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SPECTRAL PROPERTIES OF THE NON–PERMUTABILITY GRAPH OF SUBGROUPS

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Abstract. Given a finite group G and the subgroups lattice $\mathrm{L}(G)$ of G, the *non–permutability graph of subgroups* $\Gamma_{\mathrm{L}(G)}$ is introduced as the graph with vertices in $\mathrm{L}(G) \setminus \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$, where $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$ is the smallest sublattice of $\mathrm{L}(G)$ containing all permutable subgroups of G, and edges obtained by joining two vertices X, Y if $XY \neq YX$. Here we study the behaviour of the non-permutability graph of subgroups using algebraic properties of associated matrices such as the adjacency and the Laplacian matrix. Further, we study the structure of some classes of groups whose non-permutability graph is strongly regular.

1. **Introduction and statement of the main results**

We consider only finite groups in the present paper. Bianchi and others [2] introduced the permutability graph of non–normal subgroups $\Gamma_N(G)$ in 1995; all proper non–normal subgroups of G form the vertex set of $\Gamma_N(G)$ and two vertices H and K are joined if $HK = KH$. Rajkumar and Devi [10] generalized $\Gamma_N(G)$ to the permutability graph of subgroups $\Gamma(G)$, extending the vertex set to all proper subgroups of G and keeping the same criterion to join two vertices.

The references [2, 10, 11, 12, 20, 21] deal with graphs with similar properties, discussing the *non-permutability graph of subgroups* of G

$$\Gamma_{\mathrm{L}(G)} = (V(\Gamma_{\mathrm{L}(G)}), E(\Gamma_{\mathrm{L}(G)})) \tag{1.1}$$

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which is an undirected unweighted simple graph having vertices and edges

$$V(\Gamma_{\mathrm{L}(G)}) = \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big); \tag{1.2}$$

$$E(\Gamma_{\mathrm{L}(G)}) = \{(X,Y) \in V(\Gamma_{\mathrm{L}(G)}) \times V(\Gamma_{\mathrm{L}(G)}) \mid X \sim Y \iff XY \neq YX\}, \tag{1.3}$$

respectively. Here, if X is an arbitrary element of $\mathrm{L}(G)$, then

$$\mathfrak{C}_{\mathrm{L}(G)}(X) = \{Y \in \mathrm{L}(G) \mid XY = YX\} \tag{1.4}$$

is the set of all subgroups of $\mathrm{L}(G)$ commuting with X and the intersection

$$\bigcap_{X \in \mathrm{L}(G)} \mathfrak{C}_{\mathrm{L}(G)}(X) = \{Y \in \mathrm{L}(G) \mid YX = XY, \quad \forall X \in \mathrm{L}(G)\} \tag{1.5}$$

is not a sublattice of $\mathrm{L}(G)$, so we consider $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$, which is the smallest sublattice of $\mathrm{L}(G)$ containing (1.5).

It is useful to understand the behaviour of the non-permutability graph of subgroups using algebraic properties of associated matrices such as the adjacency and the Laplacian matrix. This approach is begun in the present paper for the first time.

We focus our attention on the adjacency matrix of $\Gamma_{\mathrm{L}(G)}$, that is, on the square matrix

$$A(\Gamma_{\mathrm{L}(G)}) = (a_{X,Y})_{X,Y \in V(\Gamma_{\mathrm{L}(G)})} \tag{1.6}$$

where

$$a_{X,Y} = \begin{cases} 1, & \text{if } (X,Y) \in E(\Gamma_{\mathrm{L}(G)}), \\ 0, & \text{if } (X,Y) \notin E(\Gamma_{\mathrm{L}(G)}). \end{cases}$$

Note that the degree of a vertex X in (1.1) is defined as

$$\deg(X) = \sum_{Y \in V(\Gamma_{\mathrm{L}(G)})} a_{X,Y}. \tag{1.7}$$

In particular, we say that $\Gamma_{\mathrm{L}(G)}$ is regular of degree k, or k-regular, if $\deg(X) = k$, for all $X \in V(\Gamma_{\mathrm{L}(G)})$. Up to isomorphisms, we recall from [1] that the cycle graph C_n and the complete graph K_n are the unique regular connected graphs of degree 2, and $n-1$ respectively.

Since (1.1) is undirected, (1.6) is symmetric and nonnegative, therefore all its eigenvalues are real and the spectral radius is defined as the largest of such eigenvalues. The characteristic polynomial of (1.1) is

$$p_{A(\Gamma_{\mathrm{L}(G)})}(\lambda) = \det(\lambda I_n - A(\Gamma_{\mathrm{L}(G)})), \tag{1.8}$$

where I_n denotes the usual n by n square identity matrix. The (adjacency) spectrum of (1.1) is defined as usual, that is, as

$$\mathrm{spec}(\Gamma_{\mathrm{L}(G)}) = \{\lambda_1, \ldots, \lambda_n \mid \forall i = 1, 2, \ldots, n\} \tag{1.9}$$

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and it contains the eigenvalues of (1.6) together with their multiplicities. That is, $\mathrm{spec}(\Gamma_{\mathrm{L}(G)})$ equals all the roots of the polynomial in (1.8). For basic results on the graph spectra and other graph matrices, we invite the reader to see [7, 8].

