https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 12 No. 3 (2023), pp. 131-142.

www.ui.ac.ir

THE NORMALIZED SIGNLESS LAPLACIAN ESTRADA INDEX OF GRAPHS

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Abstract. Let G be a simple connected graph of order n with m edges. Denote by $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ the normalized signless Laplacian eigenvalues of G. In this work, we define the normalized signless Laplacian Estrada index of G as $NSEE(G) = \sum_{i=1}^{n} e^{\gamma_i^+}$. Some lower bounds on $NSEE(G)$ are also established.

1. Introduction

Let G be a simple connected graph of order n with m edges and vertex set $V = \{v_1, v_{2,\ldots,}v_n\}$. Denote with d_i the degree of the vertex $v_i \in V$, $i = 1, 2, \ldots, n$. If the vertices v_i and v_j of G are adjacent, it is written as $i \sim j$.

Let $A = (a_{ij})$ be the adjacency matrix of G. Its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ represent (ordinary) eigenvalues of G [13]. The Laplacian and signless Laplacian matrices of G are defined as $L = D - A$ and $L^+ = D + A$, respectively [14, 30]. Here, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees of G. Since the graph G is connected, the matrix D is non–singular and so $D^{-1/2}$ is well defined. Then, the normalized Laplacian matrix of G is defined as [10] $\mathcal{L} = D^{-1/2}LD^{-1/2} = I - R$, where I is the unity matrix and R is the Randić matrix [4]. The eigenvalues $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- > \gamma_n^- = 0$ of $\mathcal{L}$ are the normalized Laplacian eigenvalues of G. These eigenvalues possess the following well known

Communicated by Ali Reza Ashrafi.

MSC(2010): Primary: 05C50; Secondary: 05C90

Keywords: Normalized signless Laplacian eigenvalues, Topological indices (of graph), Estrada index (of graph).

Manuscript Type: Research Paper.

Received: 24 January 2021, Accepted: 27 May 2022.

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http://dx.doi.org/10.22108/toc.2022.127155.1814 .

properties [35]:

$$\sum_{i=1}^{n-1} \gamma_i^- = n \qquad \text{and} \qquad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G)\,,$$

where,

$$R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}\,,$$

is the general Randić index of G, defined in [32] (see also [7]).

The normalized signless Laplacian matrix of G is the matrix defined by $\mathcal{L}^+ = D^{-1/2}L^+D^{-1/2} = I + D^{-1/2}AD^{-1/2} = I + R$ [10]. Its eigenvalues $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$ are called as the normalized signless Laplacian eigenvalues of G. Some basic properties regarding them are [9]:

$$\sum_{i=1}^{n} \gamma_i^+ = n \qquad \text{and} \qquad \sum_{i=1}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G). \tag{1.1}$$

Denote by $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ the eigenvalues of Randić matrix R of G [4, 27]. For $i = 1, 2, \ldots, n$, the following relation between ρ_i, γ_i^- and γ_i^+ is valid [23, 27]

$$\gamma_i^- = 1 - \rho_{n-i+1} \qquad \text{and} \qquad \gamma_i^+ = 1 + \rho_i\,. \tag{1.2}$$

The Estrada index of a graph G was defined in [19] as:

$$EE\,(G) = \sum_{i=1}^{n} e^{\lambda_i}\,. \tag{1.3}$$

This index has found important applications in chemistry [20, 21]. Its mathematical properties and bounds have also been studied in the literature. For more information, see [6, 15, 18, 28].

By analogy with Eq. (1.3), the Randić Estada index of G was introduced as [5]:

$$REE\,(G) = \sum_{i=1}^{n} e^{\rho_i}\,, \tag{1.4}$$

and the normalized Laplacian Estrada index was defined as [25]:

$$\ell EE\,(G) = \sum_{i=1}^{n} e^{\gamma_i^-}\,. \tag{1.5}$$

For details on $REE\,(G)$ and $\ell EE\,(G)$, see [5, 12, 25, 29].

Independently from the paper [25], the normalized Laplacian Estrada index was introduced in [26] as follows:

$$NEE\,(G) = \sum_{i=1}^{n} e^{\left(\gamma_i^- - 1\right)}. \tag{1.6}$$

Note that the definitions given by Eqs. (1.5) and (1.6) are equivalent since $NEE\,(G) = \frac{1}{e}\ell EE\,(G)$ [12]. Further, $NEE\,(G)$ and $REE\,(G)$ coincide for bipartite graphs [3]. Details on $NEE\,(G)$ can be found in [3, 12, 26, 33].

