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LINE GRAPHS ASSOCIATED TO ANNIHILATING-IDEAL GRAPH ATTACHED TO LATTICES OF GENUS ONE

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Abstract. Let $(L, \wedge, \vee)$ be a lattice with a least element 0. The annihilating-ideal graph of L, denoted by $\mathbb{AG}(L)$, is a graph whose vertex-set is the set of all non-trivial ideals of L and, for every two distinct vertices I and J, the vertex I is adjacent to J if and only if $I \wedge J = \{0\}$. In this paper, we characterize all lattices L whose the graph $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal.

1. Introduction

One important theorem due to Whitney about line graphs is that with one exceptional case, $\mathfrak{L}(G) = K_3$, the structure of any connected graph can be recovered from its line graph. In [14], Whitney proved there is a one-to-one correspondence between the class of connected graphs and the class of connected line graphs as follows:

Theorem 1.1. [14, Theorem 1] *Let G and G' be two connected graphs, neither of which consists of three arcs of the form ab, ac and ad. Let there be a $1-1$ correspondence between their arcs so that to any two arcs having a common vertex in one graph correspond two arcs having a common vertex in the other. Then G and G' are congruent.*

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On the other hand, the main object of topological graph theory is to embedding a graph into a surface. Therefore, the investigation of the embedding of line graphs associated to lattices into a torus can make the relationship between algebraic structures, graph theory and topological graph theory more attractive.

The concept of a annihilating-ideal graph of a lattice L, denoted by $\mathbb{AG}(L)$, is defined by Khashyarmanesh et al in [1]. Parsapour and Ahmad Javaheri completely determined all finite lattices L with projective annihilating-ideal graphs $\mathbb{AG}(L)$ in [11] and all finite lattices L with planar and projective line graphs associated to annihilating-ideal graphs $\mathbb{AG}(L)$ in [12].

$\mathbb{AG}(L)$ is a graph whose vertex-set is the set of all non-trivial ideals of L and, for every two distinct vertices I and J, the vertices I and J are adjacent if and only if $I \wedge J = 0$. In this work, we assume that L is a finite lattice and $A(L) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of L. In the second section of this paper, we study the finite lattices L with toroidal line graph associated to annihilating-ideal graphs $\mathbb{AG}(L)$.

First we summarize notations and concepts on lattices and graphs which will be needed in the next section.

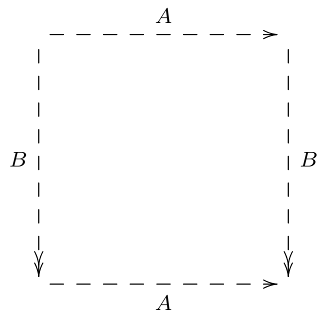
For basis facts concerning lattices we refer to [5] and [10].

Let $L = (L, \wedge, \vee)$ be a lattice with a least element 0 and I be a non-empty subset of L. We say that I is an *ideal* of L if, (i) for all $a, b \in I$, $a \vee b \in I$ and (ii) if $0 \leq a \leq b$ and $b \in I$, then $a \in I$.

For two distinct ideals I and J of a lattice L, we have $I \wedge J = \{x \wedge y \ ; \ x \in I, y \in J\}$.

Now, we recall some definitions and notation on graphs. By a graph $G = (V, E)$, we mean an undirected simple graph with vertex-set V and edge-set E. In a graph G, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. The *degree* of a vertex x, denoted by $deg(x)$, is the number of edges incident to x, and an *isolated vertex* is a vertex with zero degree. We define $G \setminus H$ to be the graph with edge-set $E(G) - E(H)$ and vertex-set $V(G) - V(H) + V(G \cap H)$. A graph with no edges (but at least one vertex) is called an *empty graph*. The graph with no vertices and no edges is the *null graph*. The complement $\overline{G}$ of a graph G has the same vertex-set as G, where vertices x and y are adjacent in $\overline{G}$ if and only if they are not adjacent in G (see [2] and [6]). A graph G is said to be *contracted* to a graph H if there exists a sequence of elementary contractions which transforms G into H, where an *elementary contraction* consists of deletion of a vertex or an edge or the identification of two adjacent vertices. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. The line graph of a graph G is the graph $\mathfrak{L}(G)$ with the edges of G as its vertices, and two edges of G are adjacent in $\mathfrak{L}(G)$ if and only if they are incident in G. In this work, we denote $w_{i,j}$ for the vertex $[v_i, v_j] \in \mathfrak{L}(G)$, where v_i and v_j are adjacent vertices in G.

Recall that a graph is said to be *planar* if it can be drawn in the plane, such that its edges intersect only at their ends. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [8]). By a *surface*, we mean a connected compact 2-dimensional real manifold without boundary, that is a connected topological space such that each point has a neighbourhood homeomorphic to an open disc. It is well-known that every compact surface is homeomorphic to a sphere, or to a connected sum of g tori (S_g), or to a connected sum of k projective planes (N_k) (see [9, Theorem 5.1]). This number g is called the *genus* of the surface. The *torus* can be thought of as a sphere with one handle. This means that the genus of torus is 1. A torus is represented in the usual way as a rectangle in which both pairs of opposite sides are identified with each other as indicated by arrows.



The canonical representation of a torus

A graph G is *embeddable* in a surface S if the vertices of G are assigned to distinct points in S such that every edge of G is a simple arc in S connecting the two vertices which are joined in G. The least number g that G can be embedded in S_g is called the genus of G and denoted by $\gamma(G)$. One easy observation is that $\gamma(H) \leq \gamma(G)$, for any subgraph H of G. One of the most remarkable theorems in topological graph theory, known as *Euler's formula*, states that if G is a finite connected graph with n vertices, m edges and genus g, then $n - m + f = 2 - 2g$, where f is the number of faces created when G is minimally embedded in a surface of genus g.

Note that graphs of genus 0 are *planar graphs* and graphs of genus 1 are *toroidal graphs*. We know that a complete graph K_n is a toroidal graph, if $n = 5, 6$ or 7, and the only toroidal complete bipartite graphs are $K_{4,4}$ and $K_{3,n}$, with $n = 3, 4, 5$ or 6 (see [3] or [13]).

By [14, Lemma 4.1],

We end this section with one of the lemmas in [4] that is very helpful in proving the theorems.

Lemma 1.2. [4, Lemma 4.1] *Let G be a simple graph and u, v be two distinct vertices of G such that $deg(u) = m$ and $deg(v) = n$. Then $\gamma(\mathfrak{L}(G)) \geq \gamma(K_m) + \gamma(K_n)$.*

2. Toroidal Line Graph Associated to Annihilating-ideal Graph of a Finite Lattice

In this paper, we assume that L is a finite lattice and $\mathbb{A}(L) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of L. The annihilating-ideal graph of a lattice L, denoted by $\mathbb{AG}(L)$, is an undirected simple graph with all non-trivial ideals of L, and two distinct vertices I and J are adjacent if and only if $I \wedge J = 0$. The main goal of this section is to determine all finite lattices L such that the graph $\mathfrak{L}(\mathbb{AG}(L))$ has genus one. We begin this section with the following notation, which is needed in the rest of the paper.