Since $\Gamma_{\mathrm{L}(G)}$ is an undirected graph without loops, the Laplace matrix of $\Gamma_{\mathrm{L}(G)}$ is the matrix

$$L(\Gamma_{\mathrm{L}(G)}) = D - A(\Gamma_{\mathrm{L}(G)}), \quad \text{where} \quad D = \mathrm{diag}(\deg(X_i)) \tag{1.10}$$

for all $X_i \in V(\Gamma_{\mathrm{L}(G)})$ and $i = 1, 2, \ldots, n = |V(\Gamma_{\mathrm{L}(G)})|$. The signless Laplace matrix of $\Gamma_{\mathrm{L}(G)}$ is the matrix

$$Q(\Gamma_{\mathrm{L}(G)}) = D + A(\Gamma_{\mathrm{L}(G)}) \tag{1.11}$$

as well known by classical references of spectral graph theory, see [6]. Note that the multiplicity of 0 as a Laplacian eigenvalue of an undirected graph $\Gamma_{\mathrm{L}(G)}$ equals the number of connected components of $\Gamma_{\mathrm{L}(G)}$. Let $\Gamma_{\mathrm{L}(G)}$ be regular of valency k, that is, every vertex of $\Gamma_{\mathrm{L}(G)}$ has the same degree and this degree is equal to k. It turns out that in this situation, k is the largest eigenvalue of $\Gamma_{\mathrm{L}(G)}$, and its multiplicity equals the number of connected components of $\Gamma_{\mathrm{L}(G)}$ (see [6, Proposition 1.3.8]).

As $\Gamma_{\mathrm{L}(G)}$ is undirected and without loops, $\Gamma_{\mathrm{L}(G)}$ is bipartite if and only if its Laplacian spectrum and its signless Laplacian spectrum coincide. This is a wel known result in spectral graph theory, available for instance in [6, Proposition 1.3.10]. If $\Gamma_{\mathrm{L}(G)}$ is in addition a complete graph with n vertices, that is, every pair of distinct vertices is connected by a unique edge, and there are exactly n vertices in $\Gamma_{\mathrm{L}(G)}$, then we use the classical notation in [4, 3, 6] denoting $\Gamma_{\mathrm{L}(G)}$ by K_n. In this situation,

$$A(\Gamma_{\mathrm{L}(G)}) = A(K_n) = J - I_n, \quad \mathrm{spec}(\Gamma_{\mathrm{L}(G)}) = \mathrm{spec}(K_n) = \{n-1, (-1)^{n-1}\}, \tag{1.12}$$

$$L(K_n) = L(\Gamma_{\mathrm{L}(G)}) = Q(\Gamma_{\mathrm{L}(G)}) = nI_n - J, \quad \mathrm{spec}(L(\Gamma_{\mathrm{L}(G)})) = \mathrm{spec}(L(K_n)) = \{0, n^{n-1}\},$$

where J denote the all-1 matrix.

From [6, Page 115] a simple undirected graph (without loops) with n vertices is *strongly regular* with parameters n, k, λ, μ whenever it is not complete or edgeless and the following three conditions are satisfied: each vertex is adjacent to k vertices; for each pair of adjacent vertices there are λ vertices adjacent to both; for each pair of nonadjacent vertices there are μ vertices adjacent to both.

We determine n such that $\Gamma_{\mathrm{L}(D_{2n})}$ is strongly regular, when D_{2n} is the dihedral group of order $2n$. This is our first main result.

Theorem 1.1. *Let $n \geq 3$ be an integer. Then $\Gamma_{\mathrm{L}(D_{2n})}$ is strongly regular if and only if $\frac{n}{2}$ is a prime number.*

Our second result generalizes Theorem 1.1 to an arbitrary group whose non–permutability graph is cocktail party. Note that a cocktail party graph of order n, also called *hyperoctahedral graph*, is the graph consisting of two rows of paired nodes in which all nodes but the paired ones are connected with a graph edge. More information can be found in [1, 6] for these graphs.

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Theorem 1.2. *Assume that G possesses a homomorphic image which is a dihedral group. Then there is a normal subgroup H of G such that $\Gamma_{\mathrm{L}(G/H)}$ is cocktail party if and only $G/H \simeq D_8$.*

Next we find applications for the *subgroup commutativity degree* of G, which is the ratio

$$\mathrm{sd}(G) = \frac{|\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|}{|\mathrm{L}(G)|^2} \tag{1.13}$$

studied in [23, 28]. Our third main result connects (1.13) with (1.9) and (1.10).

Theorem 1.3. *Let G be a non quasi-hamiltonian group. Then the subgroup commutativity degree of G is invariant under the spectrum of the adjacency and Laplacian matrices of $\Gamma_{\mathrm{L}(G)}$.*

The result above is the first kind of result, connecting the theory of the subgroup commutativity degree, discussed in the papers [13, 17, 19, 22, 24, 25, 26, 28] with the classical spectral graph theory, presented in [6, 7, 8, 9, 18, 29]. Finally we find (1.6) for dihedral groups.

