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# ADVANCES ON A CONSTRUCTION RELATED TO THE NON-ABELIAN TENSOR SQUARE OF A GROUP

RAIMUNDO BASTOS AND CARMINE MONETTA*

ABSTRACT. This is a survey on a group construction in connection with the non-abelian tensor square of groups. We report on the developments obtained in the last decade emphasizing the results from a commutator point of view.

*Dedicated to Noraí Romeu Rocco on the occasion of his 70th birthday*

## 1. Introduction

The non-abelian tensor square $G \otimes G$ of a group $G$ is defined to be the group generated by all symbols $g \otimes h$, with $g, h \in G$, subject to the relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1, h, h_1 \in G$. Here, if $x$ and $y$ are group-elements, we denote by $x^y = y^{-1}xy$ the conjugate of $x$ by $y$, while the commutator of $x$ and $y$ is the element $[x, y] = x^{-1}x^y$.

Nevertheless, from an historical point of view, the non-abelian tensor square of a group, and more generally the non-abelian tensor product of groups, appeared for the first time in the realm of algebraic topology (cfr. [14, 16, 29, 30]). For instance, in [14], Brown and Loday showed that the third homotopy group of the suspension of an Eilenberg-MacLane space $K(G, 1)$ (i.e., a topological space with a single

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*Corresponding author.
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nontrivial homotopy group, $\pi_1(K(G,1)) \cong G$) satisfies

$$\pi_3(SK(G,1)) \cong \mu(G),$$

where $\mu(G)$ denotes the kernel of the derived map $\kappa : G \otimes G \to G'$, given by $g \otimes h \mapsto [g,h]$.

Thirty years ago, inspired by the works of Sidki [45] and Brown and Loday [14], Rocco published a paper [39] in which he presented a new construction related to the non-abelian tensor square of a group, and as a result a number of tensor square's properties have been investigated (see also [23] where a similar construction was considered).

Let $G$ be a group and $\varphi : G \to G^{\varphi}$ an isomorphism ($G^{\varphi}$ is an isomorphic copy of $G$, where $g \mapsto g^{\varphi}$, for all $g \in G$). Define the group $\nu(G)$ to be

$$\nu(G) := \langle G \cup G^{\varphi} \mid [g_1, {g_2}^{\varphi}]^{g_3} = [{g_1}^{g_3}, ({g_2}^{g_3})^{\varphi}] = [g_1, {g_2}^{\varphi}]^{{g_3}^{\varphi}}, \ \ g_i \in G\rangle.$$

The relationship between the group $\nu(G)$ and the tensor square $G \otimes G$ is revealed by the commutator connection: the map $\Phi : G \otimes G \to [G, G^{\varphi}]$, defined by $g \otimes h \mapsto [g, h^{\varphi}]$, for all $g, h \in G$, is an isomorphism [39, Proposition 2.6]. Therefore, from now on we identify the non-abelian tensor square $G \otimes G$ with the subgroup $[G, G^{\varphi}]$ of $\nu(G)$, and our notation will follow the one used in [39, 40].

In this survey paper we collect improvements about the group $\nu(G)$ which in turn provide development in the study of the group $G \otimes G$, underlying the fact that the group $\nu(G)$ is reasonably close to non-abelian tensor square, which motivated its construction. The research line and the problem we addressed have been mainly inspired by the work of Kappe, Moravec, Rocco and Shumyatsky (cfr. [18, 27, 32, 44]). Indeed, we will present results concerning the structure of the group $\nu(G)$ and its derived and lower central series, exponent problems and finiteness conditions.

It is worth mentioning that the progress about this topic obtained from 1987 up to 2012 has already been collected in some survey papers. The first one is due to Kappe [27], who reported on the tensor product from 1987 up to 1997, especially underlining the commutator connection. In 2005, Morse [34] discussed about how to compute the non-abelian tensor square of a group given by a polycyclic presentation. Finally in 2012 both Nakaoka and Rocco in [35], and Blyth, Fumagalli and Morigi in [12] illustrated developments about the non-abelian tensor square and related constructions. We also mention a recent survey on exponent problems of the Schur multiplier of a group [46], where some bounds for the exponent of the non-abelian tensor square of a group are provided.