DOI: http://dx.doi.org/10.22108/toc.2022.127155.1814

By analogy with Eqs. (1.3) and (1.5), we now define the normalized signless Laplacian Estrada index of G as:

$$NSEE\left(G\right)=\sum_{i=1}^{n}e^{\gamma_i^+}. \tag{1.7}$$

Notice that,

$$NSEE\left(G\right)=\sum_{k\geq 0}\frac{1}{k!}\sum_{i=1}^{n}\left(\gamma_i^+\right)^k.$$

Here, it is assumed that $0^0=1$.

Remark 1.1. *Recall that the normalized Laplacian and the normalized signless Laplacian eigenvalues of bipartite graphs coincide [1]. Then in that case, $NSEE\left(G\right)$ is equal to $\ell EE\left(G\right)$. By Eqs. (1.2), (1.4) and (1.7), it is evident that $NSEE\left(G\right)=eREE\left(G\right)$. Therefore, the results established for $NSEE\left(G\right)$ can be directly re-stated for $REE\left(G\right)$ and vice versa.*

2. Preliminaries

In this section, we state some lemmas that will be used in our main results.

Lemma 2.1. [23] *For any connected graph G, the largest normalized signless Laplacian eigenvalue $\gamma_1^+=2$.*

Lemma 2.2. [23] *Let G be a graph of order $n\geq 2$ without isolated vertices. Then $\gamma_2^+=\gamma_3^+=\cdots=\gamma_n^+=\frac{n-2}{n-1}$ if and only if $G\cong K_n$.*

Lemma 2.3. [2] *Let G be a connected non–bipartite graph of order $n\geq 3$. Then*

$$\gamma_n^+\leq 1-\frac{2R_{-1}\left(G\right)}{n}\leq 1-\frac{1}{\Delta}\leq\frac{n-2}{n-1}.$$

Equalities hold if and only if $G\cong K_n$

Lemma 2.4. [2] *If G is a connected non–bipartite graph of order n, then $\gamma_i^+>0$, for $i=1,2,\ldots,n$.*

Lemma 2.5. [9] *Let G be a connected non-complete graph of order n. If the normalized signless Laplacian eigenvalues are ordered as $\gamma_1^+\geq\gamma_2^+\geq\cdots\geq\gamma_n^+$, then $\gamma_2^+\geq 1$.*

Lemma 2.6. [1] *If G is a bipartite graph, then the eigenvalues of $\mathcal{L}$ and $\mathcal{L}^+$ coincide.*

Lemma 2.7. [17] *Let G be a connected graph of order $n>2$. Then $\gamma_2^-=\gamma_3^-=\cdots=\gamma_{n-1}^-$ if and only if $G\cong K_n$ or $G\cong K_{p,q}$.*

DOI: http://dx.doi.org/10.22108/toc.2022.127155.1814

3. Main Results

We now establish some lower bounds on $NSEE(G)$. Some similar techniques in this section were also used in [6, 8, 22, 25, 34].

Theorem 3.1. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + e^{1-\frac{2R_{-1}(G)}{n}} + (n-2)\, e^{\frac{n-3+\frac{2R_{-1}(G)}{n}}{n-2}}, \tag{3.1}$$

or

$$REE(G) \geq e + e^{-\frac{2R_{-1}(G)}{n}} + (n-2)\, e^{\frac{\frac{2R_{-1}(G)}{n}-1}{n-2}}. \tag{3.2}$$

Equalities hold if and only if $G \cong K_n$.

Proof. By Lemma 2.1, Eq. (1.1) and arithmetic-geometric mean inequality, we have

$$\begin{aligned} NSEE(G) &= \sum_{i=1}^{n} e^{\gamma_i^+} = e^2 + e^{\gamma_n^+} + \sum_{i=2}^{n-1} e^{\gamma_i^+} \\ &\geq e^2 + e^{\gamma_n^+} + (n-2)\left(\prod_{i=2}^{n-1} e^{\gamma_i^+}\right)^{1/(n-2)} \\ &= e^2 + e^{\gamma_n^+} + (n-2)\, e^{\frac{n-2-\gamma_n^+}{n-2}}. \end{aligned}$$