Theorem 1.4. *Let $n = 2^\alpha$ with $\alpha \geq 3$ and $\sigma(2^\alpha)$ be the sum of all divisors of 2^α. Then*

$$A(\Gamma_{\mathrm{L}(D_{2^\alpha})}) = \begin{pmatrix} A(\Gamma_{\mathrm{L}(D_{2^{\alpha-1}})} \cup 2K_1) & J \\ J & A(\Gamma_{\mathrm{L}(D_{2^{\alpha-1}})} \cup 2K_1) \end{pmatrix}$$

is a square matrix of size $(\sigma(2^\alpha) - 3) \times (\sigma(2^\alpha) - 3)$, where

$$A(\Gamma_{\mathrm{L}(D_{2^3})} \cup 2K_1) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(\Gamma_{\mathrm{L}(D_{2^3})}) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and J is all-1 matrix of size $(2^\alpha - 2) \times (2^\alpha - 2)$.

After listing the main results here in Section 1, we recall some facts from the literature in Section 2 before presenting the proofs of the main results in Section 3. Applications are listed in Section 4 along with additional observations, which we made during our investigations. Concerning group theory, we follow terminology and notations of [16, 27], while we follow [3, 4, 6, 7, 9, 15] concerning graph theory and related notions of algebraic combinatorics.

2. Preliminary notions

There are several authors who devoted their investigations to spectral graph theory, but it was probably Biggs who focused more on the algebraic aspects of the spectral graph theory in the classical reference [3]. It is also useful to look at the contributions of Yin [29], Brouwer and Haemers [6], Jog and Kotambari [14], in order to observe some recent interactions between algebraic combinatorics and spectral graph theory. We mention these results, since our results are placed somehow in the middle

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between algebraic combinatorics, probabilistic group theory and spectral graph theory, as it will be clear in a moment.

The spectrum of strongly regular and cocktail party graphs are well studied (see [6]). In particular, the dihedral group D_{2n} (for an integer $n \geq 3$) is going to be explored in connection with the notion of cocktail party graph. We should mention that strongly regular graphs were introduced by Bose in 1963 (see [5]). Some elementary facts are recalled below.

Lemma 2.1. [6, Page 124] *The complement of a strongly regular graph with parameters n, k, λ, μ is again strongly regular, and has parameters $(n, n-k-1, v-2k+\mu, n-2k+\lambda)$. Moreover a cocktail party graph is obtained by removing n disjoint edges from K_{2n} and the corresponding spectrum is $\{2n-2, 0, (-2)^{n-1}\}$.*

The non–permutability graph of subgroups of the dihedral groups has been studied in [20, Section 4]. Let us begin to determine n such that $\Gamma_{\mathrm{L}(D_{2n})}$ is regular:

Lemma 2.2. *Let $n \geq 3$ be an integer. Then $\Gamma_{\mathrm{L}(D_{2n})}$ is regular if and only if either n is an odd prime, or n is of the form $n = 2p$ for some prime $p \geq 2$.*

Proof. The following Cases 1, 2 and 3 show that, if n is odd prime, or if n is of the form $n = 2p$ for some prime p, then $\Gamma_{\mathrm{L}(D_{2n})}$ is a regular graph.

Case 1. Assume that $n = 2p$ with p odd prime.

The structure of the lattice of subgroups $\mathrm{L}(D_{4p})$ of D_{4p} is well known (see [16, 27], or more specifically [20, Proofs of Lemmas 4.1 and 4.2]). In particular, one can see that $\mathrm{L}(D_{4p})$ contains p nonatomic nonnormal subgroups (the notion of *atomic subgroup* in a lattice of subgroups is well known, see [27]). In addition, we note that the triples of nonatomic nonnormal subgroups do not commute with other nonnormal subgroups of D_{4p}. This situation is described in [20, Proofs of Theorem 4.3 and 4.8]. Hence $\Gamma_{\mathrm{L}(D_{4p})}$ is a regular graph.

Case 2. Assume that $n = p$ with p odd prime.

From [20, Proofs of Theorem 4.3 and 4.8], $\mathrm{L}(D_{2p})$ contains exactly p atomic nonnormal subgroups which don't commute mutually. This means $\Gamma_{\mathrm{L}(D_{2p})}$ is regular graph.

Case 3. Assume that $n = 4$ with $p = 2$.

It is easy to see that $\mathrm{L}(D_{2n})$ contains exactly 4 atomic nonnormal subgroups. Here we have two pairs of subgroups such that the subgroups in the pair commute between themselves but don't commute with the subgroups in the other pair. This implies $\mathrm{L}(D_{2n})$ is a regular graph of order 4.

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The following Case 4 shows that, if $\Gamma_{\mathrm{L}(D_{2n})}$ is regular, then n is odd prime, or n of the form $n = 2p$ for some prime p.

Case 4. Assume that

$$n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} \tag{2.1}$$

is factorized in product of powers of distinct primes $p_j's$, $t \geq 1$ and $k_j \geq 0$ with $j = 1, \ldots, t$.