Notation. Let $i_1, i_2, \ldots, i_n$ be integers with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The notation $U_{i_1 i_2 \cdots i_k}$ stands for the following set:

$$\{I \trianglelefteq L;\ \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \subseteq I \text{ and } a_j \notin I,\ \text{for } j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}\}.$$

Note that all two distinct elements in $U_{i_1 i_2 \cdots i_k}$ are not adjacent in $\mathbb{AG}(L)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \cdots i_k}$ and $U_{j_1 j_2 \cdots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \cdots i_k} \cap U_{j_1 j_2 \cdots j_{k'}} = \emptyset$. Moreover, $V(\mathbb{AG}(L)) = \bigcup U_{i_1 i_2 \cdots i_k}$, for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Suppose that L has n atoms. Then $U_{12 \cdots n}$ consists of isolated vertices. Clearly, the isolated points do not affect toroidality. Hence, we ignore the set $U_{12 \cdots n}$ from the vertex-set of $\mathbb{AG}(L)$, and so we do not show these points in our figures.

We know that if $|A(L)| = 1$, then $\mathbb{AG}(L)$ is an empty graph, and hence $\mathfrak{L}(\mathbb{AG}(L))$ is a null graph. Hence we ignore the case that the lattice has one atom, exactly.

Now, we begin with a result which exhibits an upper bound for the number of atoms of a lattice.

Lemma 2.1. *If $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph, then the size of $A(L)$ is at most five.*

Proof. Assume to the contrary that $|A(L)| \geq 6$. Then $\mathbb{AG}(L)$ contains a copy of K_6, and so it contains two distinct vertices of degree at least 5. Hence, by [4, Lemma 4.1], the graph $\mathfrak{L}(\mathbb{AG}(L))$ has genus greater than one, which is not toroidal. □

In view of Lemma 2.1, it is sufficient to investigate the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$ in the cases that the size of $A(L)$ is $2, 3, 4$ or 5.

First we state the necessary and sufficient conditions for the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$, when $|A(L)| = 2$. To do this, we need the following lemma.

Lemma 2.2. *If $|\bigcup_{j=1}^{2} U_j| \leq 5$ or $|\bigcup_{j=1}^{2} U_j| \geq 9$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.*

Proof. First we assume that $|\bigcup_{j=1}^{2} U_j| \leq 5$. By [12, Theorem 2.3], the graph $\mathfrak{L}(\mathbb{AG}(L))$ is planar, which is not toroidal. Now, suppose that $|\bigcup_{j=1}^{2} U_j| \geq 9$. We know that as $|A(L)| = 2$, the graph $\mathbb{AG}(L)$ is a complete bipartite graph. If $\mathbb{AG}(L)$ is a star graph, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a copy of K_8, which is not toroidal. Otherwise, the graph $\mathbb{AG}(L)$ contains a copy of $K_{2,5}$. By [4, Lemma 4.1], the graph $\mathfrak{L}(K_{2,5})$ has genus greater than one, which is not toroidal. Therefore $\mathfrak{L}(\mathbb{AG}(L))$ is not a toroidal graph. □

By the above lemma, when $|A(L)| = 2$, it is sufficient to study the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$, in the following cases.

Case 1. $|\bigcup_{j=1}^{2} U_j| = 6$. In this case, the graph $\mathbb{AG}(L)$ is isomorphic to one of the graphs $K_{1,5}$, $K_{2,4}$ or $K_{3,3}$. If $\mathbb{AG}(L) \cong K_{1,5}$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ is isomorphic to K_5, which is toroidal. Now, suppose that $\mathbb{AG}(L) \cong K_{2,4}$. By [4, Example 2.14], $\gamma(\mathfrak{L}(K_{2,4})) = 1$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal. Also suppose that $\mathbb{AG}(L) \cong K_{3,3}$. By [4, Example 2.12], $\gamma(\mathfrak{L}(K_{3,3})) = 1$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal.

Case 2. $|\bigcup_{j=1}^{2} U_j| = 7$. In this case, the graph $\mathbb{AG}(L)$ is isomorphic to one of the graphs $K_{1,6}$, $K_{2,5}$ or $K_{3,4}$. If $\mathbb{AG}(L) \cong K_{1,6}$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ is isomorphic to K_6, which is toroidal. Now, suppose that $\mathbb{AG}(L) \cong K_{2,5}$. By [4, Lemma 4.1], $\gamma(\mathfrak{L}(K_{2,5})) \geq 2$, which implies that the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also suppose that $\mathbb{AG}(L) \cong K_{3,4}$. By [4, Example 2.14], $\gamma(\mathfrak{L}(K_{3,4})) = 2$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

Case 3. $|\bigcup_{j=1}^{2} U_j| = 8$. In this case, the graph $\mathbb{AG}(L)$ is isomorphic to one of the graphs $K_{1,7}$, $K_{2,6}$, $K_{3,5}$ or $K_{4,4}$. If $\mathbb{AG}(L) \cong K_{1,7}$, then $\mathfrak{L}(\mathbb{AG}(L))$ is isomorphic to K_7, which is toroidal. Now, suppose that $\mathbb{AG}(L) \cong K_{2,6}$. By [4, Lemma 2.9], $\gamma(\mathfrak{L}(K_{2,6})) \geq 2$, and so $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also suppose that $\mathbb{AG}(L)$ is isomorphic to $K_{3,5}$ or $K_{4,4}$. Clearly, the graph $\mathbb{AG}(L)$ contains a copy of $K_{3,4}$. By [4, Example 2.14], $\gamma(\mathfrak{L}(K_{3,4})) = 2$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

As a consequence of the above discussion and Lemma 2.2, we have the next theorem.

Theorem 2.3. *Suppose that $|A(L)| = 2$. Then $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph if and only if one of the following conditions holds:*

(1) $|\bigcup_{j=1}^{2} U_j| = 6$.
(2) $7 \leq |\bigcup_{j=1}^{2} U_j| \leq 8$ *and* $|U_i| = 1$, *for some unique* $i \in \{1, 2\}$.

In the following, we investigate the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$ in the case that $|A(L)| = 3$.

Lemma 2.4. *If $|\bigcup_{j=1}^{3} U_j| \geq 6$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.*

Proof. To proof this lemma, it is sufficient to consider $|\bigcup_{j=1}^{3} U_j| = 6$. Then the graph $\mathbb{AG}(L)$ is isomorphic to one of the graphs $K_{4,1,1}$, $K_{3,2,1}$ or $K_{2,2,2}$. If $\mathbb{AG}(L) \cong K_{4,1,1}$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also if $\mathbb{AG}(L) \cong K_{3,2,1}$ and the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1$, $v_3 = J_1 \in U_1$, $v_4 = \{0, a_2\}$, $v_5 = I_2 \in U_2$ and $v_6 = \{0, a_3\} \in U_3$, then the contraction of the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a copy of G_1, one of the listed graphs in [15] (see Figure 1). Therefore again $\mathfrak{L}(\mathbb{AG}(L))$ is not a toroidal graph.

