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ON THE ZAGREB INDEX OF RANDOM M-ORIENTED RECURSIVE TREES

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Abstract. The main goal of this paper is to study the modified F-indices (modified first Zagreb index and modified forgotten topological index) of random m-oriented recursive trees (RMORTs). First, through two recurrence equations, we compute the mean and the variance of these indices in our random tree model. Second, we show four convergence in probability based on these indices. Third, the asymptotic normalities, through the martingale central limit theorem, are given.

1. Introduction

Increasing trees of order n are labelled trees where the nodes are labelled by distinct integers of the set $\{1,\ldots,n\}$ in such a way that each sequence of labels along any branch starting at the root node (labelled by 1) is increasing [1]. Recursive trees are one of the most important increasing trees with applications in several fields. Recursive trees have been introduced as simple probability models for system generation, spread of contamination of organisms, pyramid scheme, stemma construction of philology, Internet interface map, stochastic growth of networks. They are related to some Internet models and some physical models; they also appeared in Hopf algebra under the name of "heap-ordered trees". The bijection between recursive trees and binary search trees not only makes the former a flexible representation of the latter but also provides a rich direction for further extensions (see [1] and references therein). Various quantities such depth, distance, degree, profile and size have been studied on this family of growing trees. For example, the quantity of depth in family genealogies indicates the same number of male offspring in a generation. The m-oriented recursive trees, as a type of recursive trees, have been less studied [2]. A random m-oriented recursive tree (RMORT) of order

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n is constructed as follows. One starts from a root node holding the label 1. At stage $2 \le i \le n$ a new node i is attached to any previous node j of outdegree d_j of the already grown tree T_{i-1} of order $i-1$ with probability $\frac{(m-1)d_j+1}{m(i-2)+1}$. When $i=2$, i.e., the tree has just the root node, then the node $i=2$ will attach to the root ($d_1 = 0$) with probability 1. Thus the higher outdegree nodes possess a higher attraction for n (see [2] and [11] for motivation of definition of this models and more information). One of the applications of this model is in the discussion of investment in stock markets. For example, investors prefer to invest in cases where the demand for stock exchange has been higher.

A topological index for a graph G is a numerical quantity invariant under automorphisms of G. These indices are used for characterizing molecular graphs. Also, they establish relationships between structure and properties of molecules and predicting biological activity of chemical compounds. Here, the *modified* topological index means that in the original definition, the degree of a node has replaced by its outdegree. Also, the meaning of modified F-index is the modified first Zagreb index and modified forgotten topological index. Thus, the modified first Zagreb index is defined as

$$M_1(G) = \sum_{v \in V(G)} d_v^2,$$

where $V(G)$ is the vertex set of a graph G. Also, the modified forgotten topological index is defined as

$$F(G) = \sum_{v \in V(G)} d_v^3.$$

Furtula and Gutman [4] raised that the predictive ability of forgotten topological index is almost similar to that of first Zagreb index and for the acentric factor and entropy, and both of them obtain correlation coefficients larger than 0.95. This fact implies the reason why forgotten topological index is useful for testing the chemical and pharmacological properties of drug molecular structures However, the motivation for studying topological indices is multifold (see [8, 9, 12]). This paper is the first report that studies the topological indices with probabilistic method in RMORT.

The paper is organized as follows. First, we show two recurrences for modified F-indices of RMORTs, from which two martingales are constructed and two first moments and covariances between of the indices are given. Then the asymptotic normalities of modified F-indices are proved as the order of the tree grows to ∞.

2. Means

Let Z_n^2 and Z_n^3 be the modified first Zagreb index and modified forgotten topological index of a RMORT of order n, respectively. Let $\mathcal{F}_n$ be the σ-field generated by the first n stages of these models [7]. Let U_n be a randomly chosen node of outdegree d_k in the tree of order n with the conditional law

$$P(U_n = k|\mathcal{F}_{n-1}) = \frac{(m-1)d_k+1}{s_{n-1,m}},\ k = 1, \dots, n-1$$

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where $s_{n,m} = m(n-1)+1$. Then,

$$Z_n^2|\mathcal{F}_{n-1} = (Z_{n-1}^2 + 2d_{U_{n-1}} + 1)|\mathcal{F}_{n-1}, \tag{2.1}$$

$$Z_n^3|\mathcal{F}_{n-1} = (Z_{n-1}^3 + 3d_{U_{n-1}}^2 + 3d_{U_{n-1}} + 1)|\mathcal{F}_{n-1}. \tag{2.2}$$

