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REMARK ON LAQUER'S THEOREM FOR CIRCULANT DETERMINANTS

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Abstract. Olga Taussky-Todd suggested the problem of determining the possible values of integer circulant determinants. To solve a special case of the problem, Laquer gave a factorization of circulant determinants. In this paper, we give a modest generalization of Laquer's theorem. Also, we give an application of the generalization to integer group determinants.

1. Introduction

For a finite group G, let x_g be an indeterminate for each $g \in G$, and let $\mathbb{Z}[x_g]$ be the multivariate polynomial ring in x_g over $\mathbb{Z}$. The group determinant $\Theta(G)$ of G is defined as follows ([3], [4]; see also [2], [6], [7]):

$$\Theta(G) := \det\left(x_{gh^{-1}}\right)_{g,h \in G} \in \mathbb{Z}[x_g].$$

When G is a cyclic group $\mathbb{Z}/n\mathbb{Z} = \left\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\right\}$, the polynomial $\Theta(G)$ is called a circulant determinant and written as $C_n(x_1, x_2, \ldots, x_n)$, where x_{i+1} denotes $x_{\overline{i}}$.

When $x_1, x_2, \ldots, x_n$ are integers, $C_n(x_1, x_2, \ldots, x_n)$ is called an integer circulant determinant. Olga Taussky-Todd suggested the problem of determining the possible values of integer circulant determinants [11]; that is, determining the set

$$S(n) := \{C_n(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{Z}\}.$$

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To solve the $n = 2p$ case, where p is an odd prime, Laquer [10, Theorem 2] gave the following theorem: *Let $n = rs$, where r and s are relatively prime. Then*

$$C_n(x_1, x_2, \ldots, x_n) = \prod_{i=0}^{s-1} C_r\left(y_1^i, y_2^i, \ldots, y_r^i\right), \quad y_j^i := \sum_{k=0}^{s-1} \zeta_s^{i(kr+j-1)} x_{kr+j},$$

where ζ_s is a primitive s-th root of unity. We call this theorem Laquer's theorem. From this theorem, when r is odd and $s = 2$, an integer circulant determinant $C_n(x_1, x_2, \ldots, x_n)$ can be written as a product of 2 integer circulant determinants. As a result of this consideration, Laquer [10, Theorem 10] showed that

$$S(2p) = (\mathbb{Z}_2^* \cup 4\mathbb{Z}) \cap (\mathbb{Z}_p^* \cup p^2\mathbb{Z}),$$

where $\mathbb{Z}_m^* := \{a \in \mathbb{Z} \mid \gcd(a, m) = 1\}$ for any positive integer m.

Laquer's theorem was proved by using a special case of Dedekind's theorem. For a finite group G, let $\widehat{G}$ be a complete set of representatives of the equivalence classes of irreducible representations of G over $\mathbb{C}$. Dedekind's theorem is as follows (e.g., [5], [9], [13], [14]): *When G is abelian, $\Theta(G)$ can be factorized into irreducible polynomials over $\mathbb{C}$ as*

$$\Theta(G) = \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g.$$

Let $\Theta(G)\,[x_g \mapsto y_g]$ be the polynomial obtained by replacing the variable x_g in $\Theta(G)$ with y_g for any $g \in G$. From Dedekind's theorem, we obtain a modest generalization of Laquer's theorem.

Theorem 1.1. *Let $G = H \times K$ be a direct product of finite abelian groups. Then we have*

$$\Theta(G) = \prod_{\chi \in \widehat{K}} \Theta(H)\left[x_h \mapsto \sum_{k \in K} \chi(k) x_{hk}\right].$$

We apply this theorem to integer group determinants. A group determinant called an integer group determinant when its variables are integers. For a finite group G, let

$$S(G) := \{\det\left(x_{gh^{-1}}\right)_{g,h \in G} \mid x_g \in \mathbb{Z}\}, \quad S(G)_{\text{even}} := S(G) \cap 2\mathbb{Z}.$$

Theorem 1.2. *Let $G = H \times (\mathbb{Z}/2\mathbb{Z})^l$, where H is a finite abelian group, and let*

$$M := \max\{m \in \mathbb{N} \mid 2^m \textit{ divides every element of } S(H)_{\text{even}}\}.$$

Then we have

$$S(G)_{\text{even}} \subset 2^{M \cdot 2^l}\mathbb{Z}.$$

It is well-known that $S(\mathbb{Z}/2\mathbb{Z})_{\text{even}} = 4\mathbb{Z}$. Also, Kaiblinger [8, Theorem 1.1, Example 3.3] showed that $S(\mathbb{Z}/2^n\mathbb{Z})_{\text{even}} \subset 2^{n+2}\mathbb{Z}$ and $S(\mathbb{Z}/2^n\mathbb{Z})_{\text{even}} \not\subset 2^{n+3}\mathbb{Z}$ for $n \geq 2$. Therefore, as a corollary of Theorem 1.2, we have the following.