## 2. New insight into the structure of $\nu(G)'$

The commutator connection between the groups $\nu(G)$ and $G \otimes G$ drives to the study of the commutator subgroups of the group $\nu(G)$. As a consequence, an efficient description of the derived and lower central series of the group $\nu(G)$ can be convenient.

The terms of the derived and lower central series of $\nu(G)$ can be expressed as semidirect product of subgroups of $\nu(G)$. This description allows to easily relate both the derived length and the nilpotency class of a group $G$ to that of the group $\nu(G)$, giving information also about $[G, G^{\varphi}]$. We summarize the aforementioned in the following.

**Theorem 2.1.** [15, Proposition 2.7], [39, Theorem A] *Let $G$ be a group.*

(i) $\nu(G)^{(k)} = ([G^{(k-1)}, (G^{\varphi})^{(k-1)}] \cdot G^{(k)}) \cdot (G^{\varphi})^{(k)}$.
(ii) *If $G$ is nilpotent of class $c$, then $\nu(G)$ is nilpotent of class at most $c+1$.*
(iii) $\gamma_{k+1}(\nu(G)) = ([\gamma_k(G), G^{\varphi}] \cdot \gamma_{k+1}(G)) \cdot \gamma_{k+1}(G^{\varphi})$.
(iv) *If $G$ is solvable of derived length $d$, then $\nu(G)$ is solvable of derived length at most $d+1$.*

**Question 2.2.** *Classify all nilpotent (soluble) groups $G$ such that $\nu(G)$ and $G$ has the same nilpotency class (derived length).*

Specializing item $(i)$ of Theorem 2.1 for $k = 1$, we have $\nu(G)' = ([G, (G^{\varphi})] \cdot G') \cdot (G')^{\varphi}$, which reveals how closely related $\nu(G)'$ and $[G, G^{\varphi}]$ are.

In a recent work [4], it has been showed that the group $\nu(G)$ contains at least other two copies of $G \otimes G$, which play a decisive role into the description of the structure of the subgroup $\nu(G)'$. Therefore, it seems convenient to denote by $\Upsilon_1(G) = [G, G^{\varphi}]$.

Let $\rho : \nu(G) \to G$ be the epimorphism defined by $\rho(g) = g = \rho(g^{\varphi})$ for any $g \in G$, and denote by $\Theta(G)$ the kernel of this map. Then, set $\Upsilon_2(G) = [\Theta(G), G]$ and $\Upsilon_3(G) = [\Theta(G), G^{\varphi}]$. We have the following.

**Theorem 2.3.** [4] *Let $G$ be a group. Then*

(a) *the subgroups $\Upsilon_2(G), \Upsilon_3(G)$ are both isomorphic to $G \otimes G$;*
(b) *the derived subgroup $\nu(G)'$ is a central product of the subgroups $\Upsilon_1(G)$, $\Upsilon_2(G)$ and $\Upsilon_3(G)$.*

*Moreover, the group $\nu(G)'$ is isomorphic to $G' \times G' \times G'$ modulo $\mu(G)$.*

We point out that the proof of item $(a)$ of Theorem 2.3 uses the definition of biderivation (or crossed pairing). Recall that if $G$ and $L$ are groups, a biderivation (or crossed pairing) from $G \times G$ to $L$ is a function $f : G \times G \to L$ such that

(i) $f(aa_1, b) = f(a^{a_1}, b^{a_1}) f(a_1, b)$
(ii) $f(a, bb_1) = f(a, b_1) f(a^{b_1}, b^{b_1})$

for all $a, a_1, b, b_1 \in G$. The biderivation $f$ lifts to a homomorphism $f^* : G \otimes G \to L$ such that $f^*(g \otimes h) = f(g, h)$ for all $g, h \in G$. This means that in order to use biderivation to compute the non-abelian tensor square of a group one has to conjecture both $L$ and $f$. For some classes of groups satisfying suitable commutator identities it can be applied easily, like for abelian groups, nilpotent groups of class 2, or 2-Engel groups (cfr. [27]). However, in the general case it could be a difficult task, especially because there is no natural candidate for $L$. Therefore we point out that in [4] the biderivation $f$ arose quite naturally from the defining relations of the group $\nu(G)$.