For $x > 0$, let us consider the function

$$f(x) = e^x + (n-2)\, e^{\frac{n-2-x}{n-2}}.$$

Then, we have

$$f'(x) = e^x - e^{\frac{n-2-x}{n-2}} \leq 0,$$

for $x \leq \frac{n-2}{n-1}$. Thus, f is decreasing for $x \leq \frac{n-2}{n-1}$. Then, by Lemma 2.3, we obtain that

$$NSEE(G) \geq e^2 + e^{1-\frac{2R_{-1}(G)}{n}} + (n-2)\, e^{\frac{n-3+\frac{2R_{-1}(G)}{n}}{n-2}}.$$

Hence, we get the lower bound in (3.1). Now we assume that the equality in (3.1) holds. Then

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+ \text{ and } \gamma_n^+ = 1 - \frac{2R_{-1}(G)}{n}.$$

Thus, using the similar way as in [2, Theorem 3.1] and considering Lemmas 2.2 and 2.3, we arrive at $G \cong K_n$. Conversely, by Lemma 2.2, one can show that the equality in (3.1) holds for $G \cong K_n$. □

Corollary 3.2. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + e^{1-\frac{1}{\Delta}} + (n-2)\, e^{\frac{n-3+\frac{1}{\Delta}}{n-2}}, \tag{3.3}$$

or

$$REE(G) \geq e + e^{-\frac{1}{\Delta}} + (n-2)\, e^{\frac{\frac{1}{\Delta}-1}{n-2}}, \tag{3.4}$$

or

$$NSEE\,(G) \geq e^2 + (n-1)\,e^{\frac{n-2}{n-1}}\,, \tag{3.5}$$

or

$$REE\,(G) \geq e + (n-1)\,e^{\frac{-1}{n-1}}, \text{ see [29]}. \tag{3.6}$$

Equalities hold if and only if $G \cong K_n$.

Remark 3.3. *By Lemma 2.3 and the proof of Theorem 3.1, we deduce that (3.1) and (3.3) are better than (3.5). This also implies that (3.2) and (3.4) improve (3.6) obtained in [29, Theorem 2.23], for connected non-bipartite graphs.*

Theorem 3.4. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + (n-1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{n-2R_{-1}(G)}{n}}}\right)^2 . \tag{3.7}$$

Equality holds if and only if $G \cong K_n$.

Proof. Let $a = (a_i)$, $i = 1, 2, \ldots, n$, be a sequence of real numbers with the property $a_1 \geq a_2 \geq \cdots \geq a_n$. In [11], it was proven that

$$\sum_{i=1}^{n} a_i \geq n \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\sqrt{a_1} - \sqrt{a_n}\right)^2 .$$

For $a_i = e^{\gamma_i^+}$, $i = 2, 3, \ldots, n$ the above inequality becomes

$$\sum_{i=2}^{n} e^{\gamma_i^+} \geq (n-1) \left(\prod_{i=2}^{n} e^{\gamma_i^+}\right)^{\frac{1}{n-1}} + \left(\sqrt{e^{\gamma_2^+}} - \sqrt{e^{\gamma_n^+}}\right)^2 . \tag{3.8}$$

By Lemma 2.1 and Eq. (1.1)

$$(n-2)\,\gamma_2^+ = \gamma_2^+ \sum_{i=2}^{n} \gamma_i^+ \geq \sum_{i=2}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G) - 4\,.$$

Therefore

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n-2}\,. \tag{3.9}$$

Considering this with Lemmas 2.1 and 2.3 and Eqs. (1.1) and (3.8), we arrive at (3.7).

If $G \cong K_n$, then $NSEE\,(G) = e^2 + (n-1)\,e^{\frac{n-2}{n-1}}$ and $\frac{n+2R_{-1}(G)-4}{n-2} = \frac{n-2R_{-1}(G)}{n}$, so the equality in (3.7) occurs.