Using the gamma function define

$$\Upsilon[n,k,m] = \frac{\Gamma\left(n-1+\frac{1}{m}\right)}{\Gamma\left(n-1+\frac{k(m-1)+1}{m}\right)},\ k \geq 2,\ n,m \geq 2\ (n > m).$$

Lemma 2.1. *For each integer $k \geq 1$,*

1)

$$\frac{\Upsilon[n-1,k,m]}{\Upsilon[n,k,m]} = 1 + \frac{k(m-1)}{s_{n-1,m}}.$$

2)

$$\frac{\Upsilon[n-1,2k,m]}{\Upsilon[n,2k,m]} = 2\frac{\Upsilon[n-1,k,m]}{\Upsilon[n,k,m]} - 1.$$

3) The function $\Upsilon[n,k,m]$ is decreasing in n and

$$\Upsilon[n,k,m] = n^{\frac{k}{m}-k}(1+\mathcal{O}(n^{-1})).$$

Proof. 1) The proof of (1) is obvious and straightforward, since $\Gamma(x+1) = x\Gamma(x)$. For part (2),

$$2\frac{\Upsilon[n-1,k,m]}{\Upsilon[n,k,m]} - 1 = 1 + \frac{2k(m-1)}{s_{n-1,m}} = \frac{\Upsilon[n-1,2k,m]}{\Upsilon[n,2k,m]}.$$

Part (3) is an immediate consequence of Stirling approximation. □

Set

$$\alpha_2[n,m] = \frac{2(n-1)}{s_{n,m}} + 1,$$

$$\alpha_3[n,m] = \frac{3}{s_{n,m}}\mathbb{E}(Z_n^2) + \frac{3(n-1)}{s_{n,m}} + 1.$$

Theorem 2.2. *For a RMORT of order n,*

$$\mathbb{E}(Z_n^\ell) = \frac{1}{\Upsilon[n,\ell,m]}\sum_{k=1}^{n-1}\Upsilon[k+1,\ell,m]\alpha_\ell[k,m],\ \ \ell = 2,3,\ \ n \geq 3.$$

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Proof. For a tree of order $n-1$, $\sum_{k=1}^{n-1} d_k = n-2$. From (2.1) and Lemma 2.1,

$$\begin{aligned}
\mathbb{E}(Z_n^2) &= \mathbb{E}(\mathbb{E}(Z_n^2|\mathcal{F}_{n-1})) \\
&= \mathbb{E}(\mathbb{E}(Z_{n-1}^2 + 2d_{U_{n-1}} + 1|\mathcal{F}_{n-1})) \\
&= \mathbb{E}\left(\mathbb{E}(Z_{n-1}^2) + 2\sum_{k=1}^{n-1} d_k P(U_n = k|\mathcal{F}_{n-1}) + 1\right) \\
&= \mathbb{E}(Z_{n-1}^2) + 2\sum_{k=1}^{n-1} d_k \frac{(m-1)d_k + 1}{s_{n-1,m}} + 1 \\
&= \left(1 + \frac{2(m-1)}{s_{n-1,m}}\right)\mathbb{E}(Z_{n-1}^2) + \alpha_2[n-1,m] \\
&= \frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\mathbb{E}(Z_{n-1}^2) + \alpha_2[n-1,m].
\end{aligned}$$

By iteration,

$$\mathbb{E}(Z_n^2) = \frac{1}{\Upsilon[n,2,m]}\sum_{k=1}^{n-1} \Upsilon[k+1,2,m]\alpha_2[k,m],$$

since $Z_1^2 = 0$. Also, From (2.2) and part (1) in Lemma 2.1,

$$\begin{aligned}
\mathbb{E}(Z_n^3) &= \mathbb{E}(\mathbb{E}(Z_n^3|\mathcal{F}_{n-1})) \\
&= \mathbb{E}(Z_{n-1}^3 + 3\mathbb{E}(d_{U_{n-1}}^2|\mathcal{F}_{n-1}) + 3\mathbb{E}(d_{U_{n-1}}|\mathcal{F}_{n-1}) + 1) \\
&= \mathbb{E}\Bigg(Z_{n-1}^3 + 3\sum_{k=1}^{n-1} d_k^2 P(U_n = k|\mathcal{F}_{n-1}) \\
&\quad + 3\sum_{k=1}^{n-1} d_k P(U_n = k|\mathcal{F}_{n-1}) + 1\Bigg) \\
&= \left(1 + \frac{3(m-1)}{s_{n-1,m}}\right)\mathbb{E}(Z_{n-1}^3) + \alpha_3[n-1,m] \\
&= \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\,\mathbb{E}(Z_{n-1}^3) + \alpha_3[n-1,m].
\end{aligned}$$

