http://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 11 No. 4 (2022), pp. 335-343.
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LINEAR CODES RESULTING FROM FINITE GROUP ACTIONS

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Abstract. In this article, we use group action theory to define some important ternary linear codes. Some of these codes are self-orthogonal having a minimum distance achieving the lower bound in the previous records. Then, we define two new codes sharing the same automorphism group isomorphic to $C_2 \times M_{11}$ where M_{11} is the Sporadic Mathieu group and C_2 is a cyclic group of two elements. We also study the natural action of the general linear group $GL(k,2)$ on the vector space F_2^k to characterize Hamming codes $H_k(2)$ and their automorphism group.

1. Introduction

Because of their important applications in theory and in practice, self-orthogonal and two-weight linear codes have received a great deal of attention in recent years. Self-orthogonal linear codes are interesting from various points of view and certainly advantageous, because the amount of computation required for coding and decoding can be considerably reduced when their automorphism group is large enough. A large automorphism group can be used to speed up the decoding process. Decoding algorithms and many other applications are facilitated by the use of error-correcting codes having a large automorphism group [1, 2, 3]. Especially, permutation decoding can be used when a code has a large automorphism group to ensure the existence of a set of automorphisms, called a PD-set [4, 5]. Self-orthogonal codes also play an important role in the construction of quantum error correcting codes [6, 7]. The study of two-weight codes have been detailed in [8]. Delsarte [9] and then Calderbank and Kantor [10] were the first to study the connections between two-weight codes, strongly regular graphs and projective point sets. Two-weight codes have also important applications in secret sharing [11] .

If C is a $k-$dimensional subspace of F_q^n where F_q is the Galois field of order q, q is a prime power, then C is called an $[n,k]_q$ linear code over F_q. A linear code C has q^k codewords and if $q = 2$, C

Communicated by Dianhua Wu.

MSC(2010): Primary: 94B05.

Keywords: Linear Code automorphism, Group Actions, Hamming codes, simplex codes.

Manuscript Type: Research Paper.

Received: 17 November 2020, Accepted: 14 January 2022.

http://dx.doi.org/10.22108/TOC.2022.126254.1786 .

is called an $[n,k]_2$ binary linear code. The two most common ways to present a linear code are with either a generator matrix or a parity check matrix. A generator matrix for an $[n,k]_q$ code C is any $k \times n$ matrix G whose rows form a basis for C [15]. Because a linear code is a subspace of a vector space, it is the kernel of some linear transformation. In particular,there is an $(n-k) \times n$ matrix H, called a parity check matrix for the $[n,k]_q$ code C, defined by:

$$C = \left\{x \in GF(q)^n | Hx^T = 0\right\} \tag{1.1}$$

The rows of a parity check matrix H are independent being considered as the rows of a generator matrix of a code, called the dual or orthogonal of C and is denoted by $C^{\perp}$. Notice that $C^{\perp}$ is an $[n, n-k]_q$ code. An alternate way to define the dual code is by using inner products. Recall that the ordinary inner product of vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in ${F_q}^n$ is $x.y = x_1.y_1 + \cdots + x_n.y_n$. Therefore from (1.1), we see that $C^{\perp}$ can also be defined by

$$C^{\perp} = \{x \in GF(q)^n |\, x.c = 0,\ \forall c \in C\} \tag{1.2}$$

If G and H are generator and parity check matrices, respectively, for C, then H and G are generator and parity check matrices, respectively, for $C^{\perp}$ [12]. We call a code C self-orthogonal if $C \subseteq C^{\perp}$ and C is called self-dual if $C = C^{\perp}$. A binary self-dual code C is called doubly even if all codewords are of weight divisible by 4. Linear codes with exactly two non-zero weights,are called two-weight codes [13].