**Question 2.4.** *Does $\nu(G)$ contain other copies of $G \otimes G$ distinct from $\Upsilon_i(G)$ for $i = 1, 2, 3$?*

The new description in Theorem 2.3 gives rise to focus on the properties of the derived subgroup of $\nu(G)$ seeking for that of $G \otimes G$.

## 3. Exponent results

A natural question arising in the study of the groups $G \otimes G$ and $\nu(G)$ is to determine the properties of these groups looking at the ones of the $p$-group $G$. Among all, one can look at the exponent of these groups. Since $\nu(G)$ is isomorphic to an extension of $G \otimes G$ by $G \times G$, it follows that $\exp(\nu(G))$ divides $\exp(G \otimes G) \cdot \exp(G)$. More precisely,

$$\exp(G \otimes G) \mid \exp(\nu(G)) \mid \exp(G \otimes G) \cdot \exp(G).$$

In 2017, de Melo and Rocco suggested to better understand the range of values of $\exp(\nu(G))$ when $G$ is a finite $p$-group. As a consequence, a number of results have been obtained (cfr. [2, 3]). Actually, this kind of question were already investigated by various authors, especially for the computation of the exponent of the group $G \otimes G$ and some of its sections (cfr. [21, 31, 32, 41, 46]). The work of Ellis [21], Lubotzky and Mann [28], and Moravec [32, 33] mostly influenced this research line. A common idea applied in this context deals with the study of the canonical generators' order to estimate the order of either $G \otimes G$ or of its sections. At the same time, of great help is the analysis of power-commutator condition of a group $G$, like powerful, potent, nilpotency, regularity, inherited by the groups $G \otimes G$ and $\nu(G)$.

Let $p$ be a prime number. A finite $p$-group $G$ is said to be *powerful* if $p > 2$ and $G' \leqslant G^p$, or $p = 2$ and $G' \leqslant G^4$. A class of $p$-groups containing powerful $p$-groups is that of *potent* $p$-groups. A finite $p$-group is said to be *potent* if $p > 2$ and $\gamma_{p-1}(G) \leqslant G^p$, or $p = 2$ and $G' \leqslant G^4$. These definitions give rise to embedding properties. Recall that a subgroup $N$ of $G$ is potently embedded in $G$ if $[N, G] \leqslant N^4$ for $p = 2$, or $[N,_{p-2} G] \leqslant N^p$ for $p$ odd ($N$ is powerfully embedded in $G$ if $[N, G] \leqslant N^4$ for $p = 2$, or $[N, G] \leqslant N^p$ for $p$ odd). More information on finite powerful and potent $p$-groups can be found in [19] and [26], respectively.

In [32], P. Moravec proved that if $G$ is a powerful $p$-group, then the non-abelian tensor square $[G, G^{\varphi}]$ and the derived subgroup $\nu(G)'$ are powerfully embedded in $\nu(G)$. Moreover, the exponent $\exp(\nu(G)')$ divides $\exp(G)$. In [3] these results have been extended to potent $p$-groups.

**Theorem 3.1.** [3] *Let $p$ be a prime and $G$ a finite potent $p$-group.*

(a) *The non-abelian tensor square $[G, G^{\varphi}]$ is potently embedded in $\nu(G)$;*
(b) *If $k \geqslant 2$, then the $k$-th term of the lower central series $\gamma_k(\nu(G))$ is potently embedded in $\nu(G)$.*

**Theorem 3.2.** [3] *Let $p$ be a prime and $G$ a $p$-group with finite exponent.*

(a) *If $G$ is potent, then $\exp(\nu(G))$ divides $p \cdot \exp(G)$;*
(b) *If $\gamma_{p-2}(G) \leq G^p$, then $\nu(G)$ is a potent $p$-group. In particular, $\exp(\nu(G)) = \exp(G)$.*

Going further, in [2] the authors obtain some asymptotic results concerning the exponent of the group $\nu(G)$. A crucial tool in the proofs is provided by the main result of Fernández-Alcober, González-Sánches and Jaikin-Zapirain in [24].