By iteration,

$$\mathbb{E}(Z_n^3) = \frac{1}{\Upsilon[n,3,m]}\sum_{k=1}^{n-1} \Upsilon[k+1,3,m]\alpha_3[k,m].$$

□

From Lemma 2.1, Part (3), we have

$$\mathbb{E}(Z_n^2) = \frac{2}{3m-2}n + \mathcal{O}(1),$$

$$\mathbb{E}(Z_n^3) = \frac{9m}{(3m-2)(4m-3)}n + \mathcal{O}(1).$$

3. Variances

Lemma 3.1. *The sequence* $\{\Upsilon[n,2,m](Z_n^2-\mathbb{E}(Z_n^2))\}_{n\geq 1}$ *is a martingale relative to the* $\mathcal{F}_{n-1}$.

Proof. We have

$$\mathbb{E}(Z_n^2|\mathcal{F}_{n-1})=\frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}Z_{n-1}^2+\alpha_2[n-1,m].$$

Thus

$$\mathbb{E}(\Upsilon[n,2,m](Z_n^2-\mathbb{E}(Z_n^2))|\mathcal{F}_{n-1})=\Upsilon[n-1,2,m](Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)),$$

and proof is completed [7]. □

From (2.1),

$$\begin{aligned}\mathbb{E}((Z_n^2-Z_{n-1}^2-1)^2)|\mathcal{F}_{n-1}) &= 4\mathbb{E}(d_{U_{n-1}}^2|\mathcal{F}_{n-1})\\ &= 4\sum_{k=1}^{n-1}d_k^2P(U_n=k|\mathcal{F}_{n-1})\\ &= \frac{4}{s_{n-1,m}}\Big((m-1)Z_{n-1}^3+Z_{n-1}^2\Big).\end{aligned}$$

Then

$$\mathbb{E}((Z_n^2-Z_{n-1}^2-1)^2)=\frac{4}{s_{n-1,m}}\Big((m-1)\mathbb{E}(Z_{n-1}^3)+\mathbb{E}(Z_{n-1}^2)\Big). \quad (3.1)$$

Also

$$\begin{aligned}\mathbb{E}((Z_n^2-Z_{n-1}^2-1)^2) &= \mathbb{E}\bigg(\Big(Z_n^2-Z_{n-1}^2-1-\mathbb{E}(Z_n^2)+\mathbb{E}(Z_n^2)-\mathbb{E}(Z_{n-1}^2+\mathbb{E}(Z_{n-1}^2)\Big)^2\bigg)\\ &= \mathbb{V}ar(Z_n^2)-2\mathbb{E}((Z_n^2-\mathbb{E}(Z_n^2))(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)))\\ &+ \mathbb{V}ar(Z_{n-1}^2)+(\mathbb{E}(Z_n^2)-\mathbb{E}(Z_{n-1}^2)-1)^2. \quad (3.2)\end{aligned}$$

From Lemma 3.1,

$$\mathbb{E}((Z_n^2-\mathbb{E}(Z_n^2))(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)))=\frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\mathbb{V}ar(Z_{n-1}^2),$$

and also,

$$(\mathbb{E}(Z_n^2)-\mathbb{E}(Z_{n-1}^2)-1)^2=\left(\frac{2(m-1)}{s_{n-1,m}}\mathbb{E}(Z_{n-1}^2)+\alpha_2[n-1,m]\right)^2.$$

From part (2) in Lemma 2.1, the relation (3.2) leads to

$$\begin{aligned}
\mathbb{E}((Z_n^2 - Z_{n-1}^2 - 1)^2) &= \mathbb{V}ar(Z_n^2) + \mathbb{V}ar(Z_{n-1}^2) - 2\frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\mathbb{V}ar(Z_{n-1}^2) \\
&+ \left(\frac{2(m-1)}{s_{n-1,m}}\mathbb{E}(Z_{n-1}^2) + \alpha_2[n-1,m]\right)^2 \\
&= \mathbb{V}ar(Z_n^2) + \left(1 - 2\frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\right)\mathbb{V}ar(Z_{n-1}^2) \\
&+ \left(\frac{2(m-1)}{s_{n-1,m}}\mathbb{E}(Z_{n-1}^2) + \alpha_2[n-1,m]\right)^2. \qquad (3.3)
\end{aligned}$$

Theorem 3.2. *Let Z_n^2 be the modified first Zagreb index of a RMORT of order n. Then*

$$\mathbb{V}ar(Z_n^2) = \frac{1}{\Upsilon[n,4,m]}\sum_{k=1}^{n-1}\Upsilon[k+1,4,m]\gamma_2[k,m],$$

where

$$\begin{aligned}
\gamma_2[n,m] &= \frac{4}{s_{n,m}}\Big((m-1)\mathbb{E}(Z_n^3) + \mathbb{E}(Z_n^2)\Big) \\
&- \left(\frac{2(m-1)}{s_{n,m}}\mathbb{E}(Z_n^2) + \alpha_2[n,m]\right)^2.
\end{aligned}$$