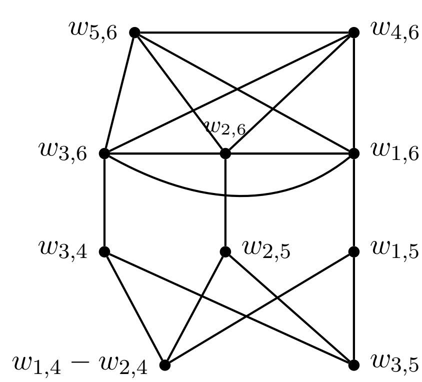


Figure 1.

Also if $\mathbb{AG}(L) \cong K_{2,2,2}$ and the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1 \in U_1$, $v_3 = \{0, a_2\}$, $v_4 = I_2 \in U_2$, $v_5 = \{0, a_3\}$, $v_6 = I_3 \in U_3$, then the contraction of the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a copy of G_2, one of the listed graphs in [15] (see Figure 2). Therefore $\mathfrak{L}(\mathbb{AG}(L))$ is not a toroidal graph.

□

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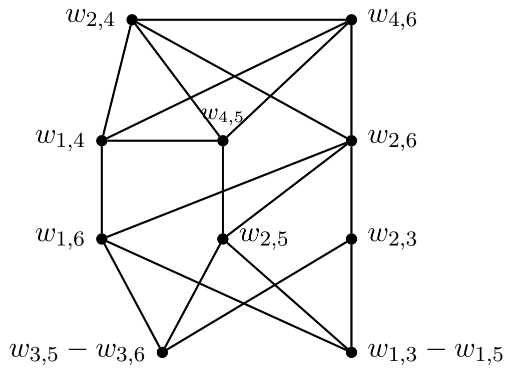

FIGURE 2.

In the following, we show an example of a lattice L whose the line graph associated to $\mathbb{AG}(L)$ is not toroidal.

Example 2.5. *A lattice L containing three atoms is pictured below (see Figure 3). We consider the graph $\mathbb{AG}(L)$ with vertices $v_1 = \{0, a_1\}$, $v_2 = I_1 = \{0, a_1, a_4\} \in U_1$, $v_3 = \{0, a_2\}$, $v_4 = I_2 = \{0, a_2, a_5\} \in U_2$, $v_5 = \{0, a_3\}$, $v_6 = I_3 = \{0, a_3, a_6\} \in U_3$. It is clear that $\mathbb{AG}(L) \cong K_{2,2,2}$ and so, by lemma 2.4, the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.*

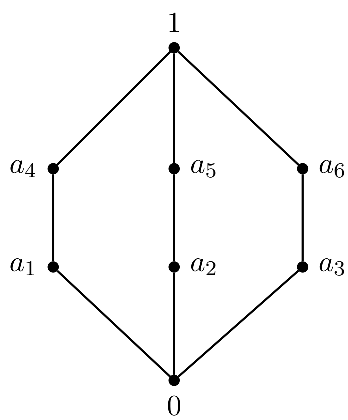

FIGURE 3.

Theorem 2.6. *Suppose that $|A(L)| = 3$. Then $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph if and only if one of the following conditions holds:*

(1) *$|\bigcup_{j=1}^{3} U_j| = 3$ and there exist unique $i, j \in \{1, 2, 3\}$, such that $3 \leq |U_{ij}| \leq 5$ and $|U_{ik}| \leq 2$ and $|U_{jk}| \leq 2$, for $1 \leq k \leq 3$.*

(2) *$|\bigcup_{j=1}^{3} U_j| = 4$ and there exists unique $i \in \{1, 2, 3\}$, such that $|U_i| = 2$.*

   (a) *$|U_{jk}| = 2$, $|U_{ij}| \leq 1$ and $|U_{ik}| \leq 1$, for $1 \leq i, j, k \leq 3$.*

   (b) *There exists unique $j \in \{1, 2, 3\}$, such that $2 \leq |U_{ij}| \leq 4$, $|U_{ik}| \leq 1$ and $|U_{jk}| \leq 1$, for $1 \leq k \leq 3$.*

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(3) $|\bigcup_{j=1}^{3} U_j| = 5$.

(a) *There exists* $i \in \{1,2,3\}$, *such that* $|U_i| = 3$. *If* $|U_{ij}| = 1$, *then* $U_{ik} = U_{jk} = \varnothing$, *with* $1 \le j,k \le 3$. *Also if* $|U_{jk}| = 1$, *then* $U_{ij} = U_{ik} = \varnothing$, *with* $1 \le j,k \le 3$.

(b) *There exists unique* $i \in \{1,2,3\}$, *such that* $|U_i| = 1$. *If* $|U_{ij}| = 1$, *then* $U_{ik} = U_{jk} = \varnothing$, *with* $1 \le j,k \le 3$. *Also if* $|U_{jk}| = 2$, *then* $U_{ij} = U_{ik} = \varnothing$, *with* $1 \le j,k \le 3$.

Proof. Firstly, we assume that $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph. By Lemma 2.4, $|\bigcup_{j=1}^{3} U_j| \le 5$. We have the following cases.

Case 1. $|\bigcup_{j=1}^{3} U_j| = 3$. In this case, if $|U_{ij}| \le 2$, for all $i,j \in \{1,2,3\}$, then, by [12, Theorem 2.5], the graph $\mathfrak{L}(\mathbb{AG}(L))$ is planar, which is not toroidal. If one of the sets U_{12}, U_{13} or U_{23} has at least six elements, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a subgraph isomorphic to K_8, which is not toroidal. Also, without loss of generality we may assume that the sets U_{12} and U_{13} have at least three elements. Then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \ge 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

Case 2. $|\bigcup_{j=1}^{3} U_j| = 4$ and $|U_i| = 2$. In this case, if $|U_{ij}| \le 1$, for all $i,j \in \{1,2,3\}$, then, by [12, Theorem 2.5], the graph $\mathfrak{L}(\mathbb{AG}(L))$ is planar, which is not toroidal. If $|U_{12}| = |U_{13}| = 2$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \ge 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also if $|U_{12}| = |U_{23}| = 2$, then the contraction of $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal. Now, we may assume that U_{12} has at least five elements. Then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a subgraph isomorphic to K_8, which is not toroidal. Also we may assume that $|U_{23}| \ge 3$. Then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \ge 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

Case 3. $|\bigcup_{j=1}^{3} U_j| = 5$. First, assume that $|U_i| = 3$, for some i, say $i = 1$, with $|U_1| = 3$. If $|U_{12}| = |U_{13}| = 1$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \ge 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also if $|U_{12}| = |U_{23}| = 1$, then the contraction of $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal. Now, we may assume that U_{12} has at least two elements and the vertices $v_1 = \{0, a_1\}, v_2 = I_1, v_3 = J_1 \in U_1$, $v_4 = \{0, a_2\} \in U_2$, $v_5 = \{0, a_3\} \in U_3$ and $v_6 = I_{12}, v_7 = I'_{12} \in U_{12}$. Then the complement of $\mathfrak{L}(\mathbb{AG}(L))$ is contained in $T5.8$, one of the listed graphs in [7] (see Figure 4). Thus it is not toroidal.