Equality in (3.8) holds if

$$e^{\gamma_3^+} = \cdots = e^{\gamma_{n-1}^+} = \sqrt{e^{\gamma_2^+} e^{\gamma_n^+}}\,,$$

that is

$$\gamma_3^+ = \cdots = \gamma_{n-1}^+ = \frac{\gamma_2^+ + \gamma_n^+}{2}.$$

DOI: http://dx.doi.org/10.22108/toc.2022.127155.1814

By Lemma 2.1 and Eq. (1.1) we have that $\gamma_2^+ + \cdots + \gamma_n^+ = n-2$. Thus

$$\gamma_2^+ + \gamma_n^+ = 2\frac{n-2}{n-1}\,.$$

Let $\gamma_2^+ = \frac{n+2R_{-1}(G)-4}{n-2}$ and $\gamma_n^+ = \frac{n-2R_{-1}(G)}{n}$. Then, we have that

$$\frac{n+2R_{-1}(G)-4}{n-2} + \frac{n-2R_{-1}(G)}{n} = 2\frac{n-2}{n-1}\,,$$

that is

$$2R_{-1}(G) = \frac{n}{n-1}.$$

The above equality holds if and only if $G \cong K_n$. Therefore equality in (3.7) holds if and only if $G \cong K_n$. □

By (3.9) and Lemma 2.3,

$$\gamma_2^+ \geq \frac{n+2R_{-1}(G)-4}{n-2} \geq \frac{n-2}{n-1} \geq 1-\frac{1}{\Delta} \geq 1-\frac{2R_{-1}(G)}{n} \geq \gamma_n^+\,.$$

Then, we have the following corollary of Theorem 3.4.

Corollary 3.5. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + (n-1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{\Delta-1}{\Delta}}}\right)^2, \tag{3.10}$$

or

$$NSEE(G) \geq e^2 + (n-1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{n-2}{n-1}}}\right)^2, \tag{3.11}$$

or

$$NSEE(G) \geq e^2 + (n-1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n-2}{n-1}}} - \sqrt{e^{\frac{n-2R_{-1}(G)}{n}}}\right)^2, \tag{3.12}$$

or

$$NSEE(G) \geq e^2 + (n-1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n-2}{n-1}}} - \sqrt{e^{\frac{\Delta-1}{\Delta}}}\right)^2. \tag{3.13}$$

Equalities hold if and only if $G \cong K_n$.

Remark 3.6. *We should note that we can obtain lower bounds on $REE(G)$ from Theorem 3.4 and Corollary 3.5. Further, as in Remark 3.3, we can deduce that these lower bounds improve (3.6) obtained in [29, Theorem 2.23], for connected non-bipartite graphs.*

Theorem 3.7. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + \sqrt{(n-1)(n-2)\,e^{2\left(\frac{n-2}{n-1}\right)} + e^{2\left(1-\frac{2R_{-1}(G)}{n}\right)} + (n-2)\,e^{2\left(\frac{n-3+\frac{2R_{-1}(G)}{n}}{n-2}\right)}}. \tag{3.14}$$

DOI: http://dx.doi.org/10.22108/toc.2022.127155.1814

Equality is attained for $G \cong K_n$.

Proof. By Eq. (1.7) and Lemma 2.1, we have

$$NSEE(G) = e^2 + \sum_{i=2}^{n} e^{\gamma_i^+},$$

and

$$\begin{aligned}\left(NSEE(G) - e^2\right)^2 &= \left(\sum_{i=2}^{n} e^{\gamma_i^+}\right)^2 \\ &= \sum_{i=2}^{n} e^{2\gamma_i^+} + 2\sum_{2\le i<j\le n} e^{\gamma_i^+} e^{\gamma_j^+}. \end{aligned} \tag{3.15}$$

Using arithmetic-geometric mean inequality, Lemma 2.1 and Eq. (1.1), we have

$$\begin{aligned} 2\sum_{2\le i<j\le n} e^{\gamma_i^+} e^{\gamma_j^+} &\ge (n-1)(n-2)\left(\prod_{2\le i<j\le n} e^{\gamma_i^+} e^{\gamma_j^+}\right)^{2/(n-1)(n-2)} \\ &= (n-1)(n-2)\left(\left(\prod_{i=2}^{n} e^{\gamma_i^+}\right)^{n-2}\right)^{2/(n-1)(n-2)} \\ &= (n-1)(n-2)\, e^{2\left(\frac{n-2}{n-1}\right)}. \end{aligned} \tag{3.16}$$