Proof. From (3.1) and (3.3) and part (2) in Lemma 2.1,

$$\mathbb{V}ar(Z_n^2) = \frac{\Upsilon[n-1,4,m]}{\Upsilon[n,4,m]}\mathbb{V}ar(Z_{n-1}^2) + \gamma_2[n-1,m].$$

By iteration, proof is completed. □

Theorem 3.3. *Set*

$$\beta[n,m] = \alpha_3[n,m]\mathbb{E}(Z_n^2) + 2\mathbb{E}(d_{U_n}Z_{n+1}^3) + \mathbb{E}(Z_{n+1}^3).$$

Let $\mathbb{C}ov(Z_n^2, Z_n^3)$ be the covariance between Z_n^2 and Z_n^3. Then

$$\mathbb{C}ov(Z_n^2, Z_n^3) = \frac{1}{\Upsilon[n,3,m]}\sum_{k=1}^{n-1}\Upsilon[k+1,3,m]\beta[k,m] - \mathbb{E}(Z_n^2)\mathbb{E}(Z_n^3).$$

Proof. From (2.1),

$$\mathbb{E}(Z_n^2 Z_n^3) = \mathbb{E}(Z_{n-1}^2 Z_n^3) + 2\mathbb{E}(d_{U_{n-1}}Z_n^3) + \mathbb{E}(Z_n^3).$$

But

$$\begin{aligned}
\mathbb{E}(Z_{n-1}^2 Z_n^3) &= \mathbb{E}(Z_{n-1}^2\mathbb{E}(Z_n^3|\mathcal{F}_{n-1})) \\
&= \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\mathbb{E}(Z_{n-1}^2 Z_{n-1}^3) + \alpha_3[n-1,m]\mathbb{E}(Z_{n-1}^2).
\end{aligned}$$

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Thus

$$\begin{aligned}\mathbb{E}(Z_n^2 Z_n^3) &= \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\mathbb{E}(Z_{n-1}^2 Z_{n-1}^3) + \alpha_3[n-1,m]\mathbb{E}(Z_{n-1}^2) + 2\mathbb{E}(d_{U_{n-1}} Z_n^3)\\ &+ \mathbb{E}(Z_n^3)\\ &= \cdots = \frac{1}{\Upsilon[n,3,m]}\sum_{k=1}^{n-1}\Upsilon[k+1,3,m]\beta[k,m].\end{aligned}$$

Proof is completed since $\mathbb{C}ov(Z_n^2, Z_n^3) = \mathbb{E}(Z_n^2 Z_n^3) - \mathbb{E}(Z_n^2)\mathbb{E}(Z_n^3)$. □

Theorem 3.4. *Let $\mathbb{C}ov(Z_n^3, Z_{n-1}^3)$ be the covariance between Z_n^3 and Z_{n-1}^3. Then*

$$\mathbb{C}ov(Z_n^3, Z_{n-1}^3) = \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\mathbb{V}ar(Z_{n-1}^3) + \frac{3\mathbb{C}ov(Z_{n-1}^2, Z_{n-1}^3)}{s_{n-1,m}}.$$

Proof. We have

$$\begin{aligned}\mathbb{C}ov(Z_n^3, Z_{n-1}^3) &= \mathbb{E}\Big[\big(Z_n^3 - \mathbb{E}(Z_n^3)\big)\big(Z_{n-1}^3 - \mathbb{E}(Z_{n-1}^3)\big)\Big]\\ &= \mathbb{E}\Bigg[\Big(Z_{n-1}^3 - \mathbb{E}(Z_{n-1}^3)\Big)\mathbb{E}\Big[\big(Z_n^3 - \mathbb{E}(Z_n^3)\big)|\mathcal{F}_{n-1}\Big]\Bigg].\end{aligned} \tag{3.4}$$

Furthermore,

$$\begin{aligned}\mathbb{E}\Big[\big(Z_n^3 - \mathbb{E}(Z_n^3)\big)|\mathcal{F}_{n-1}\Big] &= \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\big(Z_{n-1}^3 - \mathbb{E}(Z_{n-1}^3)\big)\\ &+ \frac{3}{s_{n-1,m}}\big(Z_{n-1}^2 - \mathbb{E}(Z_{n-1}^2)\big).\end{aligned} \tag{3.5}$$