The following theorems show that some power-commutator condition satisfied by $G$ are preserved by the group $\nu(G)$. In particular this enables to bound the exponent of the lower central terms $\gamma_{m+1}(\nu(G))$ in terms of the exponent of the subgroups $\gamma_m(G)$.

**Theorem 3.3.** [2] *Let $p$ be a prime and $G$ a $p$-group. Let $m$ and $s$ be positive integers such that $m \geq s$ and suppose that $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$. Then*

(1) *$\gamma_{i+s+1}(\nu(G)) = \gamma_{i+1}(\nu(G))^p$ for $i > m$;*
(2) *if $p$ is odd, then $\exp(\gamma_{m+1}(\nu(G)))$ divides $\exp(\gamma_m(G))$;*
(3) *if $p = 2$ and $\gamma_m(G)$ is powerful, then $\exp(\gamma_{m+1}(\nu(G)))$ divides $\exp(\gamma_m(G))$.*

Next, a bound for the exponent of $\nu(G)$ is provided when $G$ is a $p$-group of maximal class, that is a group of order $p^{c+1}$ and nilpotency class $c$. In the following, $\mathbf{p}$ denotes the prime $p$ if $p$ is odd, and 4 if $p = 2$.

**Corollary 3.4.** [2] *Let $p$ be a prime and $G$ a $p$-group of maximal class. Then $\exp(\nu(G))$ divides $\mathbf{p}^2 \cdot \exp(G)$.*

In some cases it is possible to bound the exponent of $\nu(G)$ by $\exp(G)^3$. For instance, one case occurs when $p$ is an odd prime, $G$ a $p$-group and $\exp(M(G))$ divides $\exp(G)^2$, where $M(G)$ is the Schur Multiplier of $G$. Indeed, by [12, Corollary 1.4], $\mu(G)$ is isomorphic to $M(G) \times \Delta(G)$, where $\Delta(G)$ is the subgroup generated by $g \otimes g$ for $g \in G$ and $exp(\Delta(G) = \exp(G)$. Moreover, $\nu(G)/\mu(G)$ is isomorphic to a subgroup of $G \times G \times G$, and the bound follows. Hence we may ask the following.

**Question 3.5.** *Let $G$ be a finite $p$-group, for $p$ an odd prime. Does $\exp(\nu(G))$ divides $\exp(G)^3$?*

In [46], Thomas conjectured that $\exp(M(G))$ divides $p \cdot \exp(G)$. Arguing as in the previous paragraph, the next bound seems more realistc.

**Question 3.6.** *Let $p \geqslant 3$. Let $G$ be a finite $p$-group. Does $\exp(\nu(G))$ divides $p \cdot \exp(G)^2$?*

**Remark 3.7.** *At the moment, the computation of $\nu(G)$ for finite $p$-groups always yields to a group whose exponent does not exceed $\exp(G)^2$. In any case, it seems natural to seek for some uniform bound for the $\exp(\nu(G))$ which only depends on $\exp(G)$ (i.e., does not depends on the nilpotency class $c$, the derived length $d$, the number of generators, ...).*

We conclude with a recent bound for the exponent of $G \otimes G$ which improves the existing ones is the following.

**Theorem 3.8.** [2] *Let $p$ be a prime and $G$ a $p$-group of nilpotency class $c$. Let $n = \lceil \log_p(c+1) \rceil$. Then $\exp(G \otimes G)$ divides $\exp(G)^n$.*

## 4. **Finiteness conditions**

Let $G$ be a group. We denote by $T_{\otimes}(G)$ the set of all tensors $g \otimes h$, with $g, h \in G$. By the commutator connection, in the group $\nu(G)$, the set $T_{\otimes}(G)$ coincides with the set of all $[g, h^{\varphi}]$ where $g, h \in G$. In particular, $T_{\otimes}(G)$ is a commutator closed subset of the group $\nu(G)$, where a subset $X$ of a group is called commutator closed if $[x, y] \in X$ for any $x, y \in X$.

In the next, we collect some results concerning the study of the non-abelian tensor square of a group by looking at behaviour of the set $T_{\otimes}(G)$.