Assume that (2.1) specializes either as $n = p_1^{k_1}$ with p_1 odd and $k_1 \geq 2$, or as $n = 2^{k_1}$ with $k_1 \geq 3$. In this situation n has at least two distinct factors, say r and r', with $r, r' \neq n$ for the first case, and $r, r' \notin \{\frac{n}{2}, n\}$ for the second case. Now suppose $r < r'$. Looking at [20, Proof of Lemma 4.1, Case (iii)], there are subgroups $H_i^{r'}$ in D_{2n} containing at least two subgroups A and B such that $AB \neq BA$. Now from [20, Theorem 4.3] we note that the degree is known by

$$\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) = \sigma(n) - x_i^r, \tag{2.2}$$

where $\sigma(n)$ denotes the sum of all divisors of n and x_i^r a positive integer, defined in [20, Lemma 4.2] and depending only on H_i^r. This implies

$$\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) > \deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^{r'}), \tag{2.3}$$

showing that the regularity fails. For any other factorization of n, we may repeat the same argument, concluding that the regularity happens only when (2.1) appears with p_1 odd, $k_1 = 1$ and $k_j = 0$ for all $j \geq 2$, or when $p_1 = 2$, $k_1 = 1$, $k_2 = 0$ and $p_j = 1$ for all $j \geq 3$.

From Case 1, Case 2, Case 3 and Case 4 above, the result follows. □

Looking at the regular graph $\Gamma_{\mathrm{L}(D_{2n})}$, we find that only one case describes a complete graph.

Corollary 2.3. *For any choice of $n \geq 3$, $\Gamma_{\mathrm{L}(D_{2n})}$ is complete if and only if n is odd prime.*

Proof. The proof follows from Lemma 2.2. □

Now we come back to the notion of strongly regular graph, and a concrete situation helps to visualize the combinatorics of strongly regular and cocktail party graphs in our context of investigation.

Example 2.4. The cycle graph C_5 and the Petersen graph are strongly regular with parameters $(5, 2, 0, 1)$ and $(10, 2, 0, 1)$, respectively. From [20, Examples 2.9, 4.7], we have that $\Gamma_{\mathrm{L}(D_8)} \simeq C_4$ and $\Gamma_{\mathrm{L}(D_6 \times \mathbb{Z}_2)}$ are both cocktail party and strongly regular graphs with parameters $(4, 2, 0, 2)$ and $(6, 4, 3, 4)$, respectively. On the other hand, $\Gamma_{\mathrm{L}(D_6 \times \mathbb{Z}_9)}$ is strongly regular with parameter $(9, 6, 3, 6)$ but not a cocktail party graph.

In order to count the number of edges of the non-permutability graph of subgroups of a group G, it was found in [20, Theorem 3.1] a numerical relation with the probability of commuting subgroups in G. Of course, the knowledge of the number of edges, as well as the degrees, is fundamental for most of the considerations which we are going to do in the present paper. Therefore we recall some basic facts of probabilistic group theory here.

Given two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$ in a group G, the *generalized subgroup commutativity degree* of G is defined by

$$gsd(G) = \frac{|\{(H,K) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid HK = KH\}|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \tag{2.4}$$

has been studied in [19]. If $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$, then $gsd(G) = sd(G)$ is exactly the subgroup commutativity degree. We can connect the theory of the subgroup commutativity degree to that of the non-permutability graph of subgroups via the following counting formula:

Lemma 2.5. *For a group G we have*

$$2\ |E(\Gamma_{\mathrm{L}(G)})| = |\mathrm{L}(G)|^2\ (1 - \mathrm{sd}(G)).$$

Proof. The proof follows from [20, Lemma 2.10] when $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$. □

Another quantity associated to a finite group G is the *factorization number*

$$F_2(G) = |\{(H,K) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid G = HK\}|, \tag{2.5}$$

which denotes the number of possible factorizations of G in the product of two subgroups H and K. There is a strong connection between $sd(G)$ and $F_2(G)$, due to Farrokhi and others [13, 17, 25, 30],

$$sd(G) = \frac{1}{|\mathrm{L}(G)|^2} \sum_{H \in \mathrm{L}(G)} F_2(H). \tag{2.6}$$

This alternative approach has led to important numerical evalutation for $sd(G)$, since $F_2(H)$ may be expressed for several linear groups via Gaussian trinomial integers.

Corollary 2.6. *For a group G we have*

$$2\ |E(\Gamma_{\mathrm{L}(G)})| = |\mathrm{L}(G)|^2\ -\ \sum_{H \in \mathrm{L}(G)} F_2(H)$$

Its proof is of course a consequence of Lemma 2.5.

3. The Proof of the Main results

Now we are ready to prove our first main result.

Proof of Theorem 1.1. If $p = 2$ or $n = 2p$ with p odd prime, then it is easy to check that $\Gamma_{\mathrm{L}(D_{2n})}$ is strongly regular graph with parameters $(4, 2, 0, 2)$ and $(3p, 3p-3, 3p-6, 3p-3)$. This concludes a first implication in the theorem.

Conversely, look at [20, Proof of Lemma 4.1, Cases (i), (ii) and (iii)] and remember that there are subgroups H_i in D_{2n} forming the vertices of the non-permutability graph of subgroups of D_{2n}. The

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description of these subgroups is made with details in [20, Lemma 4.1]. Recall that the neighborhoods of the vertices of H_i are

$$\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i) = \{H_j \in \mathrm{L}(D_{2n}) | (H_i, H_j) \in E(\Gamma_{\mathrm{L}(D_{2n})})\}, \tag{3.1}$$

and note that $\Gamma_{\mathrm{L}(D_{2n})}$ is strongly regular by assumption with parameters (t, k, λ, μ) in the sense of Lemma 2.1, where $t = V(\Gamma_{\mathrm{L}(D_{2n})})$, $k = \deg(H_i)$, for $i = 1, 2, \ldots, t$, $\lambda = |\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i) \cap \mathcal{N}_{\mathrm{L}(D_{2n})}(H_j)|$ if H_i is adjacent with H_j and $\mu = |\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i) \cap \mathcal{N}_{\mathrm{L}(D_{2n})}(H_k)|$ if H_i is non-adjacent with H_k for $i \neq j$, $i \neq k$, $j \neq k$ and $i, j, k \in \{1, 2, \ldots, t\}$.