Using the relations (3.4) and (3.5), proof is completed. □

Set

$$E[n,m] = \frac{9}{s_{n,m}}\Big((m-1)\mathbb{E}(Z_n^5) + (2m-1)\mathbb{E}(Z_n^4) + (m+1)\mathbb{E}(Z_n^3) + \mathbb{E}(Z_n^2)\Big),$$

and

$$F[n,m] = 2\Big(\frac{3(m-1)}{s_{n,m}}\mathbb{E}(Z_n^3) + \alpha_3[n,m] - 1\Big)^2 + \frac{6\mathbb{C}ov(Z_n^2, Z_n^3)}{s_{n,m}}.$$

Theorem 3.5. *Let Z_n^3 be the modified forgotten topological index of a RMORT of order n. Then*

$$\mathbb{V}ar(Z_n^3) = \frac{1}{\Upsilon[n,6,m]}\sum_{k=1}^{n-1}\Upsilon[k+1,6,m]\gamma_3[k,m],$$

where

$$\gamma_3[n,m] = E[n,m] + F[n,m].$$

Proof. We have

$$\begin{aligned}\mathbb{E}\Big[\Big(Z_n^3 - Z_{n-1}^3 - 1\Big)^2|\mathcal{F}_{n-1}\Big] &= \mathbb{E}\Big[9\Big(d_{U_{n-1}} + d_{U_{n-1}}^2\Big)^2|\mathcal{F}_{n-1}\Big] \\ &= 9\sum_{k=1}^{n-1}\Big(d_k^4 + 2d_k^3 + d_k^2\Big)\frac{(m-1)d_k + 1}{s_{n-1,m}} \\ &= \frac{9}{s_{n-1,m}}\Big((m-1)Z_{n-1}^5 + (2m-1)Z_{n-1}^4 + (m+1)Z_{n-1}^3\Big) \\ &\quad + \frac{9}{s_{n-1,m}}Z_{n-1}^2. \end{aligned} \tag{3.6}$$

Taking the expectation of both sides relation (3.6),

$$\mathbb{E}\Big[\Big(Z_n^3 - Z_{n-1}^3 - 1\Big)^2\Big] := E[n-1, m]. \tag{3.7}$$

It is not difficult to show that

$$(\mathbb{E}(Z_n^3) - \mathbb{E}(Z_{n-1}^3) - 1)^2 = \Big(\frac{3(m-1)}{s_{n-1,m}}\mathbb{E}(Z_{n-1}^3) + \alpha_3[n-1, m] - 1\Big)^2. \tag{3.8}$$

From (3.8) and Theorem 3.4,

$$\begin{aligned}\mathbb{E}\Big(Z_{3,n} - Z_{3,n-1} - 1\Big)^2 &= \mathbb{E}\Big(Z_{3,n} - \mathbb{E}(Z_{3,n}) - Z_{3,n-1} + \mathbb{E}(Z_{3,n-1})\Big)^2 \\ &\quad + \Big(\mathbb{E}(Z_{3,n}) - \mathbb{E}(Z_{3,n-1}) - 1\Big)^2 \\ &= \mathbb{E}\Big(Z_{3,n} - \mathbb{E}(Z_{3,n})\Big)^2 + \mathbb{E}\Big(Z_{3,n-1} - \mathbb{E}(Z_{3,n-1})\Big)^2 \\ &\quad - \Big(\mathbb{E}(Z_{3,n}) - \mathbb{E}(Z_{3,n-1}) - 1\Big)^2 - 2\mathbb{C}ov(Z_n^3, Z_{n-1}^3) \\ &= \mathbb{V}ar(Z_n^3) + \mathbb{V}ar(Z_{n-1}^3)\Big(1 - 2\frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\Big) \\ &\quad - \Big(\frac{3(m-1)}{s_{n-1,m}}\mathbb{E}(Z_{n-1}^3) + \alpha_3[n-1,m] - 1\Big)^2 \\ &\quad - \frac{6\mathbb{C}ov(Z_{n-1}^2, Z_{n-1}^3)}{s_{n-1,m}} \\ &= \mathbb{V}ar(Z_n^3) + \mathbb{V}ar(Z_{n-1}^3)\Big(1 - 2\frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\Big) - F[n-1,m]. \end{aligned}$$

From Lemma 2.1, relations (3.7),

$$\mathbb{V}ar(Z_n^3) = \frac{\Upsilon[n-1,6,m]}{\Upsilon[n,6,m]}\mathbb{V}ar(Z_{n-1}^3) + \gamma_3[n-1,m]. \tag{3.9}$$

By iteration, proof is completed. □

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4. Limiting rules

We use the notation $\xrightarrow{P}$ to denote convergence in probability [7].