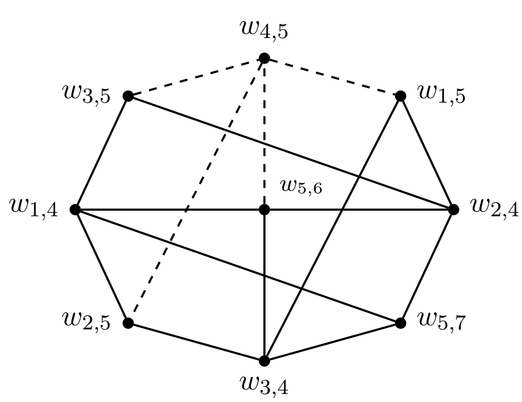

FIGURE 4.

Also if $|U_{23}| \geq 2$, then the graph $\mathbb{AG}(L)$ contains a copy of $K_{3,4}$. By [4, Example 2.14], $\gamma(\mathfrak{L}(K_{3,4})) = 2$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

Now, suppose that there exists unique U_i, with $1 \leq i \leq 3$, say U_1, such that $|U_1| = 1$. If U_{12} has at least two elements, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. If $|U_{12}| = |U_{23}| = 1$, then the contraction of $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal. And if $|U_{12}| = |U_{13}| = 1$,Then the graph $\mathbb{AG}(L)$ contains a subdivision of K_5. Clearly, $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \geq 1$. On the other hand, by Euler's formula, one can easily see that $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \neq 1$. Hence $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Also if $|U_{23}| \geq 3$, then the contraction of $\mathfrak{L}(\mathbb{AG}(L))$ is isomorphic to $K_8 \setminus K_{1,3}$. Hence it contains a copy of $D.3$, one of the listed graphs in [7], which is not a toroidal graph. Now, by the above cases, we have that if $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal,then one of the statements (1), (2) or (3) holds.

Conversely, if one of the mentioned conditions hold, then we show that $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph. First suppose that the statement (1) holds. Hence there exist unique integers i and j, with $1 \leq i, j \leq 3$, such that $3 \leq |U_{ij}| \leq 5$ and $|U_{ik}| \leq 2$ and $|U_{jk}| \leq 2$, for $1 \leq k \leq 3$. Without loss of generality, we may assume that $|U_{12}| = |U_{13}| = 2$ and $|U_{23}| = 5$. We have the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\} \in U_2$, $v_3 = \{0, a_3\} \in U_3$, $v_4 = I_{12}$, $v_5 = I'_{12} \in U_{12}$, $v_6 = I_{13}$, $v_7 = I'_{13} \in U_{13}$ and $v_8 = I_{23_1}$, $v_9 = I_{23_2}, \ldots$, $v_{12} = I_{23_5} \in U_{23}$. As Figure 5 gives an embedding of $\mathfrak{L}(\mathbb{AG}(L))$ on a torus, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal.

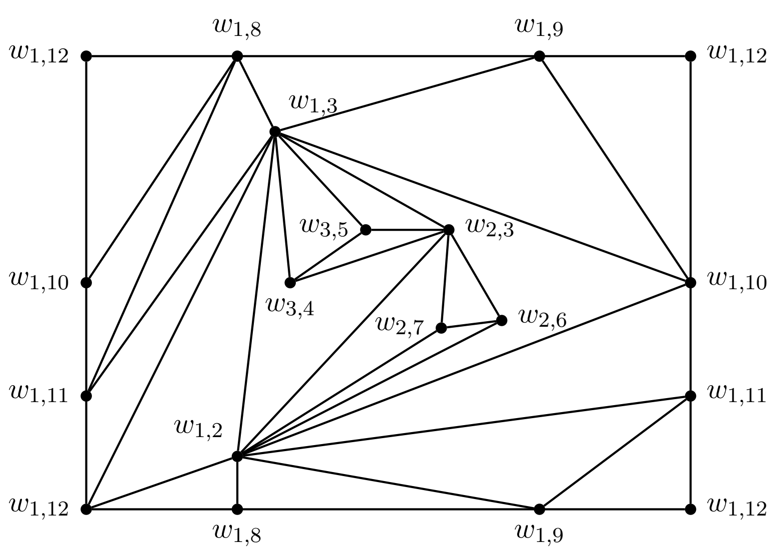

FIGURE 5.

Suppose that the first part of statement (2) holds. Without loss of generality, we may assume that $|U_1| = 2$, $|U_{12}| = |U_{13}| = 1$ and $|U_{23}| = 2$. We have the vertices $v_1 = \{0, a_1\}, v_2 = I_1 \in U_1$, $v_3 = \{0, a_2\} \in U_2$, $v_4 = \{0, a_3\} \in U_3$, $v_5 = I_{12} \in U_{12}$, $v_6 = I_{13} \in U_{13}$ and $v_7 = I_{23}$, $v_8 = I'_{23} \in U_{23}$. Therefore, the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 6, is toroidal.

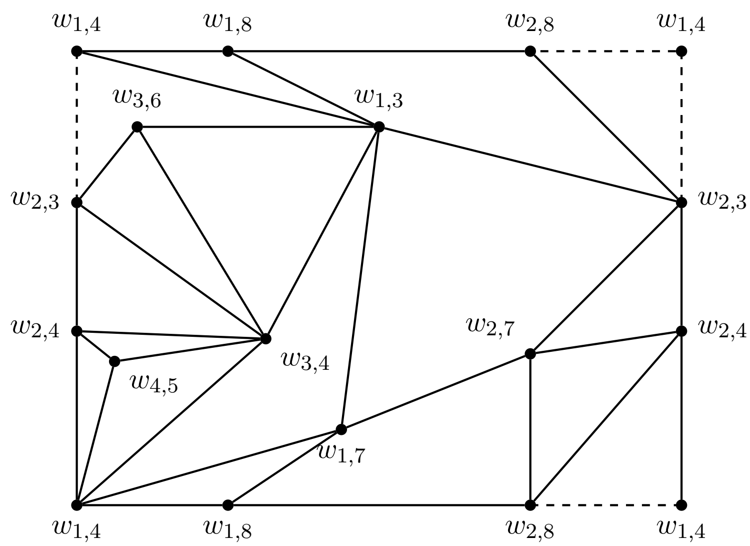

Figure 6.