Now let us determine n such that $\Gamma_{\mathrm{L}(D_{2n})}$ is strongly regular.

First of all, note that Lemma 2.2 implies $\Gamma_{\mathrm{L}(D_{2n})}$ is regular if n is odd prime or $n = 2p$ for some prime p (eventually $p = 2$), but as shown in Corollary 2.3, $\Gamma_{\mathrm{L}(D_{2n})}$ is complete and regular only if n is odd prime. This means that $t = k$, which is a contradiction to the definition of strongly regularity. Therefore we should analyse only two possibilites of n.

Case 1. Assume $n = 2p$ for $p \geq 3$.

By Lemma 2.2 we find that $\Gamma_{\mathrm{L}(D_{2n})}$ is not complete and every triple non-normal commuting subgroups are always non-commuting to each other subgroups in D_{2n}, because we may repeat the argument of the proof of Case 2 in Lemma 2.2. Therefore in this situation two arbitrary adjacent vertices H_i and H_j with $i \neq j$ have complements $\mathcal{N}'_{\mathrm{L}(D_{2n})}(H_i)$, $\mathcal{N}'_{\mathrm{L}(D_{2n})}(H_j)$, of the neighbourhoods $\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i)$, $\mathcal{N}_{\mathrm{L}(D_{2n})}(H_j)$, respectively, such that

$$|\mathcal{N}'_{\mathrm{L}(D_{2n})}(H_i)| = |\mathcal{N}'_{\mathrm{L}(D_{2n})}(H_j)| = 2, \tag{3.2}$$

$$\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i) \cap \mathcal{N}_{\mathrm{L}(D_{2n})}(H_j) = V(\mathrm{L}(D_{2n})) \setminus [\mathcal{N}'_{\mathrm{L}(D_{2n})}(H_i) \cup \mathcal{N}'_{\mathrm{L}(D_{2n})}(H_j)] \neq \emptyset. \tag{3.3}$$

Similarly, the set $\mathcal{N}_{\mathrm{L}(D_{2n})}(H_i) \cap \mathcal{N}_{\mathrm{L}(D_{2n})}(H_k)$ is also non-empty for two arbitrary non-adjacent subgroups H_i and H_k for $i \neq k$. We conclude that the result is true in this case.

Case 2. Assume $n = 4$, that is, $p = 2$.

Then one can see with a direct computation that the result is true also in this case.

□

It is appropriate to note the following fact.

Remark 3.1. Given two groups G_1 and G_2, we have $\mathrm{L}(G_1 \times G_2) \simeq \mathrm{L}(G_1) \times \mathrm{L}(G_2)$ if and only if G_1 and G_2 are of coprime orders, by a well known result of Suzuki [27, Lemma 1.6.4]. This implies that $\Gamma_{\mathrm{L}(G_1 \times G_2)} \not\simeq \Gamma_{\mathrm{L}(G_1)} \times \Gamma_{\mathrm{L}(G_2)}$ a priori. In fact $\Gamma_{\mathrm{L}(D_6 \times D_6)} \not\simeq \Gamma_{\mathrm{L}(D_6)} \times \Gamma_{\mathrm{L}(D_6)}$ shows that regularity is not preserved by direct products of groups, and so, neither by homomorphic images of groups.

We recall from [4] that $\Gamma_{\mathrm{L}(G)}$ is a *bipartite graph* with vertex classes $V_1(\Gamma_{\mathrm{L}(G)})$ and $V_2(\Gamma_{\mathrm{L}(G)})$ if $V(\Gamma_{\mathrm{L}(G)}) = V_1(\Gamma_{\mathrm{L}(G)}) \cup V_2(\Gamma_{\mathrm{L}(G)})$, $V_1(\Gamma_{\mathrm{L}(G)}) \cap V_2(\Gamma_{\mathrm{L}(G)}) = \emptyset$ and each edge joins a vertex of $V_1(\Gamma_{\mathrm{L}(G)})$ to a vertex of $V_2(\Gamma_{\mathrm{L}(G)})$. Similarly $\Gamma_{\mathrm{L}(G)}$ is *p-partite* with vertex classes $V_1(\Gamma_{\mathrm{L}(G)}), V_2(\Gamma_{\mathrm{L}(G)}), \ldots, V_p(\Gamma_{\mathrm{L}(G)})$