Theorem 4.1. *As $n \to \infty$,*

$$\frac{Z_n^2}{n} \xrightarrow{P} \frac{2}{3m-2},$$
$$\frac{Z_n^3}{n} \xrightarrow{P} \frac{9m}{(3m-2)(4m-3)}.$$

Proof. From Lemma 2.1, Part (3), and just similar to the expectations, we can show that

$$\mathbb{V}ar(Z_n^\ell) = \sigma_\ell^2(m)n + \mathcal{O}(1), \ \ell = 2, 3$$

where $\sigma_\ell^2(m)$ is a constant independent of n:

$$\sigma_2^2(m) = \frac{m(m(3m(3m(4m+1)-8)-224)+312)-108}{(2-3m)^2 m(3m-4)(4m-3)},$$
$$\sigma_3^2(m) = \frac{m\left(m\left(9m\left(m\left(24m^2+58m-213\right)+32\right)+2560\right)-2208\right)+540}{(2-3m)^2 m(4m-3)(5m-6)(6m-5)}.$$

Now, the claims are a consequence of Chebyshev's inequality. □

The table below shows some values of $\sigma_2^2(m)$ and $\sigma_3^2(m)$ for $m = 4, \ldots, 10$. These are decreasing functions of m.

m	4	5	6	7	8	9	10
$\sigma_2^2(m)$	$\frac{8797}{10400}$	$\frac{110977}{158015}$	$\frac{7781}{12544}$	$\frac{609529}{1073975}$	$\frac{298069}{561440}$	$\frac{79699}{158125}$	$\frac{911653}{1885520}$
$\sigma_3^2(m)$	$\frac{244903}{345800}$	$\frac{31181}{54587}$	$\frac{327611}{666624}$	$\frac{29792029}{67786775}$	$\frac{16542535}{41041264}$	$\frac{4942051}{13138125}$	$\frac{12477623}{35099680}$

Lemma 4.2. *For a RMORT of order n,*

$$n^{\frac{4-3m}{m}} \sum_{k=2}^{n} \frac{\Upsilon[k,2,m]^2}{s_{k-1,m}} \left(Z_{k-1}^3 - \mathbb{E}(Z_{k-1}^3)\right) \xrightarrow{P} 0.$$

Proof. It is easy to show that

$$\frac{3}{s_{n-1,m}} - \left(\frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2}\right) = \frac{\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}.$$

From Theorem 2.2, we have

$$\mathbb{E}\left(Z_n^3 - \frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2} Z_n^2 | \mathcal{F}_{n-1}\right) = \frac{\Upsilon[n-1,3,m]}{\Upsilon[n,3,m]}\left(Z_{n-1}^3 - Z_{n-1}^2\right) + k[n-1,m],$$

where

$$k[n,m] = \frac{6(n-1)}{s_{n,m}} + 2 - \left(\frac{4(n-1)}{s_{n,m}}\right)\left(\frac{-s_{n,m}+6-3m}{s_{n,m}+2m-2}\right).$$

This follows that the process $(\Upsilon[n,3,m]Z_n^*)_{n\geq 1}$ is a martingale where

$$Z_n^* = Z_n^3 - \frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2} Z_n^2 - \mathbb{E}\left(Z_n^3 - \frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2} Z_n^2\right).$$

Similarly to computation of the variance of Z_n^2 and Z_n^3 in Theorems 3.2 and 3.5, one can obtain that the variance of Z_n^* is also in the form $c_m^* n+\mathcal{O}(1)$ where c_m^* is a constant independent of n. From Lemma 2.1, part (3), there exist an obsolute constant c_0 independent of n such that $\Upsilon[n,3,m] < c_0\Upsilon[n,2,m]^{3/2}$. By Doob's inequality, we have

$$\begin{aligned}
&\mathbb{E}\bigg(\max_{2\le k\le n}(\Upsilon[k-1,3,m](Z_{k-1}^3-\mathbb{E}(Z_{k-1}^3))\bigg)^2\\
&\le 2\mathbb{E}\bigg(\max_{2\le k\le n}\Big(\Upsilon[k-1,3,m]Z_{k-1}^*\Big)^2\bigg)\\
&+2\bigg(\frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2}\bigg)^2 c_0^2\Upsilon[n,2,m]\bigg(\max_{2\le k\le n}\Big(\Upsilon[k-1,2,m](Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)\Big)^2\bigg)\\
&\le 8\Upsilon[2,3,m]\mathbb{V}ar(Z_n^*)+8\bigg(\frac{6-3m-s_{n-1,m}}{s_{n-1,m}+2m-2}\bigg)c_0^2\Upsilon[n,2,m]^3\mathbb{V}ar(Z_n^2)\\
&=\mathcal{O}(n).
\end{aligned}$$

