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Transactions on Combinatorics

ISSN (print): 2251-8657, ISSN (on-line): 2251-8665

Vol. x No. x (201x), pp. xx-xx.

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THE NORMALIZED SIGNLESS LAPLACIAN ESTRADA INDEX OF GRAPHS

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Communicated by

ABSTRACT. Let G be a simple connected graph of order n with m edges. Denote by $\gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+ \geq 0$ the normalized signless Laplacian eigenvalues of G . In this work, we define the normalized signless Laplacian Estrada index of G as $NSEE(G) = \sum_{i=1}^n e^{\gamma_i^+}$. Some lower bounds on $NSEE(G)$ are also established.

1. Introduction

Let G be a simple connected graph of order n with m edges and vertex set $V = \{v_1, v_2, \dots, v_n\}$. Denote with d_i the degree of the vertex $v_i \in V$, $i = 1, 2, \dots, n$. If the vertices v_i and v_j of G are adjacent, it is written as $i \sim j$.

Let $A = (a_{ij})$ be the adjacency matrix of G . Its eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ represent (ordinary) eigenvalues of G [13]. The Laplacian and signless Laplacian matrices of G are defined as $L = D - A$ and $L^+ = D + A$, respectively [14, 30]. Here, $D = \text{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degrees of G . Since the graph G is connected, the matrix D is non-singular and so $D^{-1/2}$ is well defined. Then, the normalized Laplacian matrix of G is defined as [10] $\mathcal{L} = D^{-1/2}LD^{-1/2} = I - R$, where I is the unity matrix and R is the Randić matrix [4]. The eigenvalues $\gamma_1^- \geq \gamma_2^- \geq \dots \geq \gamma_{n-1}^- > \gamma_n^- = 0$ of \mathcal{L} are the normalized Laplacian eigenvalues of G . These eigenvalues possess the following well known

Communicated by Ali Reza Ashrafi.

MSC(2010): Primary: 05C50; Secondary: 05C90

Keywords: Normalized signless Laplacian eigenvalues, Topological indices (of graph), Estrada index (of graph).

Manuscript Type: Research Paper.

Received: 24 January 2021, Accepted: 27 May 2022.

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<http://dx.doi.org/10.22108/TOC.2021.127953.1840> .

properties [35]:

$$\sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G),$$

where,

$$R_{-1}(G) = \sum_{i \sim j} \frac{1}{d_i d_j},$$

is the general Randić index of G , defined in [32] (see also [7]).

The normalized signless Laplacian matrix of G is the matrix defined by $\mathcal{L}^+ = D^{-1/2}L^+D^{-1/2} = I + D^{-1/2}AD^{-1/2} = I + R$ [10]. Its eigenvalues $\gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+ \geq 0$ are called as the normalized signless Laplacian eigenvalues of G . Some basic properties regarding them are [9]:

$$(1.1) \quad \sum_{i=1}^n \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^n (\gamma_i^+)^2 = n + 2R_{-1}(G).$$

Denote by $1 = \rho_1 \geq \rho_2 \geq \dots \geq \rho_n$ the eigenvalues of Randić matrix R of G [4, 27]. For $i = 1, 2, \dots, n$, the following relation between ρ_i , γ_i^- and γ_i^+ is valid [23, 27]

$$(1.2) \quad \gamma_i^- = 1 - \rho_{n-i+1} \quad \text{and} \quad \gamma_i^+ = 1 + \rho_i.$$

The Estrada index of a graph G was defined in [19] as:

$$(1.3) \quad EE(G) = \sum_{i=1}^n e^{\lambda_i}.$$

This index has found important applications in chemistry [20, 21]. Its mathematical properties and bounds have also been studied in the literature. For more information, see [6, 15, 18, 28].

By analogy with Eq. (1.3), the Randić Estrada index of G was introduced as [5]:

$$(1.4) \quad REE(G) = \sum_{i=1}^n e^{\rho_i},$$

and the normalized Laplacian Estrada index was defined as [25]:

$$(1.5) \quad \ell EE(G) = \sum_{i=1}^n e^{\gamma_i^-}.$$

For details on $REE(G)$ and $\ell EE(G)$, see [5, 12, 25, 29].

