tau(v)$ in time step $i-1$. Our goal is to find the minimum cardinality of initially infected vertices (perfect target set) such that eventually all of the vertices get infected at some time step $\ell$.

In this paper an upper bound for the convergence time of the process is given for graphs with general thresholds. Then an integer linear programming for the size of minimum perfect target set is presented. Then we give a lower bound for the size of perfect target sets with strict majority threshold on graphs which all of the vertices have even degrees. It is shown that the later bound is asymptotically tight.

## 1. Introduction

Studying the spread of influence in networks is an important part of the social networks analysis. The spread of influence has some real-world applications. For example consider the spread of a contagious disease that people can transmit infection to each other. More examples would be diffusion of innovation, viral marketing and spread of fault in distributed computing. There are two main types of influence diffusion models: the progressive (or irreversible) and the non-progressive (or reversible) models [9, 2]. In progressive models every infected (or influenced or active) individual will remain

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infected forever. But in non-progressive models it is possible that an infected vertex regarding to the number of infected and uninfected neighbors become uninfected.

Progressive models are suitable for monopolistic diffusion of an innovation among people. But in cases that two (or more) product providers are competing to get people adopt their product and people may switch between products any time they want, we need to use non-progressive models.

Progressive models have been studied in the literature extensively [3, 16, 1, 15, 10, 11, 7, 8]. Non-progressive models also have been studied in [13, 5, 8, 6] and with a little different definition in [4]. In this paper we focus on non-progressive model with threshold for vertices. The threshold of each individual has an inverse relationship with its susceptibility. We are interested in perfect target sets (dynamic monopoly or dynamo for brevity) which considering them as initially infected vertices will cause convergence to a steady state that all of the vertices are infected. Our goal is to find minimum size perfect target set.

Integer Programming approach has been used for finding a minimum size perfect target set in progressive models [14, 12, 1]. To the best of our knowledge there is no integer programming formulation for the non-progressive model. In this paper we present an integer linear programming formulation for finding a minimum perfect target set of the non-progressive model.

1.1. **Definitions.** Here we present a formal definition of the concepts. Assume that a simple graph $G$ on the vertex set $V(G)$ and the edge set $E(G)$ represents the underlying network. Denote by $N_G(v)$ the neighborhood of vertex $v$. $\deg_G(v) = |N_G(v)|$ stands for the degree of $v$. $\Delta(G)$ represents the maximum degree among the vertices of $G$.

Consider a threshold assignment $\tau : V(G) \longrightarrow \mathbb{N} \cup \{0\}$ for the vertices of $G$ such that for each vertex $v \in V(G)$ we have $\tau(v) \leq \deg_G(v)$ and it represents the resistance level of the vertex $v$ against infection. Strict majority threshold for which $\tau(v) = \left\lceil \frac{\deg(v)+1}{2} \right\rceil$ is one of the most studied threshold assignments in networks. Throughout this paper, by $(G, \tau)$ we mean a graph $G$ together with a threshold assignment $\tau$ for its vertices. The discrete time dynamic process for the non-progressive spread of influence corresponding to the threshold assignment $\tau$ is defined as follows.

The process starts with a subset $D$ of vertices which consists of the vertices having the state 1 at time step 0. We denote the set of vertices with state 1 at time step $i$ by $D_i$. So at the beginning, (i.e. time step 0), we have $D_0 = D$. Then at each time step $i + 1$, the state of each vertex $v$ becomes 1 if and only if at least $\tau(v)$ neighbors of $v$ belong to $D_i$. This is equivalent to say that

$$\forall v \in V(G) : \ \big(v \in D_{i+1} \iff |N(v) \cap D_i| \geq \tau(v)\big).$$