For any $\varepsilon>0$, by Chebyshev's inequality,

$$\begin{aligned}
&P\bigg(\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{s_{k-1,m}}\Big(Z_{k-1}^3-\mathbb{E}(Z_{k-1}^3)\Big)>4n^{\frac{3m-4}{m}}\bigg)\\
&\le\frac{1}{\varepsilon^2 n^{2\frac{3m-4}{m}}}\mathbb{E}\bigg(\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{s_{k-1,m}}(Z_{k-1}^3-\mathbb{E}(Z_{k-1}^3)\bigg)^2\\
&\le\frac{1}{\varepsilon^2 n^{2\frac{3m-4}{m}}}\bigg(\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{\Upsilon[k-1,3,m]s_{k-1,m}}\bigg)^2\mathbb{E}\bigg(\max_{2\le k\le n}(\Upsilon[k-1,3,m](Z_{k-1}^3-\mathbb{E}(Z_{k-1}^3))\bigg)^2\\
&=\mathcal{O}(n^{-1}).
\end{aligned}$$

Since

$$\lim n^{-m}\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{\Upsilon[k-1,3,m]s_{k-1,m}}<\infty,$$

proof is completed. □

Lemma 4.3. *For a RMORT of order n,*

$$n^{\frac{4-3m}{m}}\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{s_{k-1,m}}\Big(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)\Big)\xrightarrow{P}0.$$

Proof. We have

$$\sum_{k=2}^n\frac{\Upsilon[k,2,m]^2}{\Upsilon[k-1,2,m]s_{k-1,m}}=\mathcal{O}(n^{\frac{2}{m}-2}),$$

and

$$\begin{aligned}
\mathbb{E}\bigg(\max_{2\le k\le n}\Big(\Upsilon[k-1,2,m]\Big(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)\Big)\Big)^2\bigg)&\le 4\Upsilon[n,2,m]^2\mathbb{V}ar(Z_n^2)\\
&=\mathcal{O}\Big(n^{\frac{4-3m}{m}}\Big).
\end{aligned}$$

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Just silmilar to proof of Lemma 4.2 and using Chebyshev's inequality, proof is completed.

□

We use the notation $\xrightarrow{D}$ to denote convergence in distribution. The standard random variable $\mathrm{N}(\mu,\sigma^2)$ appear in the following theorem for the normal distributed with mean μ and variance σ^2.

Theorem 4.4. *As* $n\to\infty$,

$$Z_n^{2*}=\frac{Z_n^2-\frac{2}{3m-2}n}{\sqrt{n}}\xrightarrow{D} N(0,\sigma_2^2(m)).$$

Proof. The main idea to show the asymptotic normailty of the indices is to introduce an appropriate martingale difference sequence. From Lemma 3.1, we define

$$M_{k,2,m}=\Upsilon[k,2,m](Z_k^2-\mathbb{E}(Z_k^2))-\Upsilon[k-1,2,m](Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)),$$

with $M_{1,2,m}=0$. Then the process $\{M_{k,2,m}\}_{k\geq 1}$ is a martingale difference sequence. From [7] and the expression of the variance $\mathbb{V}ar(Z_n^2)$, we should prove that, for any $\varepsilon>0$,

$$\frac{1}{\Upsilon[n,2,m]^2 n}\sum_{k=2}^{n}\mathbb{E}\left(M_{k,2,m}^2 I\left(\left|\frac{M_{k,2,m}}{\Upsilon[n,2,m]\sqrt{n}}\right|>\varepsilon\right)\middle|\mathcal{F}_{k-1}\right)\xrightarrow{P}0, \tag{4.1}$$

and

$$\frac{1}{\Upsilon[n,2,m]^2\sigma_2^2(m)n}\sum_{k=2}^{n}\mathbb{E}(M_{k,2,m}^2|\mathcal{F}_{k-1}))\xrightarrow{P}1. \tag{4.2}$$

We can rewrite $M_{k,2,m}$ as

$$M_{k,2,m}=\Upsilon[k,2,m]\left(2d_{U_{k,k-1}}+1+\frac{2Z_{k-1}^2}{s_{k-1,m}}-\alpha_2[k-1,m]\right).$$