Independently from the paper [25], the normalized Laplacian Estrada index was introduced in [26] as follows:

$$(1.6) \quad NEE(G) = \sum_{i=1}^n e^{(\gamma_i^- - 1)}.$$

Note that the definitions given by Eqs. (1.5) and (1.6) are equivalent since $NEE(G) = \frac{1}{e} \ell EE(G)$ [12]. Further, $NEE(G)$ and $REE(G)$ coincide for bipartite graphs [3]. Details on $NEE(G)$ can be found in [3, 12, 26, 33].

By analogy with Eqs. (1.3) and (1.5), we now define the normalized signless Laplacian Estrada index of G as:

$$(1.7) \quad NSEE(G) = \sum_{i=1}^n e^{\gamma_i^+}.$$

Notice that,

$$NSEE(G) = \sum_{k \geq 0} \frac{1}{k!} \sum_{i=1}^n (\gamma_i^+)^k.$$

Here, it is assumed that $0^0 = 1$.

Remark 1.1. Recall that the normalized Laplacian and the normalized signless Laplacian eigenvalues of bipartite graphs coincide [1]. Then in that case, $NSEE(G)$ is equal to $\ell EE(G)$. By Eqs. (1.2), (1.4) and (1.7), it is evident that $NSEE(G) = eREE(G)$. Therefore, the results established for $NSEE(G)$ can be directly re-stated for $REE(G)$ and vice versa.

2. Preliminaries

In this section, we state some lemmas that will be used in our main results.

Lemma 2.1. [23] For any connected graph G , the largest normalized signless Laplacian eigenvalue $\gamma_1^+ = 2$.

Lemma 2.2. [23] Let G be a graph of order $n \geq 2$ without isolated vertices. Then $\gamma_2^+ = \gamma_3^+ = \dots = \gamma_n^+ = \frac{n-2}{n-1}$ if and only if $G \cong K_n$.

Lemma 2.3. [2] Let G be a connected non-bipartite graph of order $n \geq 3$. Then

$$\gamma_n^+ \leq 1 - \frac{2R_{-1}(G)}{n} \leq 1 - \frac{1}{\Delta} \leq \frac{n-2}{n-1}.$$

Equalities hold if and only if $G \cong K_n$

Lemma 2.4. [2] If G is a connected non-bipartite graph of order n , then $\gamma_i^+ > 0$, for $i = 1, 2, \dots, n$.

Lemma 2.5. [9] Let G be a connected non-complete graph of order n . If the normalized signless Laplacian eigenvalues are ordered as $\gamma_1^+ \geq \gamma_2^+ \geq \dots \geq \gamma_n^+$, then $\gamma_2^+ \geq 1$.

Lemma 2.6. [1] If G is a bipartite graph, then the eigenvalues of \mathcal{L} and \mathcal{L}^+ coincide.

Lemma 2.7. [17] Let G be a connected graph of order $n > 2$. Then $\gamma_2^- = \gamma_3^- = \dots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.

3. Main Results

We now establish some lower bounds on $NSEE(G)$. Some similar techniques in this section were also used in [6, 8, 22, 25, 34].

Theorem 3.1. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.1) \quad NSEE(G) \geq e^2 + e^{1 - \frac{2R_{-1}(G)}{n}} + (n-2)e^{\frac{n-3 + \frac{2R_{-1}(G)}{n}}{n-2}},$$

or

$$(3.2) \quad REE(G) \geq e + e^{-\frac{2R_{-1}(G)}{n}} + (n-2)e^{\frac{\frac{2R_{-1}(G)}{n} - 1}{n-2}}.$$

Equalities hold if and only if $G \cong K_n$.

Proof. By Lemma 2.1, Eq. (1.1) and arithmetic-geometric mean inequality, we have

$$\begin{aligned} NSEE(G) &= \sum_{i=1}^n e^{\gamma_i^+} = e^2 + e^{\gamma_n^+} + \sum_{i=2}^{n-1} e^{\gamma_i^+} \\ &\geq e^2 + e^{\gamma_n^+} + (n-2) \left(\prod_{i=2}^{n-1} e^{\gamma_i^+} \right)^{1/(n-2)} \\ &= e^2 + e^{\gamma_n^+} + (n-2)e^{\frac{n-2-\gamma_n^+}{n-2}}. \end{aligned}$$