Note that in this non-progressive process it is possible that the vertices with state 1 at time step $i$ change their state to 0 at time step $i + 1$ (This does not happen in progressive models). The vertices with state 1 at time step $i$ (i.e. vertices belonging to $D_i$) are said to be infected or active or influenced at that time step. By a perfect target set or a dynamic monopoly (dynamo) we mean any subset $D$ of the vertices of $G$ such that for the process starting from $D$, there exists some $\ell \in \mathbb{N}$ such that for every $i \geq \ell$ we have $D_i = V(G)$ (i.e. all vertices reach to the steady state of 1). By the size of a perfect

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target set $D$ we mean the cardinality of $D$. The smallest size of any perfect target set of $(G, \tau)$ is denoted by $\mathrm{NPPTS}_{\tau}(G)$ and also by $\mathrm{NPPTS}(G)$ if no confusion arises.

1.2. **Outline of the paper.** The outline of this paper is as follows: In section 2 we show that the activation process converges to an steady state or reaches a cycle of length two in $\mathcal{O}(|E(G)|)$ steps. In section 3 an integer programming formulation for finding a minimum perfect target set is given. In Section 4 we consider the strict majority threshold for the vertices and give a lower bound for $\mathrm{NPPTS}(G)$ where $G$ is a graph that all of its vertices have even degrees. Then it is shown that this bound is almost tight. This bound improves the lower bound proposed in [5] for graphs with no restriction on the degree of vertices. Section 5 is devoted for conclusions.

## 2. Convergence time

Any activation process on a given graph $G$ falls into a cycle (i.e. repetition) after some finite time steps. The reason is that in each time step the set of activated vertices is a subset of $V(G)$ and in order to avoid repetition the number of steps are at most $2^{|V(G)|}$ which is the cardinality of the power set of $V(G)$. In this section we show that with any threshold assignment for the vertices the activation process reaches the cycle on $\mathcal{O}(E|G|)$ time steps which is very smaller than the trivial bound $2^{|V(G)|}$.

**Definition 2.1.** *Let $(G, \tau)$ be a graph with a threshold assignment for its vertices. Suppose that for a given subset of vertices $D \subseteq V(G)$ the sequence $D_0, D_1, D_2, \ldots$ is an activation process of a non-progressive spread of influence on $(G, \tau)$ where $D_0 = D$. The* convergence time *of the activation process starting with $D$, which is denoted by $\mathrm{ct}_{(G,\tau)}(D)$ or simply by $\mathrm{ct}(D)$, is the smallest time step $T$ such that there exists $t < T$ with $D_T = D_t$.*

**Theorem 2.2.** [5] *The non-progressive spread of influence process on a graph reaches a cycle of length at most two.*

The above theorem shows that in Definition 2.1 we have either $t = T - 1$ or $t = T - 2$. Also note that if $D_0$ is a perfect target set, then $D_T = D_{T-1} = V(G)$. In the following theorem an upper bound for the convergence time of activation process in graphs with the strict majority threshold has been given in [5].

**Theorem 2.3.** [5] *For a given graph $G$ and any set $D \subseteq V(G)$, with strict majority threshold we have $\mathrm{ct}(D) = \mathcal{O}(|E(G)|)$.*

In the following theorem we extend Theorem 2.3 for general threshold $\tau$.

**Theorem 2.4.** *For a given graph $G$ and any set $D \subseteq V(G)$, with any threshold assignment $\tau$ we have $\mathrm{ct}_{(G,\tau)(D)} = \mathcal{O}(|E(G)|)$.*

*Proof.* Consider an activation process of $(G, \tau)$ starting with $D_0 = D$. We want to make a graph $H$ with strict majority threshold for its vertices such that the convergence time of activation process

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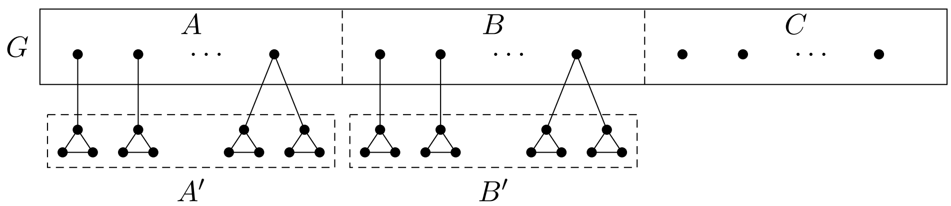


FIGURE 1. Graph $H$

on $(G,\tau)$ starting with $D_0$ is equal to the convergence time of activation process on $H$ with strict majority threshold starting with $M_0$ for some $M_0 \subseteq V(H)$. Construct $H$ from $G$ by adding some copies of complete graph $K_3$ and linking a vertex of each copy to some vertex of $G$ as follows.