Suppose that the second part of statement (2) holds. Without loss of generality, we can assume that $|U_{12}| = 4$ and $|U_{13}| = |U_{23}| = 1$. We have the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1 \in U_1$, $v_3 = \{0, a_2\} \in U_2$, $v_4 = \{0, a_3\} \in U_3$, $v_5 = I_{12}$, $v_6 = I'_{12}$, $v_7 = I''_{12}$, $v_8 = I'''_{12} \in U_{12}$, $v_9 = I_{13} \in U_{13}$ and $v_{10} = I_{23} \in U_{23}$. Therefore, the graph $\mathfrak{L}(\mathbb{AG}(L))$ is toroidal (see Figure 7).

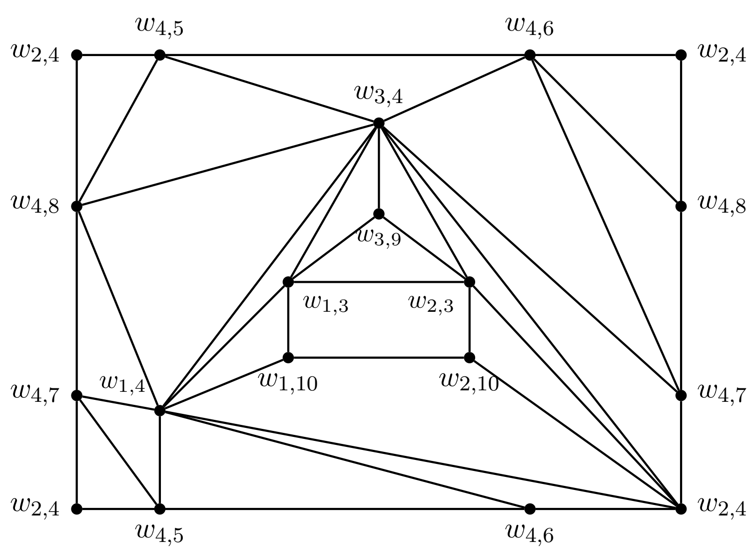

Figure 7.

Now, suppose that the statement (3)(a) holds. Then $|\bigcup_{j=1}^{3} U_j| = 5$ and without loss of generality, we assume that $|U_1| = 3$, $|U_{12}| = 1$ and $U_{13} = U_{23} = \varnothing$. So, by considering the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1$, $v_3 = I'_1 \in U_1$, $v_4 = \{0, a_2\} \in U_2$, $v_5 = \{0, a_3\} \in U_3$ and $v_6 = I_{12} \in U_{12}$, the graph $\mathfrak{L}(\mathbb{AG}(L))$ is pictured in Figure 8, which is toroidal.

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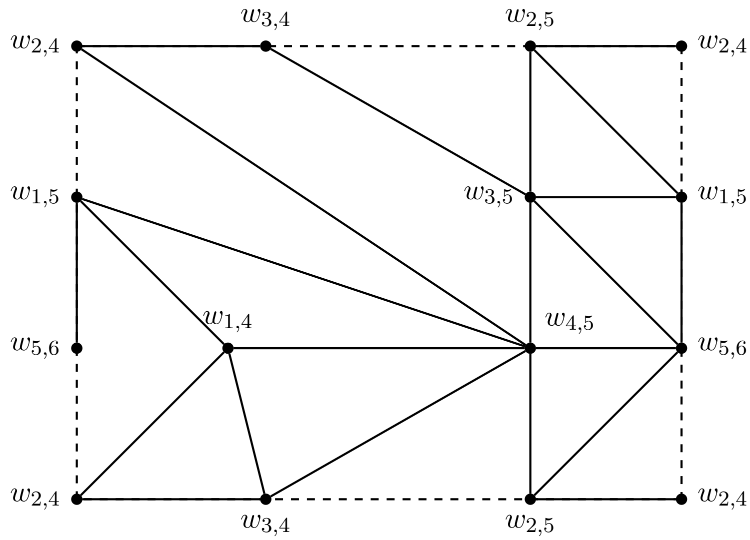

FIGURE 8.

Also suppose that $|\bigcup_{j=1}^{3} U_j| = 5$ and without loss of generality, $|U_1| = 3$, $|U_{23}| = 1$, $U_{12} = U_{13} = \varnothing$. Then, by considering the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1$, $v_3 = I_1' \in U_1$, $v_4 = \{0, a_2\} \in U_2$, $v_5 = \{0, a_3\} \in U_3$ and $v_6 = I_{23} \in U_{23}$, Figure 9 is the embedding of $\mathfrak{L}(\mathbb{AG}(L))$ on torus. Hence it is a toroidal graph.

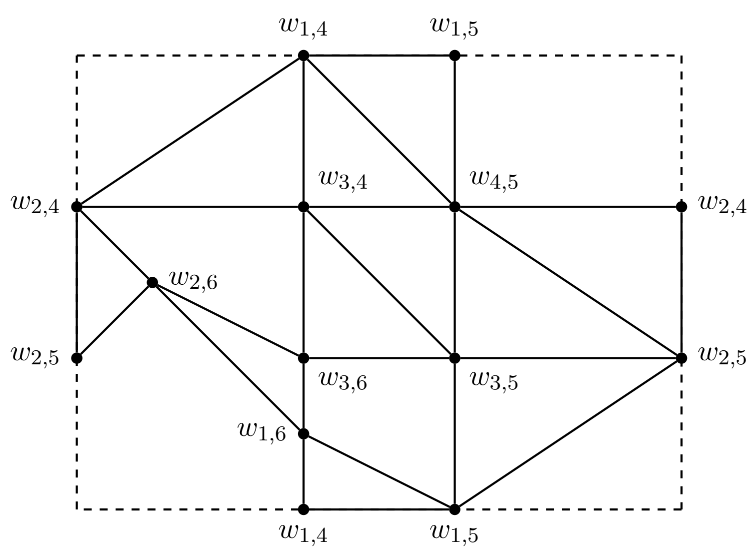

FIGURE 9.

Now, suppose that part (b) of statement (3) holds. Without loss of generality, assume that, $|U_1| = 1$, $|U_{12}| = 1$ and $U_{13} = U_{23} = \varnothing$. Then, by considering the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\}$, $v_3 = I_2 \in U_2$, $v_4 = \{0, a_3\}$, $v_5 = I_3 \in U_3$ and $v_6 = I_{12} \in U_{12}$, the graph $\mathfrak{L}(\mathbb{AG}(L))$ is pictured in Figure 10 is toroidal.