For $x > 0$, let us consider the function

$$f(x) = e^x + (n-2)e^{\frac{n-2-x}{n-2}}.$$

Then, we have

$$f'(x) = e^x - e^{\frac{n-2-x}{n-2}} \leq 0,$$

for $x \leq \frac{n-2}{n-1}$. Thus, f is decreasing for $x \leq \frac{n-2}{n-1}$. Then, by Lemma 2.3, we obtain that

$$NSEE(G) \geq e^2 + e^{1 - \frac{2R_{-1}(G)}{n}} + (n-2)e^{\frac{n-3 + \frac{2R_{-1}(G)}{n}}{n-2}}.$$

Hence, we get the lower bound in (3.1). Now we assume that the equality in (3.1) holds. Then

$$\gamma_2^+ = \gamma_3^+ = \dots = \gamma_{n-1}^+ \text{ and } \gamma_n^+ = 1 - \frac{2R_{-1}(G)}{n}.$$

Thus, using the similar way as in [2, Theorem 3.1] and considering Lemmas 2.2 and 2.3, we arrive at $G \cong K_n$. Conversely, by Lemma 2.2, one can show that the equality in (3.1) holds for $G \cong K_n$. \square

Corollary 3.2. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.3) \quad NSEE(G) \geq e^2 + e^{1 - \frac{1}{\Delta}} + (n-2)e^{\frac{n-3 + \frac{1}{\Delta}}{n-2}},$$

or

$$(3.4) \quad REE(G) \geq e + e^{-\frac{1}{\Delta}} + (n-2)e^{\frac{\frac{1}{\Delta} - 1}{n-2}},$$

or

$$(3.5) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}},$$

or

$$(3.6) \quad REE(G) \geq e + (n - 1)e^{\frac{-1}{n-1}}, \text{ see [29].}$$

Equalities hold if and only if $G \cong K_n$.

Remark 3.3. By Lemma 2.3 and the proof of Theorem 3.1, we deduce that (3.1) and (3.3) are better than (3.5). This also implies that (3.2) and (3.4) improve (3.6) obtained in [29, Theorem 2.23], for connected non-bipartite graphs.

Theorem 3.4. Let G be a connected non-bipartite graph of order $n \geq 3$. Then

$$(3.7) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{n-2R_{-1}(G)}{n}}} \right)^2.$$

Equality holds if and only if $G \cong K_n$.

Proof. Let $a = (a_i), i = 1, 2, \dots, n$, be a sequence of real numbers with the property $a_1 \geq a_2 \geq \dots \geq a_n$. In [11], it was proven that

$$\sum_{i=1}^n a_i \geq n \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}} + (\sqrt{a_1} - \sqrt{a_n})^2.$$

For $a_i = e^{\gamma_i^+}, i = 2, 3, \dots, n$ the above inequality becomes

$$(3.8) \quad \sum_{i=2}^n e^{\gamma_i^+} \geq (n - 1) \left(\prod_{i=2}^n e^{\gamma_i^+} \right)^{\frac{1}{n-1}} + \left(\sqrt{e^{\gamma_2^+}} - \sqrt{e^{\gamma_n^+}} \right)^2.$$

By Lemma 2.1 and Eq. (1.1)

$$(n - 2)\gamma_2^+ = \gamma_2^+ \sum_{i=2}^n \gamma_i^+ \geq \sum_{i=2}^n (\gamma_i^+)^2 = n + 2R_{-1}(G) - 4.$$

Therefore

$$(3.9) \quad \gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n - 2}.$$

Considering this with Lemmas 2.1 and 2.3 and Eqs. (1.1) and (3.8), we arrive at (3.7).

If $G \cong K_n$, then $NSEE(G) = e^2 + (n - 1)e^{\frac{n-2}{n-1}}$ and $\frac{n+2R_{-1}(G)-4}{n-2} = \frac{n-2R_{-1}(G)}{n}$, so the equality in (3.7) occurs.