Then there exist a positive constant c_m which only depends on m such that

$$\max_{2\leq k\leq n}|M_{k,2,m}|\leq c_m\Upsilon[n,2,m]=\mathcal{O}(\Upsilon[n,2,m]\sqrt{n}),$$

which implies that (4.1) is holds. We have

$$\begin{aligned}\mathbb{E}\left((Z_n^2-Z_{n-1}^2-1)^2|\mathcal{F}_{n-1}\right)&=\mathbb{E}\left((Z_n^2-\mathbb{E}(Z_n^2))^2|\mathcal{F}_{n-1}\right)\\&+\left(1-\frac{2\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\right)\left(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)\right)^2\\&+\frac{1}{s_{n-1,m}^2}\left(2(m-1)\mathbb{E}(Z_{n-1}^2)+4(n-2)\right)^2\\&+\frac{4(m-1)}{s_{n-1,m}^2}(2(m-1)\mathbb{E}(Z_{n-1}^2)+4(n-2)\left(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)\right)^2.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\bigg((Z_n^2-\mathbb{E}(Z_n^2))^2|\mathcal{F}_{n-1}\bigg) &= \frac{4}{s_{n-1,m}}\bigg((m-1)Z_{n-1}^3+Z_{n-1}^2\bigg)\\ &+\bigg(1-\frac{2\Upsilon[n-1,2,m]}{\Upsilon[n,2,m]}\bigg)\bigg(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)\bigg)^2\\ &-\frac{1}{s_{n-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{n-1}^2)+4(n-2)\bigg)^2\\ &-\frac{4(m-1)}{s_{n-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{n-1}^2)+4(n-2)\bigg)\\ &\times(Z_{n-1}^2-\mathbb{E}(Z_{n-1}^2)).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=2}^{n}\mathbb{E}\bigg(M_{k,2,m}^2|\mathcal{F}_{n-1}\bigg) &= \sum_{k=2}^{n}\frac{4\Upsilon[k,2,m]^2}{s_{k-1,m}}\bigg((m-1)Z_{k-1}^3+Z_{k-1}^2\bigg)\\ &-\sum_{k=2}^{n}\frac{4(m-1)^2\Upsilon[k,2,m]^2}{s_{k-1,m}^2}\bigg(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)\bigg)\\ &-\sum_{k=2}^{n}\frac{\Upsilon[k,2,m]^2}{s_{k-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{k-1}^2)+4(k-2)\bigg)^2\\ &-\sum_{k=2}^{n}\frac{4(m-1)\Upsilon[k,2,m]^2}{s_{k-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{k-1}^2)+4(k-2)\bigg)\\ &\times(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)).\end{aligned}$$

Since

$$\begin{aligned}&-\frac{1}{s_{n-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{n-1}^2)+4(n-2)\bigg)^2\\ &=\gamma_2[n-1,m]-\frac{4}{s_{n-1,m}}\bigg((m-1)\mathbb{E}(Z_n^3)+\mathbb{E}(Z_n^2)\bigg),\end{aligned}$$

we have

$$\begin{aligned}\sum_{k=2}^{n}\mathbb{E}\bigg(M_{k,2,m}^2|\mathcal{F}_{n-1}\bigg) &= \sum_{k=2}^{n}\Upsilon[k,2,m]^2\gamma_2[k-1,m]\\ &-\sum_{k=2}^{n}\frac{4\Upsilon[k,2,m]^2}{s_{k-1,m}}\bigg((m-1)(Z_{k-1}^3-\mathbb{E}(Z_{k-1}^3)\bigg)-(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2))\bigg)\\ &-\sum_{k=2}^{n}\frac{4(m-1)^2\Upsilon[k,2,m]^2}{s_{k-1,m}^2}(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2))^2\\ &-\sum_{k=2}^{n}\frac{4(m-1)\Upsilon[k,2,m]^2}{s_{k-1,m}^2}\bigg(2(m-1)\mathbb{E}(Z_{k-1}^2)+4(k-2)\bigg)(Z_{k-1}^2-\mathbb{E}(Z_{k-1}^2)).\end{aligned}$$

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Also

$$\lim_{n\to\infty} \gamma_2[n,m] = \frac{4-3m}{m}\sigma_2^2(m).$$

Now, by Lemmas 4.2 and 4.3,

$$\frac{1}{\Upsilon[n,2,m]^2\sigma_2^2(m)n}\sum_{k=2}^{n}\mathbb{E}(M_{k,2,m}^2|\mathcal{F}_{k-1}) \xrightarrow{P} 1-0-0-0=1.$$

□

Also, we can prove that

$$Z_n^{3*} = \frac{Z_n^3 - \frac{9m}{(3m-2)(4m-3)}n}{\sqrt{n}} \xrightarrow{D} \mathrm{N}(0,\sigma_3^2(m)),$$

as $n\to\infty$. Since the computations are essentially the same, but quite lengthy computations, they are omitted here.

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