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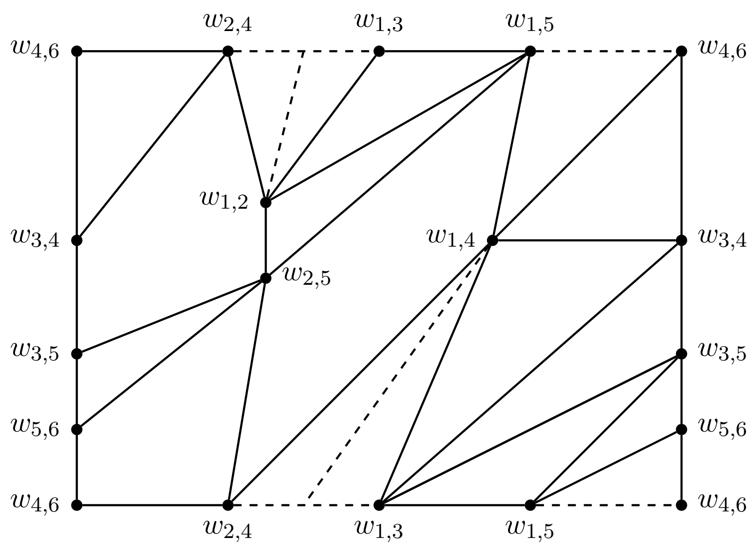

FIGURE 10.

Finally, suppose that $|\bigcup_{j=1}^{3} U_j| = 5$, $|U_1| = 1$, $|U_{23}| = 2$ and $U_{12} = U_{13} = \varnothing$. Then, by considering the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\}$, $v_3 = I_2 \in U_2$, $v_4 = \{0, a_3\}$, $v_5 = I_3 \in U_3$, $v_6 = I_{23}$ and $v_7 = I'_{23} \in U_{23}$, the graph $\mathfrak{L}(\mathbb{AG}(L))$ is pictured in Figure 11, is toroidal.

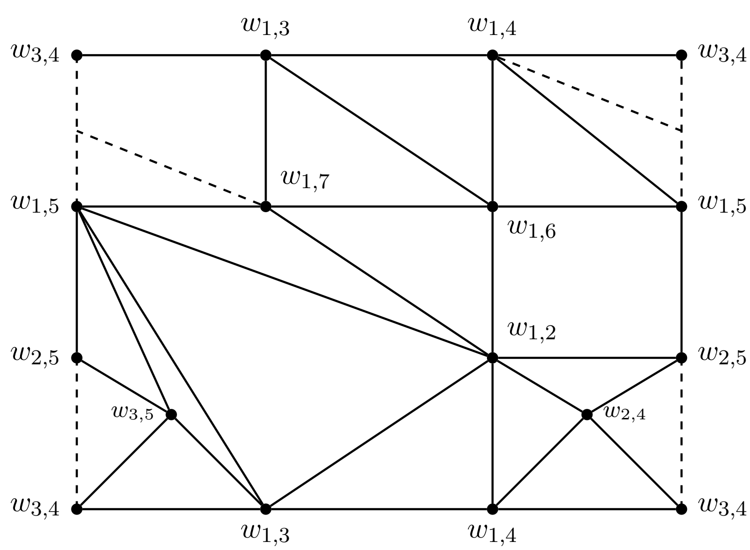

FIGURE 11.

□

Lemma 2.7. *If $|\bigcup_{j=1}^{4} U_j| \geq 6$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.*

Proof. We may assume that $|\bigcup_{j=1}^{4} U_j| = 6$. In this situation, the graph $\mathbb{AG}(L)$ is isomorphic to $K_{3,1,1,1}$ or $K_{2,2,1,1}$. By [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. □

By Lemma 2.7, it is sufficient to study the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$ in the case that $\bigcup_{j=1}^{4} U_j$ has four or five elements. So, we have the following cases.

Case 1. $|\bigcup_{j=1}^{4} U_j| = 4$.

(1) If $|U_{123}| = |U_{124}| = |U_{134}| = |U_{234}| = 1$ and $U_{12} = U_{13} = U_{14} = U_{23} = U_{24} = U_{34} = \varnothing$, then by [12, Theorem 2.7], the graph $\mathfrak{L}(\mathbb{AG}(L))$ is planar, which is not toroidal.

(2) If $|U_{123}| = 4$, $|U_{124}| = |U_{134}| = |U_{234}| = 1$ and $U_{12} = U_{13} = U_{14} = U_{23} = U_{24} = U_{34} = \varnothing$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 12, is toroidal. Consider the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\} \in U_2$, $v_3 = \{0, a_3\} \in U_3$, $v_4 = \{0, a_4\} \in U_4$, $v_5 = I_{123}$, $v_6 = I'_{123}$, $v_7 = I''_{123}$, $v_8 = I'''_{123} \in U_{123}$, $v_9 = I_{124} \in U_{124}$, $v_{10} = I_{134} \in U_{134}$ and $v_{11} = I_{234} \in U_{234}$.

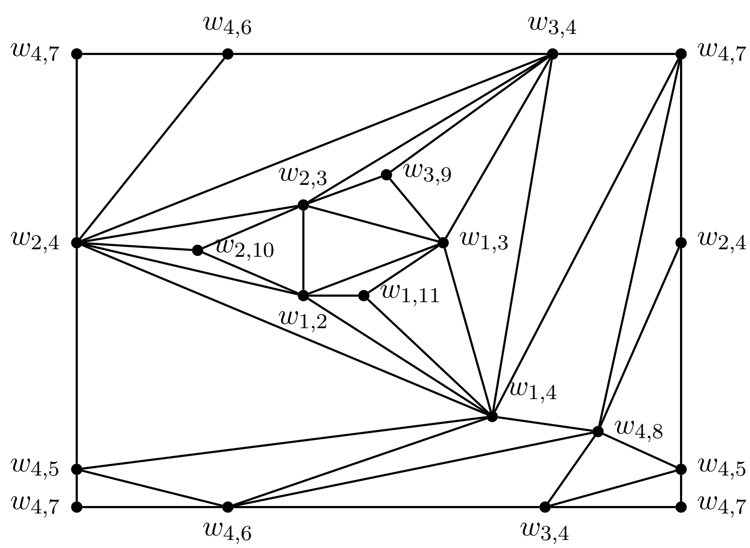

FIGURE 12.

(3) If $|U_{123}| \geq 5$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains a copy of K_8. Hence it is not a toroidal graph.

(4) If $|U_{123}| = |U_{124}| = 2$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

(5) If $|U_{12}| = |U_{123}| = |U_{134}| = |U_{234}| = 1$ and $U_{13} = U_{14} = U_{23} = U_{24} = U_{34} = U_{124} = \varnothing$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 13, is toroidal. Consider the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\} \in U_2$, $v_3 = \{0, a_3\} \in U_3$, $v_4 = \{0, a_4\} \in U_4$, $v_5 = I_{12} \in U_{12}$, $v_6 = I_{123} \in U_{123}$, $v_7 = I_{134} \in U_{134}$ and $v_8 = I_{234} \in U_{234}$.