Equality in (3.8) holds if

$$e^{\gamma_3^+} = \dots = e^{\gamma_{n-1}^+} = \sqrt{e^{\gamma_2^+} e^{\gamma_n^+}},$$

that is

$$\gamma_3^+ = \dots = \gamma_{n-1}^+ = \frac{\gamma_2^+ + \gamma_n^+}{2}.$$

By Lemma 2.1 and Eq. (1.1) we have that $\gamma_2^+ + \dots + \gamma_n^+ = n - 2$. Thus

$$\gamma_2^+ + \gamma_n^+ = 2 \frac{n - 2}{n - 1}.$$

Let $\gamma_2^+ = \frac{n+2R_{-1}(G)-4}{n-2}$ and $\gamma_n^+ = \frac{n-2R_{-1}(G)}{n}$. Then, we have that

$$\frac{n + 2R_{-1}(G) - 4}{n - 2} + \frac{n - 2R_{-1}(G)}{n} = 2 \frac{n - 2}{n - 1},$$

that is

$$2R_{-1}(G) = \frac{n}{n - 1}.$$

The above equality holds if and only if $G \cong K_n$. Therefore equality in (3.7) holds if and only if $G \cong K_n$. □

By (3.9) and Lemma 2.3,

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n - 2} \geq \frac{n - 2}{n - 1} \geq 1 - \frac{1}{\Delta} \geq 1 - \frac{2R_{-1}(G)}{n} \geq \gamma_n^+.$$

Then, we have the following corollary of Theorem 3.4.

Corollary 3.5. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.10) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{\Delta-1}{\Delta}}} \right)^2,$$

or

$$(3.11) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{e^{\frac{n-2}{n-1}}} \right)^2,$$

or

$$(3.12) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n-2}{n-1}}} - \sqrt{e^{\frac{n-2R_{-1}(G)}{n}}} \right)^2,$$

or

$$(3.13) \quad NSEE(G) \geq e^2 + (n - 1)e^{\frac{n-2}{n-1}} + \left(\sqrt{e^{\frac{n-2}{n-1}}} - \sqrt{e^{\frac{\Delta-1}{\Delta}}} \right)^2.$$

Equalities hold if and only if $G \cong K_n$.

Remark 3.6. *We should note that we can obtain lower bounds on $REE(G)$ from Theorem 3.4 and Corollary 3.5. Further, as in Remark 3.3, we can deduce that these lower bounds improve (3.6) obtained in [29, Theorem 2.23], for connected non-bipartite graphs.*

Theorem 3.7. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.14) \quad NSEE(G) \geq e^2 + \sqrt{(n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)} + e^{2\left(1 - \frac{2R_{-1}(G)}{n}\right)} + (n - 2)e^{2\left(\frac{n-3 + \frac{2R_{-1}(G)}{n}}{n-2}\right)}}.$$

Equality is attained for $G \cong K_n$.

Proof. By Eq. (1.7) and Lemma 2.1, we have

$$NSEE(G) = e^2 + \sum_{i=2}^n e^{\gamma_i^+},$$

and

$$\begin{aligned} (NSEE(G) - e^2)^2 &= \left(\sum_{i=2}^n e^{\gamma_i^+} \right)^2 \\ (3.15) \qquad &= \sum_{i=2}^n e^{2\gamma_i^+} + 2 \sum_{2 \leq i < j \leq n} e^{\gamma_i^+} e^{\gamma_j^+}. \end{aligned}$$

Using arithmetic-geometric mean inequality, Lemma 2.1 and Eq. (1.1), we have

$$\begin{aligned} 2 \sum_{2 \leq i < j \leq n} e^{\gamma_i^+} e^{\gamma_j^+} &\geq (n-1)(n-2) \left(\prod_{2 \leq i < j \leq n} e^{\gamma_i^+} e^{\gamma_j^+} \right)^{2/(n-1)(n-2)} \\ &= (n-1)(n-2) \left(\left(\prod_{i=2}^n e^{\gamma_i^+} \right)^{n-2} \right)^{2/(n-1)(n-2)} \\ (3.16) \qquad &= (n-1)(n-2) e^{2\left(\frac{n-2}{n-1}\right)}. \end{aligned}$$

Furthermore, considering the similar way in Theorem 3.1 together with Lemmas 2.1 and 2.3 and Eq. (1.1), we obtain that

$$\begin{aligned} \sum_{i=2}^n e^{2\gamma_i^+} &= e^{2\gamma_n^+} + \sum_{i=2}^{n-1} e^{2\gamma_i^+} \\ &\geq e^{2\gamma_n^+} + (n-2) \left(\prod_{i=2}^{n-1} e^{2\gamma_i^+} \right)^{1/(n-2)} \\ &= e^{2\gamma_n^+} + (n-2) e^{2\left(\frac{n-2-\gamma_n^+}{n-2}\right)} \\ (3.17) \qquad &\geq e^{2\left(1-\frac{2R-1(G)}{n}\right)} + (n-2) e^{2\left(\frac{n-3+\frac{2R-1(G)}{n}}{n-2}\right)}. \end{aligned}$$