4.1. **Finiteness results.** It is well-known that the set of all tensors $T_{\otimes}(G)$ affects the structure of the non-abelian tensor square $G \otimes G$. For example, combining [6, Theorem A] and [7, Theorem A] one obtains

**Theorem 4.1.** *Let $G$ be a group. If $T_{\otimes}(G)$ is finite with exactly $m$ elements, then the non-abelian tensor square $G \otimes G$ is finite with $m$-bounded order.*

As immediate consequence of the above result we deduce a quantitative version of the well-known result due to Brown, Johnson and Robertson [13, Proposition 5] (cf. [14, Proposition 3.3]).

**Corollary 4.2.** *Let $G$ be a finite group. Then the non-abelian tensor square $G \otimes G$ is finite (with $m$-bounded order).*

**Example 4.3.** *It is well know that the finiteness of the non-abelian tensor square $G \otimes G$ does not imply that $G$ is a finite group. For instance, the Prüfer group $C_{p^\infty}$ is an example of an infinite group such that $T_{\otimes}(C_{p^\infty}) = \{0\} = [C_{p^\infty}, (C_{p^\infty})^{\varphi}]$.*

Many authors have studied bounds on the order of $\pi_3(SK(G,1))$ (cf. [1, 11, 13, 37]). We deduce a finiteness criterion for $\pi_3(SX)$ in terms of $\pi_2(X)$ and the number of tensors $T_{\otimes}(G)$, where $\pi_1(X) \cong G$ and $SX$ is the suspension of the space $X$ (see Remark 4.5, below).

**Theorem 4.4.** [10] *Let $X$ be a connected space and $\pi_1(X) = G$. Suppose that the set of tensors $T_{\otimes}(G)$ has exactly $m$ tensors in $\nu(G)$ and $\pi_2(X)$ is finite with $|\pi_2(X)| = a$. Then $\pi_3(SX)$ is a finite group with $\{a, m\}$-bounded order.*

**Remark 4.5.** *In the above result, it is worth noting that $\pi_2(X)$ does not need to be trivial, therefore the result is more general compared to the bounds for $\pi_3(SK(G,1))$ when $X = K(G,1)$. However, in [14] it is proved that if $\pi_1(X) = G$ and $\pi_2(X)$ is trivial, then $\pi_3(SX) \cong J_2(G) = \ker(\kappa)$, where $\kappa : [G, G^{\varphi}] \to G'$ is given by $[g, h^{\varphi}] \mapsto [g, h]$.*

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Brown and Loday [14, Proposition 4.10]) prove that in the following commutative diagram, the rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_3(SK(G,1)) & \longrightarrow & G\otimes G & \longrightarrow & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_2^S(K(G,1)) & \longrightarrow & G\tilde{\wedge}G & \longrightarrow & G' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H_2(G) & \longrightarrow & G\wedge G & \longrightarrow & G' & \longrightarrow & 1
\end{array}$$

are exact, where $SK(G,1)$ is the suspension of an Eilenberg-MacLane space $K(G,1)$, $\pi_2^S(K(G,1))$ is the second stable homotopy group and $H_2(G)$ the second homology group (cf. [27, 14, 22, 40]).

According to Theorem 4.1, we deduce that if $T_\otimes(G)$ has exactly $m$ elements, then the non-abelian tensor square $G\otimes G$ is finite with $m$-bounded order. In this subsection, we will present similar quantitative results for $G\wedge G$, $H_2(G)$, $G\tilde{\wedge}G$ and $\pi_2^S(K(G,1))$.

