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EXISTENTIALLY AND κ-EXISTENTIALLY CLOSED GROUPS

BURAK KAYA AND MAHMUT KUZUCUOĞLU*

Abstract. A group G is existentially closed (algebraically closed) if every finite system of equations and in-equations that has coefficients in G and has a solution in an overgroup $H \geq G$ has a solution in G. Existentially closed groups were introduced by W. R. Scott in 1951. B. H. Neumann posed the following question in 1973: Does there exist explicit examples of existentially closed groups? Generalized version of this question is as follows: Let κ be an infinite cardinal. Does there exist explicit examples of κ-existentially closed groups? Recently an affirmative answer was given to Neumann's question and the generalized version of it, by Kaya-Kegel-Kuzucuoğlu. We give a survey of these results. We also prove that, there are maximal subgroups of κ-existentially existentially closed groups and provide some information about subgroups containing the centralizer of subgroups generated by fewer than κ-elements. This generalizes a result of Hickin-Macintyre.

1. Introduction

The motivation for the study of existentially closed groups (algebraically closed groups) comes from algebraically closed fields. Recall that a field F is called algebraically closed if K is an algebraic extension of F and if a polynomial $f(x) \in F[x]$ has a root in K, then it has a root in F. It is well known that, every field is contained in an algebraic closure of itself. As B. H. Neumann stated in [17, Page 555] that, "there are only countably many, countable algebraically closed fields one for each combination of characteristic (a positive prime number or zero) and transcendence degree (a cardinal

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*Corresponding author.

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number, in this case $\leq \aleph_0$)". The classification of algebraic extensions of finite fields of characteristic p, for any prime p, can be described by using the Steinitz numbers, see [22] and [1]. Unlike algebraically closed fields, we will see that, there are uncountably many countable existentially closed groups.

A group G is said to be **existentially closed (algebraically closed)** if every finite system

$$u_t(g_1, \dots, g_s, x_1, \dots, x_n) = 1$$
$$\vdots$$
$$v_j(g_1, \dots, g_s, x_1, \dots, x_n) \neq 1$$

of equations and in-equations in variables x_i and the group elements g_j which has a solution in an overgroup $H \geq G$ already has a solution in G.

Unlike polynomials over fields, not every system of equation or in-equation is solvable in an overgroup $H \geq G$. Indeed, let a and b be two elements of different orders, say n and m respectively. Then the equation $x^{-1}ax = b$ has no solution in any overgroup containing a and b. Similarly, $x^3 = 1, x^4 = 1, x \neq 1$ has no solution over any group. On the other hand, let $G = \langle a \mid a^3 = 1\rangle$ be a cyclic group of order 3. The equation $x^6 = 1,\ x^2 = a,\ x \neq 1$ has no solution in G but it has two solutions in the cyclic group $H = \langle c \mid c^6 = 1\rangle$ of order 6 namely $c^2 = a$ and $c^2 = a^{-1}$.

A group G is called a **weakly algebraically closed group** if every finite system of equations with coefficients in G which has solutions in an overgroup $H \geq G$, then it has a solution in G. W. R. Scott asked the question; whether every weakly algebraically closed group is algebraically closed. Neumann proved in [16] that, every weakly algebraically closed group $G \neq \{1\}$ is algebraically closed. Hence after this result, the study of non-trivial weakly algebraically closed groups turn out to be the study of existentially closed groups.

Algebraically closed groups are studied in the books by G. Higman E. Scott [5] and by R. C. Lyndon; P. E. Shupp in [12]. After the close connection with model theory and the book of G. Higman and E. Scott, algebraically closed groups started to be called **existentially closed groups.** Which properties of algebraically closed fields hold for the existentially closed groups and which does not hold?

In the case of fields; every field is contained in an algebraically closed field:

Is every group contained in an existentially closed group?

Does there exist an existentially closed group?

The answer to the above two questions is given by W. R. Scott in [19] and the following is proved.

Theorem 1.1. *Every group G can be embedded in an existentially closed group. Moreover the order of the existentially closed group is the larger of $\aleph_0$ and $|G|$.*

In particular, there are existentially closed groups of any given infinite order. This is one of the differences between existentially closed groups and κ-existentially closed groups which we will discuss in sequel.

After Scott's paper, B. H. Neumann worked on the class of existentially closed groups see [14], [15], [16], [17], and he investigated some important properties of existentially closed groups. In [15], he

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proved the following interesting result; he actually constructed uncountably many such 2-generated groups, namely:

Theorem 1.2. *There are $2^{\aleph_0}$ pairwise non-isomorphic 2-generator groups.*

By using the above theorem and by the following beautiful argument, he proved, unlike the case for algebraically closed fields, the next surprising result.