Then from Eqs. (3.15)–(3.17), we arrive at the lower bound (3.14). Moreover, by Lemma 2.2, one can easily see that the equality in (3.14) is attained for $G \cong K_n$. □

Theorem 3.8. *Let $G (\not\cong K_n)$ be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.18) \qquad NSEE(G) > e^2 + e + 2n - 5.$$

Proof. Note that $x \geq 1 + \ln x$, for $x > 0$ with equality if and only if $x = 1$ [16]. Considering this with Lemma 2.1 and Eq. (1.1), we have

$$\begin{aligned}
 NSEE(G) &= e^2 + e^{\gamma_2^+} + \sum_{i=3}^n e^{\gamma_i^+} \\
 &\geq e^2 + e^{\gamma_2^+} + n - 2 + \sum_{i=3}^n \gamma_i^+ \\
 &= e^2 + e^{\gamma_2^+} + 2(n - 2) - \gamma_2^+.
 \end{aligned}
 \tag{3.19}$$

Recall that $f(x) = e^x - x$ is increasing for $x > 0$. Then, by Lemma 2.5, we obtain that

$$NSEE(G) \geq e^2 + e + 2n - 5$$

which is the lower bound (3.18). We now assume that the equality in (3.18) holds. From the equality in (3.19), we have

$$\gamma_3^+ = \gamma_4^+ = \dots = \gamma_n^+ = 0.$$

Since G is connected non-bipartite, by Lemma 2.4, this is a contradiction. Hence we conclude that (3.18) can not become equality. \square

Theorem 3.9. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$NSEE(G) > e^2 + 2 + \sqrt{(n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11}.$$

Proof. Note that for $k \geq 2$, $\sum_{i=2}^n (2\gamma_i^+)^k \geq 4 \sum_{i=2}^n (\gamma_i^+)^k$ with equality for all $k \geq 2$ if and only if $\gamma_2^+ = \dots = \gamma_n^+ = 0$. Considering this together with Lemma 2.1 and Eq. (1.1), we get

$$\begin{aligned}
 \sum_{i=2}^n e^{2\gamma_i^+} &= \sum_{i=2}^n \sum_{k \geq 0} \frac{(2\gamma_i^+)^k}{k!} = 3n - 5 + \sum_{k \geq 2} \frac{\sum_{i=2}^n (2\gamma_i^+)^k}{k!} \\
 &\geq 3n - 5 + 4 \sum_{k \geq 2} \frac{\sum_{i=2}^n (\gamma_i^+)^k}{k!} \\
 &= -5n + 7 + 4(NSEE(G) - e^2).
 \end{aligned}$$

From Eq. (3.16), we have that $2 \sum_{2 \leq i < j \leq n} e^{\gamma_i^+} e^{\gamma_j^+} \geq (n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)}$. Therefore

$$\begin{aligned}
 (NSEE(G) - e^2)^2 &= \sum_{i=2}^n e^{2\gamma_i^+} + 2 \sum_{2 \leq i < j \leq n} e^{\gamma_i^+} e^{\gamma_j^+} \\
 &\geq 4(NSEE(G) - e^2) + (n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 7.
 \end{aligned}$$

Recall that $e^x \geq 1 + x$. Thus, if $n \geq 3$, then $(n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11 \geq (n - 3)(3n - 7) \geq 0$. Hence

$$NSEE(G) \geq e^2 + 2 + \sqrt{(n - 1)(n - 2)e^{2\left(\frac{n-2}{n-1}\right)} - 5n + 11}.$$

This leads to the lower bound (3.20). Since G is connected non-bipartite, by Lemma 2.4, (3.20) can not become equality. \square

Theorem 3.10. *Let G be a connected non-bipartite graph of order $n \geq 3$. Then*

$$(3.21) \quad NSEE(G) \geq e^2 + n - 2 + (n - 1)e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-1}}} - \sqrt{(n - 1)(n + 2R_{-1}(G) - 4)}.$$

Equality holds if and only if $G \cong K_n$.

