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UNICYCLIC GRAPHS WITH NON-ISOLATED RESOLVING NUMBER 2

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Abstract. Let G be a connected graph and $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of vertices of G. For any vertex v of G, the ordered k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the metric representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. A set W is called a resolving set for G if distinct vertices of G have distinct metric representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension denoted by $\dim(G)$. A resolving set W is called a non-isolated resolving set for G if the induced subgraph $\langle W \rangle$ of G has no isolated vertices. The minimum cardinality of a non-isolated resolving set for G is called the non-isolated resolving number of G and denoted by $nr(G)$. The aim of this paper is to find properties of unicyclic graphs that have non-isolated resolving number 2 and then to characterize all these graphs.

1. Introduction

In this section, we present some definitions and notations which are necessary to prove main results. Throughout this paper, G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v. The number of all neighbors of a vertex v is $\deg(v)$. We use symbols $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$ for a path of order n, P_n, and a cycle of order n, C_n, respectively. A unicyclic graph is a graph containing exactly one cycle.

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For an ordered subset $W = \{w_1, \ldots, w_k\}$ of $V(G)$ and a vertex v of G, the *metric representation* of v with respect to W is

$$r(v|W) = (d(v, w_1), \ldots, d(v, w_k)).$$

The set W is a *resolving set* for G if the distinct vertices of G have different metric representations, with respect to W. A resolving set W for G with minimum cardinality is a *metric basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$. A resolving set W is called a *non-isolated resolving set* for G if the induced subgraph $\langle W \rangle$ of G has no isolated vertices. A non-isolated resolving set for G with minimum cardinality is called a *non-isolated basis* for G and its cardinality is called the *non-isolated resolving number* of G and denoted by $nr(G)$.

The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [16] and by Harary and Melter [10]. Resolving sets have several applications in diverse areas such as coin weighing problems [15], network discovery and verification [5], robot navigation [13], mastermind game [7], problems of pattern recognition and image processing [14], and combinatorial search and optimization [15]. For more results about resolving sets and metric dimension see [4, 6, 7, 8, 11]. Several variations of resolving sets were introduced by imposing conditions on the subgraph induced by a resolving set. One of these variations is the concept of non-isolated resolving sets. The concepts of non-isolated resolving sets and non-isolated resolving number were introduced in [12]. The non-isolated resolving number of some families of graphs such as paths, complete graphs, bipartite graphs and some friendship graphs are obtained in [12]. For more results about non-isolated resolving sets see [1, 2, 3, 12].

It is easy to see that if G is a graph of order n, then $2 \leq nr(G) \leq n-1$. In [12] all graphs of order n and resolving number $n-1$ are characterized. Also in [12] the problem of characterization all graphs with non-isolated resolving number 2 is proposed as an open problem. An important family of graphs is the family of unicyclic graphs. Unicyclic graphs with metric dimension 2 are studied in [9]. In this paper we investigate unicyclic graphs with non-isolated resolving number 2. The aim of the paper is to find properties of these graphs and then to characterize all of them.

2. **Main Results**

The main goal of this section is to characterize all unicyclic graphs with non-isolated resolving number 2. To do this, we need to consider some properties of these graphs. By the following lemma, in a unicyclic graph with non-isolated resolving number 2 the vertices not in the cycle can not be in a minimum non-isolated resolving set.

Lemma 2.1. *Let G be a unicyclic graph with $nr(G) = 2$. Then every member of a non-isolated basis of G is on the cycle.*

Proof. Suppose, on the contrary, there exists a non-isolated basis B of G and $v \in B$ is not on the cycle. Let u be the nearest vertex of the cycle to v and u_1, u_2 be the neighbours of u on the cycle.

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Since B is non-isolated basis, for every $b \in B$ we have $d(u_1, b) = d(u_2, b)$. This contradiction completes the proof. $\square$

By the next lemma, in a unicyclic graph with non-isolated resolving number two the degree of vertices not in the cycle is at most 2.