The permutation automorphism group, denoted by $PAut(C)$, is the subgroup of the permutation group S_n acting on $(F_q)^n$ by coordinate permutation which preserves C [14]. The permutation of coordinate places represented by the $n \times n$ matrix P is in $PAut(C)$ if and only if $KG = GP$ for some nonsingular $k \times k$ matrix K and where G is a generator matrix of C. A monomial matrix is a matrix with exactly one nonzero entry in each row and column. Thus a monomial matrix over F_2 is a permutation matrix, and a monomial matrix over an arbitrary finite field F_q is $D.P$, a nonsingular diagonal matrix D times a permutation matrix P. We also recall that the group $PAut(C)$ is isomorphic to the group $PAut(C^{\perp})$.

$$PAut\,(C) \cong PAut\left(C^{\perp}\right) \tag{1.3}$$

The set of monomial matrices that map C to itself forms the group $MAut(C)$ called the monomial automorphism group of C.

A binary Hamming code $H_r(2)$ of length $n = 2^r - 1$ (r $\geq$ 2) has parity check matrix $H_{r,2}$ whose columns consist of all nonzero binary vectors of length r, each used once. $H_r(2)$ is an $[n = 2^r - 1, k = 2^r - 1 - r, d = 3]_2$ code [15], [12]. The set of all nonsingular $k \times k$ matrices over a field F_q is called the general linear group and is denoted by $GL(k,q)$. Let Γ be a finite group and X a nonempty set. We define the action of the group Γ on X in the particular case where $X = F_2^k$ or $X = F_3^k$ and $\Gamma = GL(k,2)$ or $\Gamma = C_2 \times M_{11}$ group of Order 15840 on the vector space F_3^5. A (left) natural group

http://dx.doi.org/10.22108/TOC.2022.126254.1786

action φ of the group $GL(k,q)$ on F_q^k is a function

$$\varphi\colon GL(k,q)\times F_q^k \to F_q^k$$
$$(M,V)\mapsto \varphi(M,V)=M.V \tag{1.4}$$

which satisfies the following two axioms:

- (Identity): $\forall V\in F_q^k, \varphi(I_n,V)=V$ (Here, I_n denotes the identity element of $GL(k,q)$.)
- (Compatibility): $\forall M,N\in GL(k,q), \forall V\in F_q^k, \varphi(M.N,V)=\varphi(M,\varphi(N,V))$.

The action of $GL(k,q)$ on F_q^k is Transitive since F_q^k is non-empty and for each pair U, V in F_q^k there exists a matrix M in $GL(k,q)$ such that $M.U=V$.

2. Definition of two ternary codes C_1 and $\widetilde{C}_2$

let $\Gamma=\langle A,B\rangle$ be the group generated by two following 5×5 matrices A and B defined over the Galois field F_3

$$A=\begin{pmatrix} 0&1&0&2&2\\ 1&2&2&0&1\\ 2&2&2&1&1\\ 0&1&1&1&1\\ 0&1&1&2&0 \end{pmatrix} \text{ and } B=\begin{pmatrix} 0&2&2&0&2\\ 1&0&0&2&0\\ 0&0&2&1&1\\ 1&2&0&0&0\\ 0&2&1&1&2 \end{pmatrix}.$$

The group Γ of order 15840 is isomorphic to $C_2\times M_{11}$ where M_{11} is the Sporadic Mathieu group and C_2 is a cyclic group. Under the action of Γ, $X=F_3^5$ is partitioned into three orbits $Orb_1=\{00000\}$, Orb_2 and Orb_3 of respective lengths 1, 132 and 110.