Denote by $A$, $B$ and $C$ respectively the set of all vertices $v \in V(G)$ with $\tau(v) < \left\lceil \frac{\deg_G(v)+1}{2} \right\rceil$, $\tau(v) > \left\lceil \frac{\deg_G(v)+1}{2} \right\rceil$ and $\tau(v) = \left\lceil \frac{\deg_G(v)+1}{2} \right\rceil$.

- If $v \in A$, then add $\deg_G(v) - 2\tau(v) + 1$ copies of complete graph $K_3$ and connect one of the vertices of each copy to $v$ and set $\tau'(v) = \tau(v) + (\deg_G(v) - 2\tau(v) + 1) = \deg_G(v) - \tau(v) + 1$. For the added vertices $u$ (i.e. in copies of $K_3$) set $\tau'(u) = 2$.
- If $v \in B$, then add $2\tau(v) - \deg_G(v) - 1$ copies of complete graph $K_3$ and connect one of the vertices of each copy to $v$ and set $\tau'(v) = \tau(v)$. For the added vertices $u$ (i.e. in copies of $K_3$) set $\tau'(u) = 2$.
- If $v \in C$, then set $\tau'(v) = \tau(v)$.

Figure 1 shows the graph $H$. It is easy to see that $\tau'$ is strict majority threshold for the vertices of $H$. Note that since adding each copy of $K_3$ adds 4 edges to the graph and $0 \leq \tau(v) \leq \deg_G(v)$ we have

$$
\begin{aligned}
|E(H)| &= |E(G)| + 4\sum_{v\in A}(\deg_G(v) - 2\tau(v) + 1) + 4\sum_{v\in B}(2\tau(v) - \deg_G(v) - 1) \\
&\leq |E(G)| + 4\sum_{v\in A}(\deg_G(v) + 1) + 4\sum_{v\in B}(\deg_G(v) - 1) \\
&\leq |E(G)| + 4\sum_{v\in V(G)}(\deg_G(v) + 1) \\
&= 4|V(G)| + 9|E(G)| \\
&= \mathcal{O}(|E(G)|).
\end{aligned}
$$

In the last line of above inequality we have assumed that $G$ does not have isolated vertices because otherwise the isolated vertices have threshold zero and their convergence time is at most 2. Now suppose that $M_0 = D_0 \cup A'$ where $A'$ consists of the vertices of all copies of $K_3$ attached to a vertex in $A$. By considering activation processes $D_0, D_1, D_2, \ldots$ and $M_0, M_1, M_2, \ldots$ respectively on $(G,\tau)$ and $(H,\tau')$ it is easy to show that for all $i \geq 0$ we have $M_i = D_i \cup A'$. So we conclude that $\mathrm{ct}_{(G,\tau)}(D_0) = \mathrm{ct}_{(H,\tau')}(M_0)$. Using Theorem 2.3 we have $\mathrm{ct}_{(H,\tau')}(M_0) = \mathrm{ct}(M_0) = \mathcal{O}(|E(H)|)$. So we have $\mathrm{ct}_{(G,\tau)}(D_0) = \mathcal{O}(|E(H)|) = \mathcal{O}(|E(G)|)$. This completes the proof. □

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Theorem 2.4 shows that starting from any perfect target set as the initially infected vertices will infect entire the graph in $\mathcal{O}(|E(G)|)$ time steps (i.e. $D_\ell = V(G)$ where $\ell = \mathcal{O}(|E(G)|)$). This will be used in the next section for finding a minimum perfect target set.