Theorem 1.3. *There are $2^{\aleph_0}$ mutually non-isomorphic countable existentially closed groups.*

The argument is as follows:

"There are $2^{\aleph_0}$ mutually non-isomorphic 2-generator groups and each by Theorem 1.1 can be embedded in a countable existentially closed group. But each countable group contains only countably many pairs of elements and thus only countably many 2-generator groups. Hence $2^{\aleph_0}$ countable existentially closed groups are needed to accommodate all 2-generator groups".

One of the important property of existentially closed groups is the following result.

Theorem 1.4 (B. H. Neumann)**.** *An existentially closed group is simple.*

Then by using the above theorem Neumann proved the following in [17].

Theorem 1.5. *An existentially closed group can not be finitely generated.*

The proof of this is quite interesting, informative and short. Let's have a look at the proof.

Proof. Let G be an existentially closed group. Let $X = \{g_1, g_2, \ldots, g_n\}$ be a finite subset of G. We can solve the equations

$$x^{-1}g_1x = g_1,\ x^{-1}g_2x = g_2, \ldots,\ x^{-1}g_nx = g_n, \quad x \neq 1$$

in the direct product $G \times H$ where H is a non-trivial group and hence in G. So every finitely generated subgroup of G has a non-trivial centralizer in G. But by Theorem 1.4, an existentially closed group is simple. So G cannot be finitely generated. □

Theorem 1.6. [17, Lemma 2.5] *Every existentially closed group contains, cyclic groups of every order.*

It follows that an existentially closed group cannot be periodic.

One can see from the definition that, every existentially closed group contains an isomorphic copy of every finitely presented simple group. So every existentially closed group contains an isomorphic copy of every finite group. This also can be established by the argument that, every finite group can be uniquely determined by its multiplication table and one can write the multiplication table of a finite group with finitely many equations and in-equations. By using the existentially closed groups and existentially closed Lie algebras, the construction of some groups and Lie algebras with some prescribed properties is obtained in [20]. In [21], it is shown that, in every variety of G-groups, every G-existentially closed element satisfies Nullstellensatz for finite consistent systems of equations. For set theoretical arguments and results see [6].

2. κ-Existentially Closed Groups

Let κ be an infinite cardinal. The generalization of existentially closed groups to κ-existentially closed groups are indicated in the paper of W. R. Scott [19] and studied in [13] and in different aspect in [2]. κ-existentially closed groups are the analogs of existentially closed groups, allowing the number of equations and the number of in-equations to be infinite.

Definition. Let κ be an infinite cardinal. A group G with $|G| \geq \kappa$ is called **κ-existentially closed** if every system of less than κ-many equations and in-equations with coefficients in G which has a solution in some overgroup $H \geqslant G$ already has a solution in G.
One may also define κ-existentially closed groups using free product of groups, see [10, Page 299].

Observe that existentially closed groups defined by W. R. Scott are $\aleph_0$-existentially closed groups.

As we discussed above, by Theorem 1.3, there are uncountably many countable, $\aleph_0$-existentially closed groups. Using the structure of the centralizers of countable subgroups, in [4, Theorem 2], Hickin and Macintyre proved that, there are $2^{\aleph_1}$ pairwise non-isomorphic $\aleph_0$-existentially closed groups of cardinality $\aleph_1$.

The situation in the uncountable case is actually quite different. Namely, if κ is an uncountable cardinal, we prove in [10, Theorem 2.7] that, up to isomorphism κ-existentially groups of cardinality κ are unique.

Theorem 2.1. *Let G and H be two κ-existentially closed groups of uncountable cardinality κ. Then G is isomorphic to H.*

The following Lemma will be used to characterize the κ-existentially closed groups, see [10, Lemma 2.6].

Lemma 2.2. *If κ is uncountable and G is a κ-existentially closed group, then isomorphic copy of every group A of order $|A| < \kappa$ is contained in G.*

Moreover if κ is an uncountable cardinal, then isomorphic copy of every group of cardinality κ is contained in G.

We give the following characterization of κ-existentially closed groups in [10, Proposition 2.8].

Proposition 2.3. *Let G be a group and κ be an uncountable cardinal. Then G is κ-existentially closed if and only if*

(i) *G contains an isomorphic copy of every group of cardinality less than κ, and*

(ii) *every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.*

We use the above characterization of κ-existentially closed groups in Proposition 2.3, to construct explicit examples of κ-existentially closed groups and in the proof of the existence of κ-existentially closed groups of cardinality $\lambda \geq \kappa$, see [7, Theorem 4].