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if $V(\Gamma_{\mathrm{L}(G)}) = V_1(\Gamma_{\mathrm{L}(G)}) \cup \cdots \cup V_p(\Gamma_{\mathrm{L}(G)})$, $V_i(\Gamma_{\mathrm{L}(G)}) \cap V_j(\Gamma_{\mathrm{L}(G)}) = \emptyset$ whenever $1 \leq i < j \leq p$, and no edge joins two vertices in the same class (see [4]). We also recall from [4] that if $\Gamma_{\mathrm{L}(G)}$ and $\Gamma_{\mathrm{L}(H)}$ are graphs, then their union is the graph $\Gamma_{\mathrm{L}(G)} \cup \Gamma_{\mathrm{L}(H)}$ having $V(\Gamma_{\mathrm{L}(G)} \cup \Gamma_{\mathrm{L}(H)}) = V(\Gamma_{\mathrm{L}(G)}) \cup V(\Gamma_{\mathrm{L}(H)})$, and $E(\Gamma_{\mathrm{L}(G)} \cup \Gamma_{\mathrm{L}(H)}) = E(\Gamma_{\mathrm{L}(G)}) \cup E(\Gamma_{\mathrm{L}(H)})$. If $V(\Gamma_{\mathrm{L}(G)}) \cap V(\Gamma_{\mathrm{L}(H)}) = \emptyset$, then $\Gamma_{\mathrm{L}(G)}$ and $\Gamma_{\mathrm{L}(H)}$ are disjoint graphs. If $\Gamma_{\mathrm{L}(G)}$ and $\Gamma_{\mathrm{L}(H)}$ are disjoint graphs, then their join is the graph $\Gamma_{\mathrm{L}(G)} + \Gamma_{\mathrm{L}(H)}$ having $V(\Gamma_{\mathrm{L}(G)} + \Gamma_{\mathrm{L}(H)}) = V(\Gamma_{\mathrm{L}(G)}) \cup V(\Gamma_{\mathrm{L}(H)})$, $E(\Gamma_{\mathrm{L}(G)} + \Gamma_{\mathrm{L}(H)}) = E(\Gamma_{\mathrm{L}(G)}) \cup E(\Gamma_{\mathrm{L}(H)}) \cup \{XY : X \in V(\Gamma_{\mathrm{L}(G)}) \text{ and } Y \in V(\Gamma_{\mathrm{L}(H)})\}$ (see [3, page 67]), and we will use these notions in the following proof.

Proof of Theorem 1.2. Assume that we have a normal subgroup H such that $G/H \simeq D_8$. Then the result follows from Example 2.4.

Conversely suppose $\Gamma_{\mathrm{L}(G/H)}$ is a cocktail party graph for some normal subgroup H of G such that $G/H \simeq D_{2n}$. Of course, G/H cannot be quasihamiltonian of order ≥ 6, since otherwise $\Gamma_{\mathrm{L}(G/H)}$ would be the null graph. Since $\Gamma_{\mathrm{L}(G/H)}$ is of even order, we may assume $|V(\Gamma_{\mathrm{L}(G/H)})| = |V(\Gamma_{\mathrm{L}(D_{2n})})| = 2t$ for some integer $t \geq 1$. From Lemma 2.2, we have $\Gamma_{\mathrm{L}(D_{2n})}$ is regular if $n = 2p$ or $n = p$, where p is prime. However when n is odd prime, the graph is complete, which contradicts the definition. Again if $n = 2p$, then $\Gamma_{\mathrm{L}(D_{2n})}$ is p-partite. But since the degree of each vertex is equal to $2t - 2$, and this fails if $p \geq 3$, so $\Gamma_{\mathrm{L}(D_{2n})}$ can never be cocktail party, except for the case $G/H \simeq D_8$.

□

In order to see the proof of our third main result, we begin to note a relation between $\Gamma_{\mathrm{L}(G)}$ and the counting formula shown in Lemma 2.5.

Proposition 3.2. *Let G be a non quasi-hamiltonian group and $a_{X,Y}$ in (1.6). Then*

$$sd(G) = 1 - \frac{1}{|\mathrm{L}(G)|^2} \sum_{X,Y \in V(\Gamma_{\mathrm{L}(G)})} a_{X,Y}.$$

Proof. By looking at the definition of $A(\Gamma_{\mathrm{L}(G)})$ in (1.6), we immediately get

$$2\ |E(\Gamma_{\mathrm{L}(G)})| = \sum_{X,Y \in V(\Gamma_{\mathrm{L}(G)})} a_{X,Y}. \tag{3.4}$$

Now we combine Lemma 2.5 with (3.4). The result follows. □

We have all that we need, in order to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose $|V(\Gamma_{\mathrm{L}(G)})| = n$. In this situation $A(\Gamma_{\mathrm{L}(G)}) = (a_{X,Y})_{X,Y \in V(\Gamma_{\mathrm{L}(G)})}$ and $L(\Gamma_{\mathrm{L}(G)}) = (a_{X,Y})_{X,Y \in V(\Gamma_{\mathrm{L}(G)})}$ are the adjacency and Laplacian matrices of $\Gamma_{\mathrm{L}(G)}$ with spectra $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{\mu_1, \mu_2, \ldots, \mu_n\}$, respectively. From [6, Lemma 14.4.3] we have

$$|E(\Gamma_{\mathrm{L}(G)})| = \frac{1}{2}\sum_{i=0}^{n} \lambda_i^2 = \frac{1}{2}\sum_{i=0}^{n} \mu_i. \tag{3.5}$$

Then using Lemma 2.5 we have

$$sd(G) = 1 - \frac{1}{|\mathrm{L}(G)|^2}\sum_{i=0}^{n}\lambda_i^2 = 1 - \frac{1}{|\mathrm{L}(G)|^2}\sum_{i=0}^{n}\mu_i, \tag{3.6}$$

which shows that the subgroup commutativity degree is invariant under the spectrum of the adjancency matrix and under the spectrum of the Laplacian matrix of the non-permutability graph of subgroups of G. □

The generalized subgroup commutativity degree is also invariant under the spectrum of the Laplacian matrix of the subgraph of $\Gamma_{\mathrm{S}(G)}$, when we assign a sublattice $\mathrm{S}(G)$ of $\mathrm{L}(G)$.