## 3. Integer Programming Approach

In this section we give an integer linear programming formulation for the size of minimum perfect target set of graphs with general threshold $\tau$ for the vertices of the graph. In our formulation we assume that the process takes place in time period $k$ which $k$ is a predefined constant since in many applications the time period is bounded as the *latency* of the process. In the case that there is no predefined latency we can use Theorem 2.4 and be sure that in $k = \mathcal{O}(|E(G)|)$ time steps process reaches to a convergence.

We assume that the binary variable $x_{vi}$ shows the state of the vertex $v$ in time step $i$ (i.e. $x_v = 1$ if and only if $v$ is infected in time step $i$). Here we present an integer linear programming formulation for the problem:

$$\begin{array}{lll} & \min \sum_{v\in V(G)} x_{v0} & \\ \text{s.t.} & & \\ (3.1) & (\deg(v)+1-\tau(v))x_{vi} - \sum_{u\in N(v)} x_{u(i-1)} \geq 1-\tau(v) & \forall v\in V(G)\ \forall i=1,\dots,k \\ & \sum_{u\in N(v)} x_{u(i-1)} - \tau(v)x_{vi} \geq 0 & \forall v\in V(G)\ \forall i=1,\dots,k \\ & x_{vk}=1 & \forall v\in V(G) \\ & x_{vi}\in\{0,1\} & \forall v\in V(G)\ \forall i=0,\dots,k. \end{array}$$

**Theorem 3.1.** *The IP formulation (3.1) gives the size of minimum perfect target set with latency $k$.*

*Proof.* Activation process in non-progressive spread of influence has two rules:

(1) If the number of active neighbors of vertex $v$ in time step $i-1$ is greater than or equal to $\tau(v)$, then $v$ becomes active in time step $i$. This is equivalent to:

$$\sum_{u\in N(v)} x_{u(i-1)} \geq \tau(v) \implies x_{vi} = 1. \tag{3.2}$$

(2) If the number of active neighbors of vertex $v$ in time step $i-1$ is strictly less than $\tau(v)$, then $v$ becomes inactive in time step $i$. This is equivalent to:

$$\sum_{u\in N(v)} x_{u(i-1)} < \tau(v) \implies x_{vi} = 0. \tag{3.3}$$

Now we give linear constraints to guarantee these two rules. For the first one consider the following constraint:

$$(\deg(v)+1-\tau(v))x_{vi} - \sum_{u\in N(v)} x_{u(i-1)} \geq 1-\tau(v). \tag{3.4}$$

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Note that in case $x_{vi} = 1$, the constraint (3.4) reduces to $\deg(v) \geq \sum_{u\in N(v)} x_{u(i-1)}$ which is trivial and gives us nothing. In case $x_{vi} = 0$, the constraint (3.4) reduces to $\sum_{u\in N(v)} x_{u(i-1)} \leq \tau(v) - 1$. This means that

$$x_{vi} = 0 \implies \sum_{u\in N(v)} x_{u(i-1)} \leq \tau(v) - 1.$$

This is the contrapositive of (3.2) and so equivalent to it. So constraint (3.4) guarantees first rule of the activation process.

For the second rule consider the following constraint:

$$\sum_{u\in N(v)} x_{u(i-1)} - \tau(v)x_{vi} \geq 0 \tag{3.5}$$

Note that in case $x_{vi} = 0$, the constraint (3.5) reduces to $\sum_{u\in N(v)} x_{u(i-1)} \geq 0$ which is trivial and gives us nothing. In case $x_{vi} = 1$, the constraint (3.5) reduces to $\sum_{u\in N(v)} x_{u(i-1)} \geq \tau(v)$. This means that

$$x_{vi} = 1 \implies \sum_{u\in N(v)} x_{u(i-1)} \geq \tau(v).$$

This is the contrapositive of (3.3) and so equivalent to it. So constraint (3.5) guarantees second rule of the activation process.