For the existence of κ-existentially closed groups of cardinality $\lambda \geq \kappa$ we have the following result.

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Theorem 2.4. *Let $\kappa \leq \lambda$ be uncountable cardinals with $\lambda^{<\kappa} = \lambda$. Then there exists a κ-existentially closed group of cardinality λ.*

For an infinite cardinal λ the cofinality of λ will be denoted by $cf(\lambda)$.

Proposition 2.5. *(GCH) If there exists a κ-existentially closed group of cardinality $\lambda \geq \kappa$, then $cf(\lambda) \geq \kappa$.*

By a theorem of W. R. Scott, $\aleph_0$-existentially closed groups exist for any given infinite cardinal λ. But for uncountable cardinal κ, the κ-existentially closed groups of cardinality κ we have the following.

Corollary 2.6. *If κ is a singular cardinal, there does not exist a κ-existentially closed group of cardinality κ.*

Assuming the Generalized Continuum Hypothesis (GCH), one can completely characterize the cardinals $\kappa \leq \lambda$ for which there exists a κ-existentially closed group of cardinality λ by Proposition 2.5.

Corollary 2.7 (GCH)**.** *Let $\kappa \leq \lambda$ be uncountable cardinals. Then there exists a κ-existentially closed group of cardinality λ if and only if $cf(\lambda) \geq \kappa$.*

In particular, if λ is a successor cardinal, then there exists a κ-existentially closed group of cardinality λ.

In [17, Page 555], B. H. Neumann stated that "However, no existentially closed (algebraically closed) group is explicitly known, the existence proof being highly non-constructive. This stem in part from the fact that there is no useful criterion known that tells one what sentences are or are not consistent over a given group". Neumann's question is also mentioned in [12, Page 230]. We will give as is stated in [7], explicit examples of $\aleph_0$-existentially closed groups and the generalized version namely; explicit examples of κ-existentially closed groups for large cardinals. Hence we answer Neumann's question and the generalized version of it affirmatively.

3. Construction of κ-Existentially Closed Groups

The main ingredient of the construction of κ-existentially closed groups is the Cayley's Theorem. The regular representation has the following very interesting property discovered by P. Hall in [3, Lemma 1] and in unpublished notes of P. Hall by B. H. Neumann in [18, Page 537, Lemma] see also in [11, Lemma 2(c)].

Lemma 3.1. *If A and B are subgroups of K, then the corresponding right regular representations $\rho(A)$ and $\rho(B)$ are conjugate in the group $Sym(K)$ if and only if (i) A and B are isomorphic and (ii) A and B have the same index in K.*

In finite groups this corresponds to the following.

Lemma 3.2. *If K and K^* are isomorphic subgroups of a finite group G, then the right regular representations $\rho(K)$ and $\rho(K^*)$ are conjugate in $Sym(G)$.*

Observe that the above Lemma proves that, the image under right regular representation ρ of any two elements of the same order are conjugate, in the symmetric group, $Sym(G)$. But note that inside the symmetric group, any two elements of the same order may not be conjugate.

For the explicit examples of the κ-existentially closed groups; observe first that, for an infinite set Ω, the group $Sym(\Omega)$ is not existentially closed.

An uncountable regular cardinal κ is called an **inaccessible cardinal** if for every cardinal $\nu < \kappa$, we have $2^\nu < \kappa$. If κ is an inaccessible cardinal, the explicit example of a κ-existentially closed group of cardinality κ is given in [10, Corollary 4.1]. Actually this is the unique κ-existentially closed group of uncountable cardinality κ, obtained as a direct limit of groups embedded to the next one, by the right regular representation.

For any group G and any infinite cardinal κ, there exists a κ-existentially closed group containing G. Using the same technique as in [10], we prove that every group G is contained in a κ-existentially closed group, but the cardinality of the κ-existentially closed group might be larger, as there exists no κ-existentially closed group of cardinality κ for a singular cardinal κ. Indeed, let κ be any infinite cardinal. Start with an arbitrary group G_1 with $|G_1| \geq 3$ and embed G_1 into $Sym(G_1) = G_2$ by right regular representation and embed G_2 into $Sym(G_2) = G_3$ by right regular representation and continue like this and obtain the direct limit group $G_\omega = \bigcup_{i=1}^{\infty} G_i$. Then the group G_ω is the group constructed by P. Hall in [3], that is the unique countably infinite universal locally finite group. Then embed G_ω into $Sym(G_\omega) = G_{\omega+1}$ by right regular representation when α is a limit ordinal, then the direct limit group $G_\alpha = \bigcup_{i<\alpha} G_i$ continue like this for successor ordinals until we reach the group G_{2^κ}. Then the group G_{2^κ} is a κ-existentially closed group.