Lemma 2.2. *Let G be a unicyclic graph with $nr(G) = 2$. Then every vertex out of the cycle is of degree at most 2.*

Proof. Suppose, on the contrary, that there exists a vertex u out of the cycle with degree at least three. If B is a non-isolated basis of G, then by Lemma 2.1 every vertex in B is on the cycle. Since $\deg(u) \geq 3$, u has three neighbours u_1, u_2, u_3 such that for every $b \in B$, $d(u_2, b) = d(u_1, b) + 2 = d(u_3, b)$, which is a contradiction. Therefore every vertex out of the cycle is of degree at most 2. $\square$

Lemma 2.3. *Let G be a unicyclic graph with $nr(G) = 2$. Then $\deg(v) \leq 3$, for every $v \in V(G)$.*

Proof. By Lemma 2.2 it is sufficient to prove that all vertices on the cycle are of degree at most 3. If u is a vertex on the cycle of degree at least 4, then there are two neighbours u_1, u_2 of u, out of the cycle. Note that by Lemma 2.1 all vertices of a non-isolated basis for G are on the cycle. But for each vertex v on the cycle we have $d(u_1, v) = d(u, v) + 1 = d(u_2, v)$. This contradiction implies that every vertex on the cycle is of degree at most 3. $\square$

The following lemma implies that in a unicyclic graph with non-isolated resolving number 2, there are at most three vertices of degree more than 2.

Lemma 2.4. *Let G be a unicyclic graph with $nr(G) = 2$ and C_m be the unique cycle of G. If m is odd, then there are at most three vertices of degree 3 in G and the distance between every two vertices of degree 3 is at most two. If m is even, then there are at most two vertices of degree 3 in G and they are adjacent.*

Proof. By Lemma 2.2, all vertices of degree 3 are on the cycle. Let $C_m = (v_1, v_2, \ldots, v_m, v_1)$ be the unique cycle of G and $B = \{v_1, v_2\}$ be a non-isolated basis of G. Clearly for every $1 \leq i, j \leq m$, $d(v_i, v_j) \leq \lfloor \frac{m}{2} \rfloor = d$. Note that if for some $1 \leq i \leq m$ both entries of $r(v_i|B) = (r_1, r_2)$ are less than d, then v_i has a neighbour on C_m with metric representation $(r_1 + 1, r_2 + 1)$. On the other hand, if $u_i \notin V(C_m)$ is adjacent to v_i, then $r(u_i|B) = (r_1 + 1, r_2 + 1)$ with respect to B. Hence a vertex of C_m has a neighbour out of C_m if and only if at least one of the entries of its metric representation with respect to B is d. Therefore when m is odd all vertices of degree 3 must be in the set $\{v_{d+1}, v_{d+2}, v_{d+3}\}$, because vertices in this set are all vertices on C_m with distance d to v_1 or v_2. When m is even all vertices of degree 3 must be in the set $\{v_{d+1}, v_{d+2}\}$, because vertices in this set are all vertices on C_m with distance d to v_1 or v_2. Clearly the distance between each pair of these vertices is at most 2. $\square$

To state our characterization we need the following definition.

Definition 2.5. *If $n \geq 3$ is an odd number, then define the family $\mathcal{O}_n$ of graphs G obtained from a cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and paths $P_r = (a_1, a_2, \ldots, a_r)$, $P_s = (b_1, b_2, \ldots, b_s)$, and $P_t = (c_1, c_2, \ldots, c_t)$, $r, s, t \geq 0$, by identifying vertices v_1 to a_1, v_2 to b_1 and v_n to c_1.*
If $n \geq 4$ is an even number, then define the family $\mathcal{E}_n$ of graphs G obtained from a cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$, and paths $P_r = (a_1, a_2, \ldots, a_r)$ and $P_s = (b_1, b_2, \ldots, b_s)$, $r, s \geq 0$, by identifying vertices v_1 to a_1 and v_2 to b_1.

The next theorem characterizes all unicyclic graphs with non-isolated resolving number 2.

Theorem 2.6. *Let G be a unicyclic graph. Then $nr(G) = 2$ if and only if G belongs to $\mathcal{O}_n$ or $\mathcal{E}_n$, for some integer n.*

Proof. If $G \in \mathcal{O}_n$, let $m = \frac{n-1}{2}$. We prove that $B = \{v_{m+1}, v_{m+2}\}$ is a non-isolated basis for G. Since v_{m+1} is adjacent to v_{m+2} it is enough to prove that B is a resolving set for G. Note that

$$d(v_i, v_{m+1}) = \begin{cases} i-(m+1) & \text{if } m+2 \leq i \leq n, \\ m+1-i & \text{if } 1 \leq i \leq m+1. \end{cases}$$

$$d(v_i, v_{m+2}) = \begin{cases} i-(m+2) & \text{if } m+2 \leq i \leq n, \\ m+2-i & \text{if } 1 < i \leq m+1, \\ m & \text{if } i = 1. \end{cases}$$