Furthermore, considering the similar way in Theorem 3.1 together with Lemmas 2.1 and 2.3 and Eq. (1.1), we obtain that

$$\begin{aligned} \sum_{i=2}^{n} e^{2\gamma_i^+} &= e^{2\gamma_n^+} + \sum_{i=2}^{n-1} e^{2\gamma_i^+} \\ &\ge e^{2\gamma_n^+} + (n-2)\left(\prod_{i=2}^{n-1} e^{2\gamma_i^+}\right)^{1/(n-2)} \\ &= e^{2\gamma_n^+} + (n-2)\, e^{2\left(\frac{n-2-\gamma_n^+}{n-2}\right)} \\ &\ge e^{2\left(1-\frac{2R_{-1}(G)}{n}\right)} + (n-2)\, e^{2\left(\frac{n-3+\frac{2R_{-1}(G)}{n}}{n-2}\right)}. \end{aligned} \tag{3.17}$$

Then from Eqs. (3.15)–(3.17), we arrive at the lower bound (3.14). Moreover, by Lemma 2.2, one can easily see that the equality in (3.14) is attained for $G \cong K_n$. □

Theorem 3.8. *Let* G $(\ncong K_n)$ *be a connected non–bipartite graph of order* $n \ge 3$. *Then*

$$NSEE(G) > e^2 + e + 2n - 5. \tag{3.18}$$

Proof. Note that $x \geq 1 + \ln x$, for $x > 0$ with equality if and only if $x = 1$ [16]. Considering this with Lemma 2.1 and Eq. (1.1), we have

$$\begin{aligned} NSEE(G) &= e^2 + e^{\gamma_2^+} + \sum_{i=3}^{n} e^{\gamma_i^+} \\ &\geq e^2 + e^{\gamma_2^+} + n - 2 + \sum_{i=3}^{n} \gamma_i^+ \qquad (3.19) \\ &= e^2 + e^{\gamma_2^+} + 2(n-2) - \gamma_2^+ . \end{aligned}$$

Recall that $f(x) = e^x - x$ is increasing for $x > 0$. Then, by Lemma 2.5, we obtain that

$$NSEE(G) \geq e^2 + e + 2n - 5$$

which is the lower bound (3.18). We now assume that the equality in (3.18) holds. From the equality in (3.19), we have

$$\gamma_3^+ = \gamma_4^+ = \cdots = \gamma_n^+ = 0.$$

Since G is connected non-bipartite, by Lemma 2.4, this is a contradiction. Hence we conclude that (3.18) can not become equality. □

Theorem 3.9. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) > e^2 + 2 + \sqrt{(n-1)(n-2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11}\,. \qquad (3.20)$$

Proof. Note that for $k \geq 2$, $\sum_{i=2}^{n} \left(2\gamma_i^+\right)^k \geq 4 \sum_{i=2}^{n} \left(\gamma_i^+\right)^k$ with equality for all $k \geq 2$ if and only if $\gamma_2^+ = \cdots = \gamma_n^+ = 0$. Considering this together with Lemma 2.1 and Eq. (1.1), we get

$$\begin{aligned} \sum_{i=2}^{n} e^{2\gamma_i^+} &= \sum_{i=2}^{n} \sum_{k\geq 0} \frac{\left(2\gamma_i^+\right)^k}{k!} = 3n - 5 + \sum_{k\geq 2} \frac{\sum_{i=2}^{n} \left(2\gamma_i^+\right)^k}{k!} \\ &\geq 3n - 5 + 4 \sum_{k\geq 2} \frac{\sum_{i=2}^{n} \left(\gamma_i^+\right)^k}{k!} \\ &= -5n + 7 + 4\left(NSEE(G) - e^2\right). \end{aligned}$$

From Eq. (3.16), we have that $2 \sum_{2\leq i<j\leq n} e^{\gamma_i^+} e^{\gamma_j^+} \geq (n-1)(n-2)e^{2\left(\frac{n-2}{n-1}\right)}$. Therefore

$$\begin{aligned} \left(NSEE(G) - e^2\right)^2 &= \sum_{i=2}^{n} e^{2\gamma_i^+} + 2 \sum_{2\leq i<j\leq n} e^{\gamma_i^+} e^{\gamma_j^+} \\ &\geq 4\left(NSEE(G) - e^2\right) + (n-1)(n-2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 7 . \end{aligned}$$

Recall that $e^x \geq 1 + x$. Thus, if $n \geq 3$, then $(n-1)(n-2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11 \geq (n-3)(3n-7) \geq 0$. Hence

$$NSEE(G) \geq e^2 + 2 + \sqrt{(n-1)(n-2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11}\,.$$

DOI: http://dx.doi.org/10.22108/toc.2022.127155.1814

This leads to the lower bound (3.20). Since G is connected non-bipartite, by Lemma 2.4, (3.20) can not become equality. □

Theorem 3.10. *Let G be a connected non–bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + n - 2 + (n-1)\, e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}} - \sqrt{(n-1)(n+2R_{-1}(G)-4)}\,. \tag{3.21}$$

Equality holds if and only if $G \cong K_n$.