In order to take a short path, start with an arbitrary countably infinite group $G = G_0$ embed G_0 into $Sym(G_0) = G_1$ and for limit ordinal α let $G_\alpha = \bigcup_{\nu<\alpha} G_\nu$. Then the group G_{2^κ} satisfies the first property of Proposition 2.3, as $cf(2^\kappa) > \kappa$ every group of order less than κ can be embedded into G_{2^κ}. Moreover by Lemma 3.2 under the right regular representation isomorphic subgroups of order less than κ is contained in a subgroup G_α for some $\alpha < 2^\kappa$ as $cf(2^\kappa) > \kappa$ hence they are conjugate in G_{2^κ} by Lemma 3.1 and [11, Lemma 2(c)].

Hence for any group G and any infinite cardinal κ, there exists explicit example of a κ-existentially closed group containing isomorphic copy of G. Since every κ-existentially closed group is also an $\aleph_0$-existentially closed group, the above examples, G_{2^κ} of κ-existentially closed groups are also explicit examples of $\aleph_0$-existentially closed groups. Moreover the same argument implies that, even for singular cardinal $\mu < \kappa$, there exists μ-existentially closed groups. But recall that when μ is a singular cardinal μ-existentially closed groups of cardinality μ does not exist by [7, Corollary 7]. Then the following question still remains open.

Open Question (A). Let κ be an infinite cardinal which is not an uncountable inaccessible cardinal. Does there exist explicit example of a κ-existentially closed group of cardinality κ?

In particular explicit examples of countable, $\aleph_0$-existentially closed group is still open.

4. Automorphisms of κ-existentially Closed Groups

It was proved by Macintyre in [13, Page 56] that every countable, $\aleph_0$-existentially closed group has $2^{\aleph_0}$ automorphisms.

Question What can we say about the cardinality of automorphism groups of κ-existentially closed groups of cardinality $\lambda \geq \kappa$?

Definition. Let G be a group. An automorphism $\varphi \in Aut(G)$ is called κ-*inner* if for every subgroup $X \subseteq G$ with $|X| < \kappa$, there exists an element $g \in G$ such that $\iota_g(x) = \varphi(x)$ for all $x \in X$.

Obviously every inner automorphism is a κ-inner automorphism. Let κ-$Inn(G)$ denote the set of all κ-inner automorphisms of G.

We clearly have

$$Inn(G) \trianglelefteq \kappa\text{-}Inn(G) \trianglelefteq Aut(G)$$

Moreover, the inclusion on right is indeed an equality for κ-existentially closed groups.

Proposition 4.1. [8] *Let κ be uncountable and let G be κ-existentially closed. Then every automorphism of G is κ-inner. i.e. κ-$Inn(G) = Aut(G)$.*

We have seen in the explicit examples of κ-existentially closed groups that the construction has 2^κ levels. We now recall the notion of a level preserving automorphism from [8].

Let $C \subseteq \kappa$. An automorphism $\varphi \in Aut(G)$ is called C-*level preserving* if

$$\varphi[G_\alpha] = G_\alpha$$

for all $\alpha \in C$. The set of C-level preserving automorphisms of G is denoted by $Aut_C(G)$. We have $Aut_\emptyset(G) = Aut(G)$ and

$$Aut_C(G) \leqslant Aut_D(G) \text{ whenever } D \subseteq C$$

Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is closed unbounded (club subset) subset of κ if C is unbounded in κ and if it contains all its limit points less than κ. Observe that every club subset has cardinality κ.

Corollary 4.2. [8] *Let κ be inaccessible and let G be the unique κ-existentially closed group of cardinality κ, with countable base. Then*

$$Aut(G) = \bigcup_{\substack{C \subseteq \kappa \\ C \text{ is club}}} Aut_C(G) = \bigcup_{\alpha < \kappa} Aut_{\{\alpha\}}(G)$$

A κ-existentially closed group G of cardinality κ has κ levels. So it is natural to ask whether, for each ordinal μ in the set of transfinite unbounded sequences of subsets of κ the group G_μ is preserved by every automorphism of G.