Hence, if for some $1 \leq i < j \leq n$, we have $d(v_i, v_{m+1}) = d(v_j, v_{m+1})$, then $i = 2m+2-j$. Therefore $i \leq m$ and $m+2 \leq j \leq n$. This implies $d(v_j, v_{m+2}) = j-(m+2)$. if $i = 1$, clearly $d(v_i, v_{m+2}) = m \neq j-(m+2)$. Hence $i \neq 1$ and $d(v_i, v_{m+2}) = i-(m+2)$. This means $d(v_i, v_{m+2}) \neq d(v_j, v_{m+2})$, otherwise $i = 2m+4-j$, which contradicts $i = 2m+2-j$. Therefore $r(v_i|B) \neq r(v_j|B)$, for $1 \leq i \neq j \leq n$. On the other hand,

$$r(a_i|B) = (i+m-1, i+m-1),\ 1 \leq i \leq r,$$

$$r(b_i|B) = (i+m-2, i+m-1),\ 1 \leq i \leq s,$$

$$r(c_i|B) = (i+m-1, i+m-2),\ 1 \leq i \leq t.$$

Therefore $r(x|B) \neq r(y|B)$ for all $x, y \in V(G) \setminus V(C_n)$. Also if $v_i \in V(C_n)$ and $r(v_i|B) = (r_1, r_2)$, then $r_1 + r_2 \leq 2m$, while for every $u \in V(G) \setminus V(C_n)$ the summation of entries of $r(u|B)$ is at least $2m+1$, because $a_1, b_1, c_1 \in V(C_n)$. Hence B is a non-isolated basis of G.

If $G \in \mathcal{E}_n$, let $m = \frac{n}{2}$. We prove that $B = \{v_{m+1}, v_{m+2}\}$ is a non-isolated basis for G. Since v_{m+1} is adjacent to v_{m+2} it is enough to prove that B is a resolving set for G. Note that

$$d(v_i, v_{m+1}) = \begin{cases} i-(m+1) & \text{if } m+2 \leq i \leq n, \\ m+1-i & \text{if } 1 \leq i \leq m+1. \end{cases}$$

$$d(v_i, v_{m+2}) = \begin{cases} i-(m+2) & \text{if } m+2 \leq i \leq n, \\ m+2-i & \text{if } 1 < i \leq m+1, \\ m-1 & \text{if } i = 1. \end{cases}$$

Hence, if for some $1 \leq i < j \leq n$, we have $d(v_i, v_{m+1}) = d(v_j, v_{m+1})$, then $i = 2m + 2 - j$. Therefore $i \leq m$ and $m + 2 \leq j \leq n$. This implies $d(v_j, v_{m+2}) = j - (m + 2)$. if $i = 1$, clearly $d(v_i, v_{m+2}) = m-1 \neq j-(m+2)$. Hence $i \neq 1$ and $d(v_i, v_{m+2}) = i-(m+2)$. This means $d(v_i, v_{m+2}) \neq d(v_j, v_{m+2})$, otherwise $i = 2m + 4 - j$, which contradicts $i = 2m + 2 - j$. Therefore $r(v_i|B) \neq r(v_j|B)$, for $1 \leq i \neq j \leq n$. On the other hand,

$$r(a_i|B) = (i + m - 1, i + m - 2),\ 1 \leq i \leq r,$$

$$r(b_i|B) = (i + m - 2, i + m - 1),\ 1 \leq i \leq s,$$

Therefore $r(x|B) \neq r(y|B)$ for all $x, y \in V(G) \setminus V(C_n)$. Also if $v_i \in V(C_n)$ and $r(v_i|B) = (r_1, r_2)$, then $r_1 + r_2 \leq 2m - 1$, while for every $u \in V(G) \setminus V(C_n)$ the summation of entries of $r(u|B)$ is at least $2m + 1$, because $a_1, b_1 \in V(C_n)$. Hence B is a non-isolated basis of G.

For the converse let G be a unicyclic graph with $nr(G) = 2$, Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ be the unique cycle of G. By Lemma 2.2, G consists of C_n and some paths that connect to C_n by identifying one of their leaves and a vertex of C_n. By Lemma 2.3 at most one path can connect to each vertex of C_n. If n is odd, then Lemma 2.4 concludes the number of paths is at most 3 and they must connect to vertices with distance at most 2 on C_n. Hence by a relabeling vertices of C_n we can let paths are connected to some vertices in the set $\{v_n, v_1, v_2\}$. This means for odd numbers n, G belongs to $\mathcal{O}_n$. If n is even, then Lemma 2.4 implies that the number of paths is at most 2 and they must connect to vertices with distance at most 1 on C_n. Hence by a relabeling vertices of C_n we can let paths are connected to some vertices in the set $\{v_1, v_2\}$. This means G belongs to $\mathcal{E}_n$. □

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