Corollary 3.3. *Let G be a non quasi-hamiltonian group and $\mathrm{S}(G), \mathrm{T}(G)$ two sublattices of $\mathrm{L}(G)$. If $\mathrm{S}(G) = \mathrm{T}(G)$, $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G)) \subseteq \mathrm{S}(G)$ and $\Gamma_{\mathrm{S}(G)}$ is a subgraph of $\Gamma_{\mathrm{L}(G)}$, then*

$$gsd(G) = 1 - \frac{1}{|\mathrm{S}(G)|^2}\sum_{i=0}^{n}\mu_i.$$

where $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is the spectrum of the Laplacian matrix of $\Gamma_{\mathrm{S}(G)}$.

Proof. The proof follows from [20, Lemma 2.10] and Theorem 1.3. □

A very nice description is possible for cocktail party graphs.

Corollary 3.4. *Let G be a non quasi-hamiltonian group such that $\Gamma_{\mathrm{L}(G)}$ is cocktail party of type $K_{2\times n}$. Then*

$$sd(G) = 1 - \frac{(2n-2)^2}{|\mathrm{L}(G)|^2} - \frac{(-2)^{2(n-1)}}{|\mathrm{L}(G)|^2}.$$

Proof. Since the spectrum of the adjacency matrix of cocktail party graph $K_{2\times n}$ is $\{2n-2, 0, (-2)^{n-1}\}$, the result follows from Theorem 1.3. □

Finally we can see the proof of Theorem 1.4 below.

Proof of Theorem 1.4. From [20] we have $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(D_{2n})\big) = N(\mathrm{L}(D_{2n})$, then the complement of the graph $\Gamma_{\mathrm{L}(D_{2n})}$ is $\Gamma_N(D_{2n})$ and the structure of $\Gamma_{\mathrm{L}(D_{2n})}$ for some values of n is given in [10, Theorem 4.5]. Now if $n = 2^\alpha$, $\alpha \geq 2$, $\Gamma_{\mathrm{L}(D_{2^\alpha})}$ has the structure

$$\big(\Gamma_{\mathrm{L}(D_{2^{\alpha-1}})} \cup 2K_1\big) + \big(\Gamma_{\mathrm{L}(D_{2^{\alpha-1}})} \cup 2K_1\big) \tag{3.7}$$

where $\Gamma_{\mathrm{L}(D_{2^3})} = C_4$. Hence the result follows. □

4. Applications

Thanks to the main results, proved in Section 3, we may give more details on the spectral aspects of the non–permutability graph of dihedral groups.

Corollary 4.1. *Let $n \geq 3$ be an integer. Then the adjacency matrix of $\Gamma_{\mathrm{L}(D_{2n})}$ is given below.*

(i). *If* $n = p^\alpha$*, where* p *is odd prime and* $\alpha \geq 1$*, then*

$$A(\Gamma_{\mathrm{L}(D_{2p^\alpha})}) = \begin{pmatrix} A(\Gamma_{\mathrm{L}(D_{2p^{\alpha-1}})} \cup K_1) & J & \cdots & J \\ J & A(\Gamma_{\mathrm{L}(D_{2p^{\alpha-1}})} \cup K_1) & \cdots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & A(\Gamma_{\mathrm{L}(D_{2p^{\alpha-1}})} \cup K_1) \end{pmatrix}$$

is a square matrix of size $(\sigma(p^\alpha) - 1) \times (\sigma(p^\alpha) - 1)$*, where* J *is the all 1-matrix of size* $\sigma(2^{\alpha-1}) \times \sigma(2^{\alpha-1})$,

$$A(\Gamma_{\mathrm{L}(D_{2p})} \cup K_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad A(\Gamma_{\mathrm{L}(D_{2p})}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(ii). *If* $n = 2p$*, where* p *is odd prime, then*

$$A(\Gamma_{\mathrm{L}(D_{2n})}) = \begin{pmatrix} O & J & \cdots & J \\ J & O & \cdots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \cdots & O \end{pmatrix}$$

is square matrix of size $(\sigma(2p) - 3) \times (\sigma(2p) - 3)$*, where* O *is the all 0-matrix* 3×3 *and* J *the all 1-matrix* 3×3.

(iii). *If* $n = pq$*, where* p, q *are distinct primes with* $2 < p < q$*, then*

$$A(\Gamma_{\mathrm{L}(D_{2n})}) = \begin{pmatrix} U & O & W_1 & W_2 & \cdots & W_q \\ O^t & T & T & T & \cdots & T \\ W_1^t & T & T & J & \cdots & J \\ W_2^t & T & J & T & \cdots & J \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ W_q^t & T & J & J & \cdots & T \end{pmatrix}$$

is the square matrix of size $(\sigma(pq) - 1) \times (\sigma(pq) - 1)$*, where we have the following block matrices, whose size is reported with the right index at the bottom*

$$U = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{p \times p}, \quad O = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{p \times q}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{q \times q}$$

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$$W_2 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{q\times q}, \quad W_3 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{q\times q}, \cdots,$$

$$W_q = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{q\times q} \quad T = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}_{q\times q} \quad \textit{and} \quad J = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{q\times q}.$$

Proof. Again as the complement of the graph $\Gamma_{\mathrm{L}(D_{2n})}$ is $\Gamma_N(D_{2n})$, and the structure of $\Gamma_{\mathrm{L}(D_{2n})}$ for some values of n is given in [10, Theorem 4.5], then $\Gamma_{\mathrm{L}(D_{2n})}$ has the following structure.