2.1. Linear code C_1 induced by the second orbit Orb_2.

The second orbit Orb_2 is a list of the following 132 row vectors of F_3^5:

00001	02210	01221	01201	10001	00112	12220	21112	02022	11101	20201	20002
00222	12110	11102	21102	22121	22110	02112	01011	11020	21220	20012	01000
21110	21000	22202	00011	00221	22201	12201	21221	00111	01120	01001	22011
20010	02102	22212	11212	01102	22221	10102	10021	02000	02011	12221	22010
00002	21201	11022	01110	00211	20202	02110	02212	12000	22211	02122	22021
11220	02002	10020	10000	02211	12112	10212	20011	02220	22102	21020	01222
01112	01121	00010	11121	00200	01021	10211	00022	12102	00122	10101	01101
01022	21121	02121	21120	11120	01212	12211	02200	00020	02222	01220	00201
20200	02201	11112	20121	10022	11201	12010	02111	20000	01122	02010	00220
10201	01111	21010	02221	11122	01211	11012	12212	12210	22210	21122	02202
01020	20022	20102	12020	01100	20122	00100	02012	00102	00110	10011	10100

Orb_2 is a 132×5 matrix and let N_1 be its transposed matrix (this way of notation is consistent with Gap commands [17]). Code C_1 with generator matrix N_1 is of length $n_1=132$, dimension $k=5$ and minimum distance $d=72$. The generator matrix N_1 verifies $N_1 0^t N_1=I_n$ so the code C_1 is self-orthogonal. C_1 is a two-weight ternary codes, indeed the possible weights of non-zero codewords are either $w_1=72$ (22 codewords) or $w_2=90$ (220 codewords). Note that some row vectors of the orbit Orb_2 are proportional. The row vectors are then chosen in the projective space $PG(4,3)$ over the field F_3 so that two row vectors are not proportional (i.e. no row is equal to the multiplication of another by a scalar of F_3). We thus obtain a matrix K_1 with the only following 66 row vectors:

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10001	01221	01011	00112	11102	12201	11101	10102
10021	01000	12220	11212	11220	00001	01001	11022
10020	01201	12000	00011	01120	12102	01110	00122
10101	12221	11121	01102	11112	10212	10022	11201
12010	01222	00111	12110	01220	01121	11122	01211
11012	12212	01212	12210	11120	12211	11020	01112
00010	00100	01021	12112	10211	00102	01111	10000
01020	00110	01100	01022	10100	12020	01101	10011
10201	01122						

Let $S_1 = {}^tK_1$ be the transposed matrix of K_1 and L_1 be the ternary code defined by the generating matrix S_1. The ternary code L_1 is of length 66, of dimension 5 and minimum distance 36. L_1 is a two-weight ternarynary codes, indeed the possible weights of non-zero codewords are either $w_1 = 36$ (22 codewords) or $w_2 = 45$ (220 codewords).

2.2. **Linear code $\widetilde{C}_2$ induced by the third orbit Orb_3.**

The third orbit Orb_3 is a list of the following 110 row vectors of F_3^5:

00012	20112	12120	22100	20210	21012	12011	22122	20110	12021	20120	11000
20021	11200	10222	11021	21111	01010	21022	12121	20101	01002	01012	11222
11010	00021	22022	21101	00212	22222	10122	02101	22002	12111	10221	20001
02120	20222	22220	20020	20212	12001	12002	22200	11001	20100	00101	21210
00202	00120	22112	22101	21100	02100	11202	12012	10200	12100	22020	02021
10002	10120	02020	12200	12122	21021	10202	00210	22001	10112	10111	11221
01200	21222	11210	12222	20211	11211	10220	10012	11002	21202	22012	00121
21211	10110	21002	20111	01210	10010	20221	12022	22111	21001	10210	22000
21200	20220	02001	11110	12101	11011	21212	12202	01202	22120	21011	11111
11100	10121										

Orb_3 is a 110×5 matrix and let N_2 be its transposed matrix. The code $\widetilde{C}_2$ with generator matrix N_2 is a code of length $n_1 = 110$, dimension $k = 5$ and minimum distance $d = 72$ which achieves the lower $Lb(110, 5) = 72$, and upper bound $Ub(110, 5) = 72$ in the previous records [16].