Proof. We first recall the following inequality from [24]

$$(3.22) \quad \left(\frac{1}{p} \sum_{i=1}^p a_i^r\right)^{1/r} \leq \left(\frac{1}{p} \sum_{i=1}^p a_i^k\right)^{1/k},$$

where a_1, a_2, \dots, a_p are non-negative real numbers and $r \leq k$ with $r, k \neq 0$. The equality in (3.22) holds if and only if $a_1 = a_2 = \dots = a_p$. Then, for $k \geq 2, r = 2, p = n - 1, a_i = \gamma_i^+, i = 2, \dots, n$, the inequality (3.22) becomes

$$\sum_{i=2}^n (\gamma_i^+)^k \geq (n - 1) \left(\frac{1}{n - 1} \sum_{i=2}^n (\gamma_i^+)^2\right)^{k/2}.$$

From the above, Lemma 2.1 and Eq. (1.1), we have that

$$(3.23) \quad \sum_{i=2}^n (\gamma_i^+)^k \geq (n - 1) \left(\sqrt{\frac{n + 2R_{-1}(G) - 4}{n - 1}}\right)^k,$$

which is the equality for $k = 2$. Further, for $k \geq 3$, the equality holds if and only if $\gamma_2^+ = \dots = \gamma_n^+$. By Lemma 2.1 and Eqs. (1.1) and (3.23), we get

$$\begin{aligned} NSEE(G) &= e^2 + \sum_{k \geq 0} \frac{1}{k!} \sum_{i=2}^n (\gamma_i^+)^k \\ &= e^2 + 2n - 3 + \sum_{k \geq 2} \frac{1}{k!} \sum_{i=2}^n (\gamma_i^+)^k \\ &\geq e^2 + 2n - 3 + (n - 1) \sum_{k \geq 2} \frac{1}{k!} \left(\sqrt{\frac{n + 2R_{-1}(G) - 4}{n - 1}}\right)^k \\ &= e^2 + 2n - 3 + (n - 1) \left(-\sqrt{\frac{n + 2R_{-1}(G) - 4}{n - 1}} - 1 + e^{\sqrt{\frac{n + 2R_{-1}(G) - 4}{n - 1}}}\right) \\ &= e^2 + n - 2 + (n - 1)e^{\sqrt{\frac{n + 2R_{-1}(G) - 4}{n - 1}}} - \sqrt{(n - 1)(n + 2R_{-1}(G) - 4)}. \end{aligned}$$

Hence the lower bound (3.21) holds. The equality in (3.21) holds if and only if the lower bound (3.23) for $\sum_{i=2}^n (\gamma_i^+)^k$ is achieved for $k \geq 3$. Then, by Lemma 2.2, $G \cong K_n$. \square

Considering the similar techniques in Theorem 3.10 with Lemmas 2.6 and 2.7, we get:

Theorem 3.11. Let G be a connected bipartite graph of order $n \geq 3$. Then

$$(3.24) \quad \begin{aligned} NSEE(G) = \ell EE(G) &\geq e^2 + n - 1 + \\ &+ (n-2) e^{\sqrt{\frac{n+2R_{-1}(G)-4}{n-2}}} - \sqrt{(n-2)(n+2R_{-1}(G)-4)}. \end{aligned}$$

Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.

Theorem 3.12. Let G be a connected graph of order $n \geq 3$. Then

$$(3.25) \quad NSEE(G) \geq e^2 + \frac{5}{2}n + R_{-1}(G) - 5.$$

Proof. In monograph [31], the following inequality can be found

$$e^x \geq 1 + x + \frac{x^2}{2},$$

where x is an arbitrary real number. For $x = \gamma_i^+$ the above inequality becomes

$$e^{\gamma_i^+} \geq 1 + \gamma_i^+ + \frac{(\gamma_i^+)^2}{2}.$$

Summing the above inequality over i , for $i = 2, \dots, n$ and using Lemma 2.1 and Eq. (1.1), we obtain

$$\begin{aligned} \sum_{i=2}^n e^{\gamma_i^+} &\geq \sum_{i=2}^n \left(1 + \gamma_i^+ + \frac{(\gamma_i^+)^2}{2} \right) = (n-1) + (n-2) + \frac{1}{2}(n+2R_{-1}(G)-4) \\ &= \frac{5}{2}n + R_{-1}(G) - 5, \end{aligned}$$

wherefrom (3.25) is obtained. □

Acknowledgments

This paper was partly supported by the Serbian Ministry of Education, Science and Technological Development.

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