Proof. We first recall the following inequality from [24]

$$\left(\frac{1}{p}\sum_{i=1}^{p} a_i^r\right)^{1/r} \leq \left(\frac{1}{p}\sum_{i=1}^{p} a_i^k\right)^{1/k}, \tag{3.22}$$

where $a_1, a_2, \dots, a_p$ are non-negative real numbers and $r \leq k$ with $r, k \neq 0$. The equality in (3.22) holds if and only if $a_1 = a_2 = \cdots = a_p$. Then, for $k \geq 2$, $r = 2$, $p = n-1$, $a_i = \gamma_i^+$, $i = 2, \dots, n$, the inequality (3.22) becomes

$$\sum_{i=2}^{n} \left(\gamma_i^+\right)^k \geq (n-1)\left(\frac{1}{n-1}\sum_{i=2}^{n}\left(\gamma_i^+\right)^2\right)^{k/2}.$$

From the above, Lemma 2.1 and Eq. (1.1), we have that

$$\sum_{i=2}^{n} \left(\gamma_i^+\right)^k \geq (n-1)\left(\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}\right)^k, \tag{3.23}$$

which is the equality for $k = 2$. Further, for $k \geq 3$, the equality holds if and only if $\gamma_2^+ = \cdots = \gamma_n^+$. By Lemma 2.1 and Eqs. (1.1) and (3.23), we get

$$\begin{aligned}
NSEE(G) &= e^2 + \sum_{k\geq 0}\frac{1}{k!}\sum_{i=2}^{n}\left(\gamma_i^+\right)^k \\
&= e^2 + 2n - 3 + \sum_{k\geq 2}\frac{1}{k!}\sum_{i=2}^{n}\left(\gamma_i^+\right)^k \\
&\geq e^2 + 2n - 3 + (n-1)\sum_{k\geq 2}\frac{1}{k!}\left(\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}\right)^k \\
&= e^2 + 2n - 3 + (n-1)\left(-\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}} - 1 + e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}}\right) \\
&= e^2 + n - 2 + (n-1)\, e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}} - \sqrt{(n-1)(n+2R_{-1}(G)-4)}\,.
\end{aligned}$$

Hence the lower bound (3.21) holds. The equality in (3.21) holds if and only if the lower bound (3.23) for $\sum_{i=2}^{n}\left(\gamma_i^+\right)^k$ is achieved for $k \geq 3$. Then, by Lemma 2.2, $G \cong K_n$. □

Considering the similar techniques in Theorem 3.10 with Lemmas 2.6 and 2.7, we get:

Theorem 3.11. *Let G be a connected bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) = \ell EE(G) \geq e^2 + n - 1 + (n-2)\, e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{(n-2)(n+2R_{-1}(G)-4)}\,. \tag{3.24}$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 3.12. *Let G be a connected graph of order $n \geq 3$. Then*

$$NSEE(G) \geq e^2 + \frac{5}{2}n + R_{-1}(G) - 5\,. \tag{3.25}$$

Proof. In monograph [31] , the following inequality can be found

$$e^x \geq 1 + x + \frac{x^2}{2}\,,$$

where x is an arbitrary real number. For $x = \gamma_i^+$ the above inequality becomes

$$e^{\gamma_i^+} \geq 1 + \gamma_i^+ + \frac{\left(\gamma_i^+\right)^2}{2}\,.$$

Summing the above inequality over i, for $i = 2, \ldots, n$ and using Lemma 2.1 and Eq. (1.1), we obtain

$$\begin{aligned} \sum_{i=2}^{n} e^{\gamma_i^+} &\geq \sum_{i=2}^{n} \left(1 + \gamma_i^+ + \frac{\left(\gamma_i^+\right)^2}{2}\right) = (n-1) + (n-2) + \frac{1}{2}\left(n + 2R_{-1}(G) - 4\right) \\ &= \frac{5}{2}n + R_{-1}(G) - 5\,, \end{aligned}$$

wherefrom (3.25) is obtained. □

Acknowledgments

This paper was partly supported by the Serbian Ministry of Education, Science and Technological Development.

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