In order to have a perfect target set eventually (i.e. at the time $k$), all of the vertices should be activated. So for each vertex $v \in V(G)$ we add the following constraint:

$$x_{vk} = 1. \tag{3.6}$$

Our goal is to find a perfect target set of minimum size. So clearly the objective function should be $\sum_{v\in V(G)} x_{v0}$. □

Here at the end of this section we state the following proposition which gives a valid inequality for our integer programming formulation.

**Proposition 3.2.** *Adding the following inequalities for $i = 1, \ldots, k$ to the IP formulation (3.1) does not change the optimum value.*

$$\sum_{v\in V(G)} x_{v0} \leq \sum_{v\in V(G)} x_{vi} - 1.$$

*Proof.* In case that for some dynamic target set we have $\sum_{v\in V(G)} x_{vi} \geq \sum_{v\in V(G)} x_{v0}$ which $(x_{v0})_{v\in V(G)}$ is the characteristic vector of the dynamic target set of $G$, we can replace $x_{v0}$ by $x_{vi}$ to get new dynamic target set whose size does not increase and the convergence time decreases. □

## 4. Strict Majority Threshold

In this section we consider strict majority threshold in which for all vertices of the given graph we have $\tau(v) = \left\lceil \frac{\deg(v)+1}{2} \right\rceil$. With this threshold assignment we give a lower bound for the size of non-progressive perfect target set of graphs which all of the vertices have even degrees. This bound is asymptotically two times greater than the bound given in [5] for general graphs. We start with following theorem for bipartite graphs.

**Theorem 4.1.** [5] *For every bipartite graph $G$ of order $n$ whose maximum degree is $\Delta$ we have*

$$\mathrm{NPPTS}(G) \geq \frac{2n}{\Delta+1}.$$

In the following theorem we improve the lower bound of Theorem 4.1 for bipartite graphs which all of the vertices have even degrees.

**Theorem 4.2.** *Let $G$ be a bipartite graph of order $n$ and maximum degree $\Delta$. Suppose that all vertices of $G$ have even degrees. Then we have*

$$\mathrm{NPPTS}(G) \geq \frac{4n}{\Delta+2}.$$

*Proof.* From the proof of Theorem 4.1 (see [5]) we have

$$\sum_{v\in W}(2\tau(v)-\deg(v)) \leq \sum_{v\in B}(\deg(v)-\tau(v)) \tag{4.1}$$

where $B$ is the set of initially active vertices (i.e. infected vertices at time step 0) and $W = V(G)\setminus B$. By the assumption that every vertex $v \in V(G)$ has even degree we have:

$$\tau(v) = \left\lceil \frac{\deg(v)+1}{2} \right\rceil = \frac{\deg(v)+2}{2}.$$

Therefore $2\tau(v) - \deg(v) = 2$ and $\deg(v) - \tau(v) = \frac{\deg(v)-2}{2}$. So from inequality (4.1) we get:

$$2|W| \leq \sum_{v\in B} \frac{\deg(v)-2}{2}.$$

And so

$$2|W| \leq \frac{\Delta-2}{2}\left(|B|\right).$$

Since $|B| + |W| = n$, we conclude that

$$|B| \geq \frac{4n}{\Delta+2},$$

as desired. ∎

The following proposition has been stated in the proof of [5, Theorem 1].

**Proposition 4.3.** *Let $G$ be a graph vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and threshold assignment $\tau$ for its vertices. Construct bipartite graph $H$ with partite sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ and edge set $E(H) = \{x_iy_j \mid v_iv_j \in E(G)\}$. Define threshold assignment $\tau'(x_i) = \tau'(y_i) = \tau(v_i)$. Then we have*

$$\mathrm{NPPTS}_{\tau'}(H) = 2\mathrm{NPPTS}_{\tau}(G).$$

Now using the previous proposition we extend the bound of Theorem 4.2 for general graphs.