(6) If $|U_{12}| = 1$ and $|U_{123}| = 2$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal.

(7) $|U_{12}| = 1$ and $|U_{134}| = 2$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal.

(8) If $|U_{12}| = |U_{123}| = |U_{124}| = 1$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

(9) If $|U_{12}| = |U_{13}| = |U_{234}| = 1$ and $U_{14} = U_{23} = U_{24} = U_{34} = U_{123} = U_{124} = U_{134} = \varnothing$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 14, is toroidal. Consider the vertices

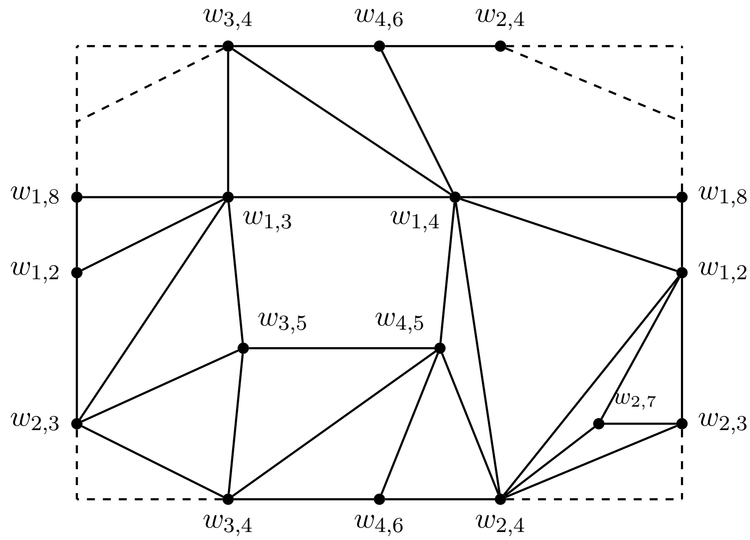

FIGURE 13.

$v_1 = \{0, a_1\} \in U_1,\ v_2 = \{0, a_2\} \in U_2,\ v_3 = \{0, a_3\} \in U_3,\ v_4 = \{0, a_4\} \in U_4,\ v_5 = I_{12} \in U_{12}$, $v_6 = I_{13} \in U_{13}$ and $v_7 = I_{234} \in U_{234}$.

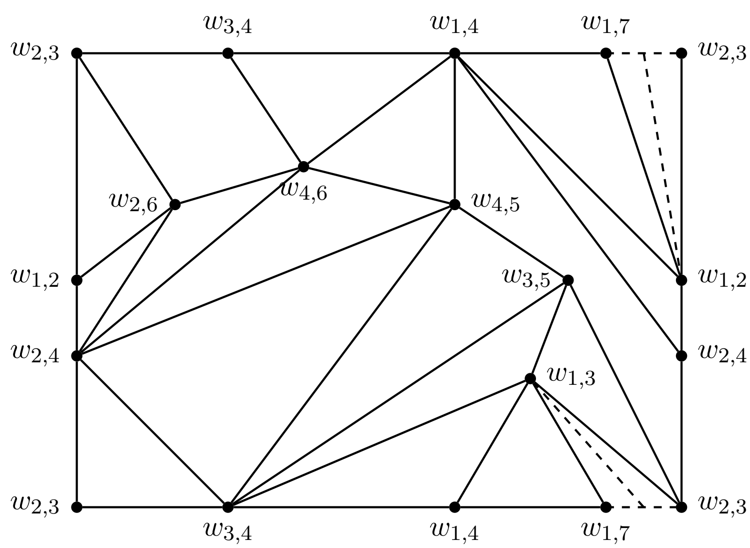

FIGURE 14.

(10) If $|U_{12}| = |U_{34}| = 1$ and $U_{13} = U_{14} = U_{23} = U_{24} = U_{123} = U_{124} = U_{134} = U_{234} = \varnothing$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 15, is toroidal. Consider the vertices $v_1 = \{0, a_1\} \in U_1$, $v_2 = \{0, a_2\} \in U_2$, $v_3 = \{0, a_3\} \in U_3$, $v_4 = \{0, a_4\} \in U_4$, $v_5 = I_{12} \in U_{12}$ and $v_6 = I_{34} \in U_{34}$.

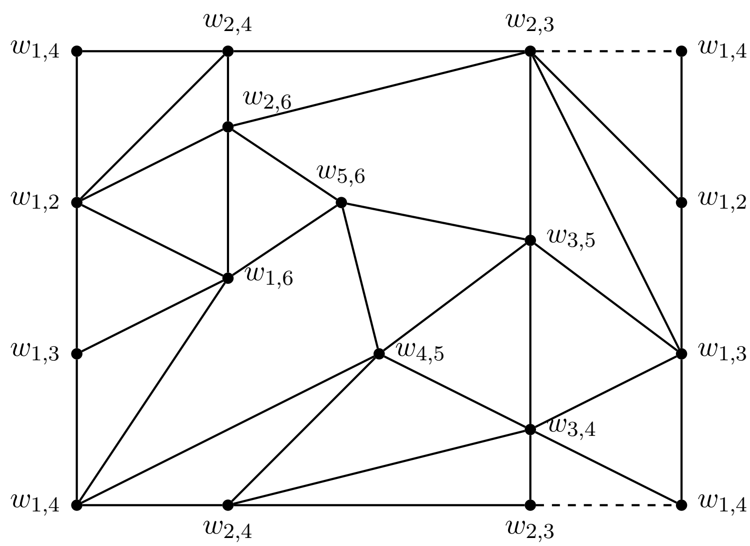

FIGURE 15.

(11) If $|U_{12}| = 2$, then, by [4, Lemma 4.1], $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \geq 2$. Hence the graph $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

Case 2. $|\bigcup_{j=1}^{4} U_j| = 5$ and $|U_1| = 2$.

In this case, $\mathbb{AG}(L) \cong K_5 \setminus K_2$, and so the graph $\mathfrak{L}(\mathbb{AG}(L))$, which is pictured in Figure 16, is toroidal. Consider the vertices $v_1 = \{0, a_1\}$, $v_2 = I_1 \in U_1$, $v_3 = \{0, a_2\} \in U_2$, $v_4 = \{0, a_3\} \in U_3$ and $v_5 = \{0, a_4\} \in U_4$.