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Corollary 4.3. [8] *Let κ be inaccessible and let G be the unique κ-existentially closed group of cardinality κ. Then $|Aut(G)| = 2^{\kappa}$.*

Not much known about the structure of the automorphism group of κ-existentially closed groups.

Open Question (B). Let G be a κ-existentially closed group of cardinality $\lambda \geq \kappa$ where κ is a regular cardinal. Determine the structure of the $Aut(G)$.

We prove that for a κ-existentially closed group of cardinality κ the $|Aut(G)| = 2^{\kappa}$, see [9].

5. Maximal subgroups in κ-existentially closed groups

Recall that a group satisfies Min (the minimum condition) if every descending chain of subgroups terminates after finitely many steps.

The following theorem generalizes [4, Theorem 3] from $\aleph_0$-existentially closed groups to κ-existentially closed groups.

Theorem 5.1. *Let G be a κ-existentially closed group and B be a subgroup of G satisfying Min and B is the intersection of finitely many subgroups generated by fewer than κ-elements. Then for every M with*

$$C_G(B) \leq M \leq G$$

there exists a subgroup $H \leq B$ such that $C_G(H) \leq M \leq N_G(H)$.

It follows that, if B is a finite group, then there are only finitely many subgroups of G containing $C_G(B)$.

Proof. Let M be a subgroup of G containing the subgroup $C_G(B)$. We prove the theorem by induction on the subgroups of B.

If B is the intersection of finitely many, finitely generated subgroups, then the result holds by [4, Theorem 3]. If $C_G(B) \leq M \leq N_G(B)$, then we are done. So suppose that, there exists $y \in M \setminus N_G(B)$. Then $B \cap B^y = B_0 \lneqq B$ because $B \neq B^y$. Then we have

$$M \geq \langle\ C_G(B), y\ \rangle \geq \langle\ C_G(B), C_G(B^y)\ \rangle$$

Let $B = B_1 \cap \cdots \cap B_n$ where B_i's are generated by fewer than κ elements. Then for each i, $C_G(B_i) \leq C_G(B_1 \cap \cdots \cap B_n) = C_G(B) \leq M$ and $C_G(B_i^y) \leq C_G(B_1 \cap \cdots \cap B_n)^y = C_G(B^y) \leq M$. As $B_0 = B \cap B^y = B_1 \cap B_1^y \cap \cdots \cap B_n \cap B_n^y$ we have by [10, Lemma 3.3] $M \geq C_G(B_0) = \langle C_G(B_1), \ldots, C_G(B_n), C_G(B_1^y), \ldots, C_G(B_n^y)\ \rangle$. By induction assumption there exists a subgroup H in B_0 such that

$$C_G(H) \leq M \leq N_G(H).$$

Hence we are done. □

Corollary 5.2. *Let G be a κ-existentially closed group and B be a subgroup of G satisfying Min and B is the intersection of finitely many subgroups generated by fewer than κ-elements and B is characteristically simple. Then $N_G(B)$ is a maximal proper subgroup of G.*

Proof. Assume if possible that, there exists a subgroup M satisfying $N_G(B) \leq M \leq G$. By Theorem 5.1 there exists a subgroup H in B satisfying

$$C_G(H) \leq M \leq N_G(H).$$

Since G is κ-existentially closed group by [10, Lemma 2.4] every automorphism of B is induced by an element in $N_G(B)$.

Let θ be an automorphism of B. Then there exists an element $n \in N_G(B)$ such that $b^\theta = b^n$ for some $n \in N_G(B)$. But $N_G(B) \leq M \leq N_G(H)$ implies for any $h \in H$, $h^\theta = h^n \in H$, as $n \in N_G(H)$. Hence H is a characteristic subgroup of B. Then either $H = 1$ and so $M = G$ or $H = B$ and $M = N_G(B)$. Hence $N_G(B)$ is a maximal subgroup of G.

□

In particular normalizer of every subgroup of order p for a prime p is a maximal subgroup of G. Hence centralizers of involutions are maximal subgroups of G. Recall that Prüfer groups and Tarski groups (that is infinite simple groups whose proper non-trivial subgroups have prime order) and their finitely many direct products satisfy Min. If κ is an uncountable cardinal, then every κ-existentially closed group contains isomorphic copies of such groups. Hence their centralizers satisfies the properties of the above Theorem 5.1.

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Burak Kaya

Department of Mathematics, Middle East Technical University, 06800, Ankara, Turkey

Email: `burakk@metu.edu.tr`

Mahmut Kuzucuoğlu

Department of Mathematics, Middle East Technical University, 06800, Ankara, Turkey

Email: `matmah@metu.edu.tr`