**Theorem 4.4.** *Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Suppose that all of the vertices of $G$ have even degrees and strict majority thresholds. Then we have*

$$\mathrm{NPPTS}(G) \geq \frac{4n}{\Delta+2}.$$

*Proof.* Suppose that $\tau$ is strict majority threshold for $G$. Construct bipartite graph $H$ with the threshold assignment $\tau'$ for its vertices like in Proposition 4.3. Clearly $\tau'$ is strict majority threshold for $H$. Using Proposition 4.3 and Theorem 4.2 for $H$ we have:

$$\begin{aligned}\mathrm{NPPTS}(G) &= \frac{1}{2}\mathrm{NPPTS}(H)\\ &\geq \frac{1}{2}\times\frac{4|V(H)|}{\Delta(H)+2}\\ &= \frac{1}{2}\times\frac{8|V(G)|}{\Delta(G)+2}\\ &= \frac{4|V(G)|}{\Delta(G)+2}\end{aligned}$$

as desired. □

The following theorem shows that the bound given in Theorem 4.4 is asymptotically sharp.

**Theorem 4.5.** *For infinitely many $n$ and for any $0<\epsilon<1$ and any integer $d>1$ there exists a $2d$-regular graph with $n$ vertices for which we have*

$$\mathrm{NPPTS}(G) < \frac{1}{1-\epsilon}\left(\frac{4n}{\Delta+2}\right).$$

*Proof.* Suppose that $k$ is a positive integer such that $\left(\frac{d-1}{d+1}\right)^{k+1} < \epsilon$. $G$ be a graph with vertex set $V(G)$ which is a disjoint union of the form $V(G) = V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ where $V_1$ consists of $n_1 = c(d+1)^{k+1}$ vertices and for $1 \leq i \leq k$, $V_{i+1}$ is a set of vertices of the size $n_{i+1} = \frac{d-1}{d+1}n_i$. Consider a bipartite graph $H_i$ for $1 \leq i \leq k$ with partite sets $V_i$ and $V_{i+1}$ which every vertex in $V_i$ has degree $d-1$ and every vertex in $V_{i+1}$ has degree $d+1$. Construct regular graphs $G_1$ and $G_{k+1}$ on vertex sets $V_1$ and $V_{k+1}$ respectively where $G_1$ is $(d+1)$-regular and $G_{k+1}$ is $(d-1)$-regular. Now suppose that $E(G) = \left(\bigcup_{i=1}^{k} E(H_i)\right) \cup E(G_1) \cup E(G_{k+1})$. Clearly $G$ is a $2d$-regular graph and so $\Delta = 2d$.

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The graph $G$ has

$$\begin{aligned} n &= \sum_{i=1}^{k+1} n_i \\ &= n_1 \left( \frac{1 - \left(\frac{d-1}{d+1}\right)^{k+1}}{1 - \frac{d-1}{d+1}} \right) \\ &= \frac{n_1}{2} \left( (d+1) - (d+1) \left( \frac{d-1}{d+1} \right)^{k+1} \right) \\ &> \frac{n_1}{2} \left( (d+1) - (d+1)\epsilon \right) \\ &= \frac{n_1}{2} (d+1)(1-\epsilon) \end{aligned}$$

vertices. It is easy to see that $V_1$ is a perfect target set for $G$. So for the size of minimum perfect target set of $G$ we have

$$\begin{aligned} \text{NPPTS}(G) &\leq n_1 \\ &< \frac{2n}{(d+1)(1-\epsilon)} \\ &= \frac{1}{1-\epsilon} \left( \frac{4n}{\Delta + 2} \right). \end{aligned}$$

This inequality completes the proof. □

## 5. **Conclusion**

This paper studies the minimum target set selection problem in the non-progressive model of spread of influence with general and strict majority thresholds. Our main results include upper bound for convergence time of the activation process and integer linear programming formulation for the problem and also a lower bound for the size of perfect target sets in case that every vertex has even degree.

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