Corollary 1.3. *For any $n \geq 2$, the following hold:*

(1) *For $G = (\mathbb{Z}/2\mathbb{Z})^n = \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{n-1}$, we have $S(G)_{\text{even}} \subset 2^{2 \cdot 2^{n-1}}\mathbb{Z} = 2^{2^n}\mathbb{Z}$;*

(2) *For* $G = \mathbb{Z}/2^n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*, we have* $S(G)_{\mathrm{even}} \subset 2^{2(n+2)}\mathbb{Z}$*.*

For the $n = 2$ and 3 cases of G in (1), $S(G)$ were determined in [1] and [12], respectively, and for the $n = 2$ case of G in (2), $S(G)$ was determined in [12]:

$$S((\mathbb{Z}/2\mathbb{Z})^2) = \{4m + 1,\ 2^4(2m + 1),\ 2^6m \mid m \in \mathbb{Z}\},$$
$$S((\mathbb{Z}/2\mathbb{Z})^3) = \{8m + 1,\ 2^8(4m + 1),\ 2^{12}m \mid m \in \mathbb{Z}\},$$
$$S(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \{8m + 1,\ 2^8m \mid m \in \mathbb{Z}\}.$$

These results imply that the upper inclusions in Corollary 1.3 (1) and (2) are best possible in the sense that

$$S((\mathbb{Z}/2\mathbb{Z})^n)_{\mathrm{even}} \not\subset 2^{2^n+1}\mathbb{Z}, \quad S(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})_{\mathrm{even}} \not\subset 2^9\mathbb{Z},$$

where $n = 2, 3$. Furthermore, for $G = (\mathbb{Z}/2\mathbb{Z})^n$ with an arbitrary n, putting $x_e = 2$ and $x_g = 0$ for all $g \neq e$, where e is the unit element of G, we have $\Theta(G) = 2^{2^n}$. That is, $S(G)_{\mathrm{even}} \not\subset 2^{2^n+1}\mathbb{Z}$.

2. **Proof of Theorems**

Proof of Theorem 1.1. From Dedekind's theorem and $\widehat{G} = \widehat{H} \times \widehat{K}$, we have

$$\begin{aligned}
\Theta(G) &= \prod_{\chi \in \widehat{G}} \sum_{g \in G} \chi(g) x_g \\
&= \prod_{\chi \in \widehat{H}} \prod_{\chi' \in \widehat{K}} \sum_{h \in H} \sum_{k \in K} \chi(h)\chi'(k) x_{hk} \\
&= \prod_{\chi' \in \widehat{K}} \prod_{\chi \in \widehat{H}} \sum_{h \in H} \chi(h) \left(\sum_{k \in K} \chi'(k) x_{hk} \right) \\
&= \prod_{\chi' \in \widehat{K}} \Theta(H) \left[x_h \mapsto \sum_{k \in K} \chi'(k) x_{hk} \right].
\end{aligned}$$

□

We show that Laquer's theorem is a special case of Theorem 1.1. Let $G = \mathbb{Z}/n\mathbb{Z}$ and $n = rs$ where r and s are relatively prime. Since

$$G = \langle s \rangle \times \langle r \rangle = \left\{ \overline{as} + \overline{br} \mid 0 \leq a \leq r - 1,\ 0 \leq b \leq s - 1 \right\},$$

from Theorem 1.1, we have

$$C_n(x_{\overline{0}}, x_{\overline{1}}, \ldots, x_{\overline{n-1}}) = \prod_{i=0}^{s-1} \prod_{l=0}^{r-1} \sum_{a=0}^{r-1} \zeta_r^{l(as)} y_{\overline{as}}^i, \quad y_{\overline{as}}^i := \sum_{b=0}^{s-1} \zeta_s^{i(br)} x_{\overline{as+br}}.$$

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From $\{as+br \mid 0 \le a \le r-1,\ 0 \le b \le s-1\} = \{j-1+kr \mid 1 \le j \le r,\ 0 \le k \le s-1\}$,

$$\begin{aligned}\sum_{a=0}^{r-1} \zeta_r^{l(as)} y_{\overline{as}}^{i} &= \sum_{a=0}^{r-1}\sum_{b=0}^{s-1} \zeta_r^{l(as+br)} \zeta_s^{i(as+br)} x_{\overline{as+br}} \\ &= \sum_{j=1}^{r}\sum_{k=0}^{s-1} \zeta_r^{l(j-1+kr)} \zeta_s^{i(j-1+kr)} x_{\overline{j-1+kr}} \\ &= \sum_{j=1}^{r} \zeta_r^{l(j-1)} \sum_{k=0}^{s-1} \zeta_s^{i(j-1+kr)} x_{\overline{j-1+kr}}.\end{aligned}$$

Therefore, denoting $x_{\overline{i}}$ by x_{i+1}, we obtain Laquer's theorem.

Proof of Theorem 1.2. Let $K := (\mathbb{Z}/2\mathbb{Z})^l$ and let $x_g \in \mathbb{Z}$ for all $g \in G$. For any $\chi \in \widehat{K}$, since $\operatorname{Im}(\chi) \in \{\pm 1\}$, $\alpha_\chi := \Theta(H)\left[x_h \mapsto \sum_{k \in K} \chi(k) x_{hk}\right]$ is an integer group determinant and

$$\alpha_\chi \equiv \alpha_{\chi_1} \pmod 2,$$

where χ_1 is the trivial character of K. Suppose that $\Theta(G)$ is even. Then from Theorem 1.1 and the definition of M, α_χ are divisible by 2^M for all $\chi \in \widehat{K}$. Therefore, $\Theta(G) = \prod_{\chi \in \widehat{K}} \alpha_\chi$ is divisible by $(2^M)^{|K|} = (2^M)^{2^l}$, where $|K|$ denotes the order of K. □

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