In case (i)

$$\Gamma_{\mathrm{L}(D_{2p^\alpha})} \simeq \mathop{+}_{i=1}^{p} \left(\Gamma_{\mathrm{L}(D_{\frac{n}{p}})} \cup K_1\right)_i \tag{4.1}$$

where $\Gamma_{\mathrm{L}(D_{2p})} = K_p$.

In case (ii), we have

$$\Gamma_{\mathrm{L}(D_{2n})} \simeq K_{\underbrace{3,3,\ldots,3}_{p-\text{times}}}, \tag{4.2}$$

which is a p-partite complete graph.

Finally in case (iii), $\Gamma_{\mathrm{L}(D_{2n})}$ is the complement of the graph $\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} G_{ij}$, where for each $i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q,\ G_{ij}$ is a complete graph with vertex set $\{u_i, v_j, w_{ij}\}$. Hence the result follows. □

We may also describe the characteristic polynomials, in our present context of study.

Corollary 4.2. *Let $n \geq 3$ be an odd prime integer. Then the characteristic polynomial of $\Gamma_{\mathrm{L}(D_{2n})}$ of the adjacency matrix is*

$$\phi(A(\Gamma_{\mathrm{L}(D_{2n})}), \lambda) = \left(\sum_{i=2}^{n-1} (i-1)\binom{n}{n-i}\lambda^{n-1}\right) - 12(n-1).$$

Proof. Since n is odd prime, Corollary 2.3 implies that $\Gamma_{\mathrm{L}(D_{2n})}$ is complete and from [20, Lemma 4.1] we have $V(\Gamma_{\mathrm{L}(D_{2n})}) = \sigma(n) - 1 = n$, so $A(\Gamma_{\mathrm{L}(D_{2n})})$ has size $(\sigma(n) - 1) \times (\sigma(n) - 1)$. Hence the result follows by direct calculation of the determinant of $\lambda I - A(\Gamma_{\mathrm{L}(D_{2n})})$, where I is an identity matrix of the same size of $A(\Gamma_{\mathrm{L}(D_{2n})})$. □

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Corollary 4.3. *Let $n \geq 3$ be an odd prime integer, then the characteristic polynomial of $\Gamma_{\mathrm{L}(D_{2n})}$ with Laplacian matrix $L(\Gamma_{\mathrm{L}(D_{2n})})$, is*

$$\phi(L(\Gamma_{\mathrm{L}(D_{2n})}), \theta) = \binom{n-1}{0}\theta^n - n\binom{n-1}{1}\theta^{n-1} + n^2\binom{n-1}{2}\theta^{n-2} - n^3\binom{n-1}{3}\theta^{n-3} + \cdots + (-1)^i n^{n-1}\binom{n-1}{n-1}\theta.$$

Proof. From [20, Lemma 4.1], the degree of H_i is equal to $\sigma(n) - 2$ for all $H_i \in V(\Gamma_{\mathrm{L}(G)})$, $i = 1, 2, \ldots, \sigma(n) - 1$. Then,

$$L(\Gamma_{\mathrm{L}(D_{2n})}) = \begin{pmatrix} \sigma(n)-2 & 1 & \cdots & 1 \\ 1 & \sigma(n)-2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \sigma(n)-2 \end{pmatrix} \tag{4.3}$$

has size $(\sigma(n) - 1) \times (\sigma(n) - 1)$. Again as n is odd prime integer we have $\sigma(n) - 2 = n - 1$ and hence the result follows by calculating the determinant of $\theta I - L(\Gamma_{\mathrm{L}(D_{2n})})$ and its roots. □

We end with some concrete computations, whose methods are new after the results which we have just seen.

Example 4.4. From Example 2.4 we have $\Gamma_{\mathrm{L}(D_8)} \simeq C_4$, which is a cocktail party graph. Then the spectrum of the adjacency matrix $A(\Gamma_{\mathrm{L}(D_8)})$ of $\Gamma_{\mathrm{L}(D_8)}$ is $\{2, 0, -2\}$ and since $|\mathrm{L}(D_8)| = 10$ we can apply Corollary 3.4, obtaining

$$sd(D_8) = \frac{100 - \big((2)^2 + (-2)^2\big)}{100} = 0.92. \tag{4.4}$$

Further if n is odd prime integer, then $\Gamma_{\mathrm{L}(D_{2n})} \simeq K_n$ and we may easily compute the spectrum of $A(\Gamma_{\mathrm{L}(D_{2n})})$ by Corollary 4.3, hence

$$sd(D_{2n}) = \frac{((\sigma(n) + \tau(n))^2 - n^2 + n}{(\sigma(n) + \tau(n))^2}. \tag{4.5}$$

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