The *diagonal* subgroup of $[G,G^\varphi]$, denoted by $\Delta(G)$, is given by $\Delta(G)=\langle [g,g^\varphi] \,:\, g\in G\rangle$. In [20], the factor group $\nu(G)/\Delta(G)$ is denoted by $\tau(G)$, the subgroup $[G,G^\varphi]_\tau=[G,G^\varphi]/\Delta(G)$ is isomorphic with the non-abelian exterior square $G\wedge G$; we write $[g,h^\varphi]_\tau$ for the coset $[g,h^\varphi]\Delta(G)$, so that $g\wedge h$ is identified with $[g,h^\varphi]_\tau$. We denote by $\tilde{\Delta}(G)$ the subgroup $\langle [g,h^\varphi][h,g^\varphi] \,:\, g,h\in G\rangle$ and by $\tilde{\tau}(G)$ the quotient $\nu(G)/\tilde{\Delta}(G)$; the subgroup $[G,G^\varphi]_{\tilde{\tau}}=[G,G^\varphi]/\tilde{\Delta}(G)$ is isomorphic with the so called symmetric product $G\tilde{\wedge}G$; we write $[g,h^\varphi]_{\tilde{\tau}}$ for the coset $[g,h^\varphi]\tilde{\Delta}(G)$. An element $\alpha\in\tau(G)$ (resp. $\alpha\in\tilde{\tau}(G)$) is called a *exterior tensor* if $\alpha=[a,b^\varphi]_\tau$ (resp. *symmetric tensor* if $\alpha=[a,b^\varphi]_{\tilde{\tau}}$) for suitable $a,b\in G$. Write $T_\wedge(G)=\{[a,b^\varphi]_\tau \,:\, a,b\in G\}$ and $T_{\tilde{\wedge}}(G)=\{[a,b^\varphi]_{\tilde{\tau}} \,:\, a,b\in G\}$.

**Theorem 4.6.** [7] *Let $G$ be a group. Assume that $|T_\wedge(G)|=m$. Then $H_2(G)$ and $G\wedge G$ are finite with $m$-bounded order.*

**Example 4.7.** *It is straightforward to see that the finiteness of the set of exterior tensors $T_\wedge(G)$ does not imply the finiteness of the set $\{g\otimes h \,:\, g,h\in G\}$. For instance, if $G$ is the infinite metacyclic group $\langle a,b \,:\, a^k=1, [a,b]=a^{1-r}\rangle$, where $k$ is a positive integer and $r\in U(k)$, then the non-abelian exterior square $G\wedge G$ is isomorphic with $C_k$, while $\mu(G)\cong\pi_3(SK(G,1))$ is infinite and so, $[G,G^\varphi]$ is infinite (see* [11, Theorems 5.1 and 5.2]*).*

**Theorem 4.8.** [7] *Let $G$ be a group. Assume that $|T_{\tilde{\wedge}}(G)|=m$. Then $G\tilde{\wedge}G$ and $\pi_2^S(K(G,1))$ are finite with $m$-bounded order.*

4.2. **BFC-results.** Among all finiteness conditions, the property to have finite conjugacy classes of bounded size can be considered. More precisely, a group $G$ is a BFC-group if every conjugacy class of $G$ contains at most $n$ elements, for a positive integer $n$. In [36], BFC-groups have been charcterized by Neumann as groups with finite derived subgroup, which occurs if and only if the group contains only finitely many commutators. A quantitative version was provided by Wiegold: if $G$ is a group containing exactly $m$ commutators, then the order of the derived subgroup $G'$ is finite with $m$-bounded order [47, Theorem 4.7], and the best known bound was obtained in [25].

In these subsection we report on a new sufficient condition for a group to be a BFC-group in terms of conjugacy classes of tensors. If we denote by $x^G = \{x^g \mid g \in G\}$ the conjugacy class of $x$ in $G$, by the defining relations of the group $\nu(G)$, for any tensor $\alpha \in T_{\otimes}(G)$, we have

$$\alpha^{\nu(G)} = \alpha^G = \alpha^{G^{\varphi}}.$$

In [18], Dierings and Shumyatsky proved that if $\left|x^G\right| \leq n$ for every commutator $x$ in $G$, then the second derived subgroup $G''$ is finite with $n$-bounded order. A similar result in the context of Rocco's construction was demonstrated in [5].

**Theorem 4.9.** [5] *Let $n$ be a positive integer and $G$ a group. Suppose that the size of the conjugacy class $\left|\alpha^{\nu(G)}\right| \leq n$ for every $\alpha \in T_{\otimes}(G)$. Then the second derived subgroup $\nu(G)''$ is finite with $n$-bounded order.*

Following the work of [25, 47], the next question arises naturally:

**Question 4.10.** *Find a function $f : \mathbb{N} \to \mathbb{N}$ such that if $G$ is a group and $n \in \mathbb{N}$ such that $\left|x^{\nu(G)}\right| \leq n$ for every $x \in T_{\otimes}(G)$, then $|\nu(G)''| \leq f(n)$.*

Sometimes it is possible to obtain information about a group looking at its non-abelian tensor square. In this direction, Theorem 4.9 leads to a sufficient condition for a group $G$ to be a BFC-group in terms of tensors' conjugacy classes.