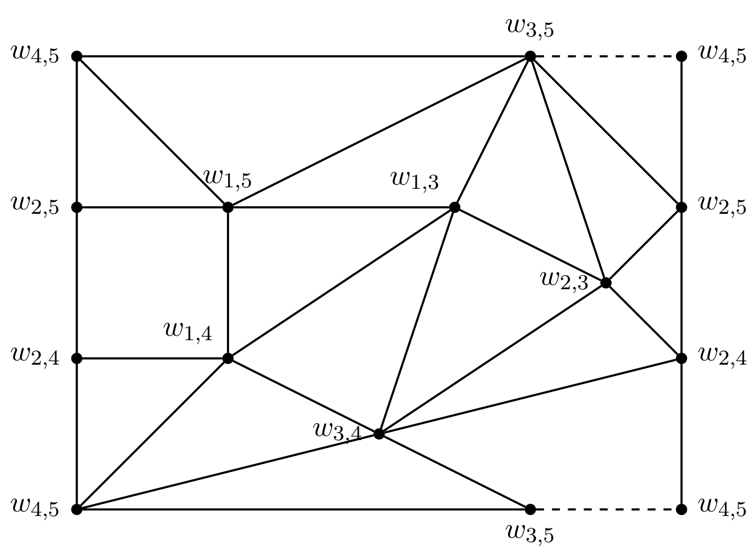

FIGURE 16.

In addition, we may consider the following situations.

(1) If $|U_{123}| \geq 1$, then the graph $\mathfrak{L}(\mathbb{AG}(L))$ contains G_1, one of the listed graphs in [15], which is not toroidal.

(2) If $|U_{234}| \geq 1$, then the graph $\mathbb{AG}(L)$ contains a subdivision of K_5. Clearly, $\gamma(\mathfrak{L}(\mathbb{AG}(L)) \geq 1$. On the other hand, by Euler's formula, one can easily see that $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \neq 1$. Hence $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal.

As a consequence of the above discussion and Lemma 2.7, we state necessary and sufficient conditions for toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$, when $|A(L)| = 4$.

Theorem 2.8. *Suppose that $|A(L)| = 4$. Then $\mathfrak{L}(\mathbb{AG}(L))$ is a toroidal graph if and only if one of the following conditions holds.*

(1) $|\bigcup_{j=1}^{4} U_j| = 4$ *and one of the following conditions holds.*
   - (a) *There exist unique integers $i, j, k \in \{1, 2, 3, 4\}$, such that $2 \leq |U_{ijk}| \leq 4$ and $|U_{i'j'k'}| \leq 1$, for all $1 \leq i', j', k' \leq 4$ with $i'j'k' \neq ijk$, and also $U_{ij} = \varnothing$, for all $1 \leq i, j \leq 4$.*
   - (b) *There exist unique integers $i, j, k \in \{1, 2, 3, 4\}$, such that $|U_{ij}| = 1$ and $|U_{ijk}|, |U_{ikk'}|, |U_{jkk'}| \leq 1$, where $\{k'\} = \{1, 2, 3, 4\} \setminus \{i, j, k\}$, and also $U_{i'j'} = U_{ijk'} = \varnothing$, where $1 \leq i', j' \leq 4$ and $i'j' \neq ij$.*
   - (c) *There exist unique integers $i, j, k, k' \in \{1, 2, 3, 4\}$ with $i, j, k' \neq k$, such that $1 \leq |U_{ij}| + |U_{ik}| \leq 2$ and $|U_{jkk'}| \leq 1$, and also $U_{i'j'} = U_{ii'j'} = \varnothing$, for all $i'j' \notin \{ij, ik\}$ and $ii'j' \neq jkk'$, with $1 \leq i', j' \leq 4$.*
   - (d) *There exist unique integers $i, i', j, j' \in \{1, 2, 3, 4\}$, such that $1 \leq |U_{ij}| + |U_{i'j'}| \leq 2$, for $i'j' \neq ij$, and $U_{i''j''} = \varnothing$, for all $i''j'' \notin \{ij, i'j'\}$, where $1 \leq i'', j'' \leq 4$, and also $U_{ijk} = \varnothing$, for all $1 \leq i, j, k \leq 4$.*

(2) $|\bigcup_{j=1}^{4} U_j| = 5$, *there exists unique i, with $1 \leq i \leq 4$ such that $|U_i| = 2$ and $U_{ij} = U_{ijk} = \varnothing$, for all $1 \leq i, j, k \leq 4$.*

In what follows, we investigate the toroidality of the graph $\mathfrak{L}(\mathbb{AG}(L))$, when $|A(L)| = 5$. To do this, suppose that $|\bigcup_{j=1}^{5} U_j| = 5$. Then the graph $\mathbb{AG}(L)$ is isomorphic to K_5. Clearly, $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \geq 1$. On the other hand, by Euler's formula, one can easily see that $\gamma(\mathfrak{L}(\mathbb{AG}(L))) \neq 1$. Hence $\mathfrak{L}(\mathbb{AG}(L))$ is not toroidal. Therefore, we have the following theorem.

Theorem 2.9. *There is no finite lattice L, when $|A(L)| = 5$, with toroidal graph $\mathfrak{L}(\mathbb{AG}(L))$.*

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References

[1] M. Afkhami, S. Bahrami, K. Khashyarmanesh and F. Shahsavar, The annihilating-ideal graph of a lattice, *Georgian Math. J.*, **23** (2016) 1–7.

[2] J. A. Bondy and U. S. R. Murty, Graph theory with applications, New York, NY: American Elsevier Publishing, (1976) 264 p.

[3] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, *J. Combin. Theory Ser. B*, **24** (1978) 24–33.

[4] H. J. Chiang-Hsieh, P. F. Lee and H. J. Wang, The embedding of line graphs associated to the zero-divisor graphs of commutative rings, *Israel J. Math.*, **180** (2010) 193–222.

[5] B. A. Davey and H. A. Priestley: Introduction to lattices and order, Cambridge University Press, (2002) 298 p.

[6] C. D. Godsil and G. Royle, Algebraic graph theory, Springer-Verlag, New York, (2001) 439 p.

[7] A. L. Hlavacek, 9-vertex irreducible graphs for the torus, PhD. thesis, University of Ohio State, 1997.

[8] K. Kuratowski, Sur le probléme des courbes gauches en topologie, *Fund. Math.*, **15** (1930) 271–283.

[9] W. Massey, Algebraic Topology: An Introduction, Harcourt, Brace & World, New York, 1967.

[10] J. B. Nation, Notes on Lattice Theory, Cambridge studies in advanced mathematics, **60**, Cambridge University Press, Cambridge, 1998.

[11] A. Parsapour and Kh. Ahmad Javaheri, The embedding of annihilating-ideal graphs associated to lattices in the projective plane, *Bull. Malays. Math. Sci. Soc.*, **42** (2019) 1625–1638.

[12] A. Parsapour and Kh. Ahmad Javaheri, When a line graph associated to annihilating-ideal graph of a lattice is planar or projective, *Czech. Math. J.*, **68** (2018) 19–34.

[13] G. Ringel, Map Color Theorem, Springer-Verlag, New York/Heidelberg, 1974.

[14] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.*, **54** (1932) 150–168.

[15] D. Zeps, Forbidden minors for projective plane are free-toroidal or non-toroidal, IUUK-CE-ITI series, 2009.

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