**Corollary 4.11.** [5] *Let $n$ be a positive integer. Let $G$ be a group in which the derived subgroup $G'$ is finitely generated. Assume that the size of the conjugacy class $|\alpha^{\nu(G)}| \leq n$ for every $\alpha \in T_{\otimes}(G)$. Then $G$ is a BFC-group.*

Notice that, the previous corollary is no longer true if either $G'$ is not finitely generated, or the hypothesis "$|\alpha^{\nu(G)}| \leq n$ for every $\alpha \in T_{\otimes}(G)$" is replaced by "$|x^G| \leq n$ for every $x$ commutator of $G$" (cfr. [5, Examples 3.7-3.8]).

4.3. **Non-abelian tensor square of residually finite groups.** According to the Zel'manov's solution of the Restricted Burnside Problem [50, 51] every residually finite group of finite exponent is locally finite. Recall that a group is said to be residually finite if the intersection of all its subgroups of finite index is trivial, while a group is locally finite (nilpotent) if every finitely generated subgroup is finite (nilpotent). Another interesting result in this context, due to Shumyatsky [44] states that if $G$ is a residually finite group satisfying a non-trivial identity and generated by a normal commutator-closed set of $p$-elements, then $G$ is locally finite. The next results were obtained in [8].

**Theorem 4.12.** [8] *Let $p$ a prime. Let $m$ be a positive integer and $G$ a residually finite group satisfying some non-trivial identity $f \equiv 1$. Suppose that for every $x, y \in G$ there exists a positive* *integer $q = q(x, y)$ dividing $p^m$ such that the tensor $[x, y^{\varphi}]^q = 1$. Then the derived subgroup $\nu(G)'$ is locally finite. In particular, the non-abelian tensor square $[G, G^{\varphi}]$ is locally finite.*

**Question 4.13.** *Let $m$ be a positive integer and $G$ a residually finite group. Assume that $[x, y^{\varphi}]^m = 1$ for every $x, y \in G$. Is then the non-abelian tensor square $[G, G^{\varphi}]$ necessarily locally finite?*

The previous question is related to Shumyatsky's problem [43, Problem 2].

Let $G$ be a group. For elements $x, y$ of $G$ we define $[x, {}_1y] = [x, y]$ and $[x, {}_{i+1}y] = [[x, {}_iy], y]$ for $i \geq 1$. An element $y \in G$ is called *left $n$-Engel* if for any $x \in G$ we have $[x, {}_ny] = 1$. The group $G$ is called a *left $n$-Engel group* if $[x, {}_ny] = 1$ for all $x, y \in G$.

Another consequence of the solution of the Restricted Burnside Problem is that any residually finite $n$-Engel group is locally nilpotent (J. S. Wilson, [48]). Later Shumyatsky proved that if $G$ is a residually finite group in which every commutator $[x, y]$ is left $n$-Engel, then the derived subgroup $G'$ is locally nilpotent [42, 43]. We recall that a group $G$ is locally virtually nilpotent if every finitely generated subgroup of $G$ has a normal nilpotent subgroup of finite index.

In the present paper we establish the following related result.

**Theorem 4.14.** [8] *Let $m, n$ be positive integers and $p$ a prime. Suppose that $G$ is a residually finite group in which for every $x, y \in G$ there exists a positive integer $q = q(x, y)$ dividing $p^m$ such that the element $[x, y^{\varphi}]^q$ is left $n$-Engel in $\nu(G)$. Then the non-abelian tensor square $G \otimes G$ is locally virtually nilpotent.*

A natural question arising in the context of Theorem 4.14 is whether the theorem remains valid when $q$ is allowed to be an arbitrary natural number, rather than a $p$-power.


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**Raimundo Bastos**

Departamento de Matemática, Universidade de Brasília, Brasilia-DF Brazil

Email: bastos@mat.unb.br

**Carmine Monetta**

Dipartimento di Matematica, Università di Salerno, Salerno, Italy

Email: cmonetta@unisa.it

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