n/k	1	2	3	4	5	6	7	8	9	10
101	101	75	69	67	66	$64-65$	63	$60-63$	$59-62$	$57-61$
102	102	76	70	68	67	$65-66$	64	$61-63$	$60-63$	$57-62$
103	103	77	71	69	68	66	65	$62-64$	$60-63$	$58-63$
104	104	78	72	69	69	$66-67$	66	$63-65$	$60-64$	$59-63$
105	105	78	72	70	69	67	66	$63-66$	$61-65$	$60-64$
106	106	79	72	71	70	68	67	$63-66$	$62-66$	$60-65$
107	107	80	73	72	71	69	68	$64-67$	$63-66$	$60-65$
108	108	81	74	72	72	69	69	$65-68$	$63-67$	$61-66$
109	109	81	75	72	72	70	69	$66-68$	$63-67$	$62-67$
110	110	82	75	73	72	71	70	$66-69$	$64-68$	$63-67$

Some bounds on the minimum distance of linear codes over F_3

The generator matrix N_2 verifies $N_2 0^t N_2 = I_n$ so the code $\widetilde{C}_2$ is self-orthogonal. $\widetilde{C}_2$ is a two-weight ternary codes, the possible weights of non-zero codewords are either $m_1 = 72$ (220 codewords) or $m_2 = 90$ (22 codewords). Note also that some row vectors of the Orb_3 orbit are proportional. The row vectors are then chosen in the projective space $PG(4, 3)$ over the field F_3 so that two row vectors are not proportional. We thus obtain a matrix K_2 of only 55 row vectors which are listed in the following table:

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11000	10220	12222	10012	01012	11222	01010	12120	10122	01202
11010	10221	10120	12001	12002	11100	11001	12111	11200	11211
12012	10200	12100	00101	12021	12011	10210	11221	01200	11002
11210	10002	00012	10222	11021	01210	10010	10112	10110	12022
00120	12121	10202	01002	12202	11202	11110	12200	12122	11011
00121	11111	10111	12101	10121					

Let $S_2 = {}^tK_2$ be the transposed matrix of K_2 and L_2 be the ternary code defined by the generator matrix S_2. The ternary code L_2 is of length 55, dimension 5 and minimum distance 36. This ternary code is exceptional since it reaches the lower $Lb(55,5) = 36$, and upper bound $Ub(55,5) = 36$ of the minimum distance in the previous records [16].

n/k	1	2	3	4	5	6	7	8	9	10
51	51	38	35	33	32	31	30	$28-30$	$27-29$	$26-28$
52	52	39	36	34	33	32	$30-31$	$29-30$	$27-30$	$27-29$
53	53	39	36	35	34	33	$30-32$	$30-31$	$28-30$	$27-30$
54	54	40	36	36	35	34	$31-33$	$30-32$	$29-31$	$27-30$
55	55	41	37	36	36	35	$32-33$	$30-32$	$30-31$	$28-31$
56	56	42	38	36	36	36	$33-34$	$31-33$	$30-32$	$29-31$
57	57	42	39	36	36	36	$33-35$	$32-34$	$30-33$	$30-32$
58	58	43	39	36	36	36	$34-35$	$33-35$	$31-33$	$30-33$
59	59	44	40	37	37	36	$35-36$	$33-35$	$32-34$	$30-33$
60	60	45	41	38	38	$36-37$	36	$33-36$	$33-35$	$31-34$

Some bounds on the minimum distance of linear codes over $GF(3)$.

2.3. Codes C_1 and $\widetilde{C}_2$ share the same group of automorphisms.

Theorem 2.1.

1) *The group $C_2 \times M_{11}$ where M_{11} is the Sporadic Mathieu group and C_2 is a cyclic group of two elements is an automorphism group of self-orthogonal codes C_1 and $\widetilde{C}_2$.*
2) *The group $C_2 \times M_{11}$ is a monomial automorphism group of the code L_i for $i = 1, 2$.*

Proof. 1) Consider the application defined by

$$\varphi : C_2 \times M_{11} \to PAut\,(C_1)$$
$$D \to \varphi\,(D) = P \quad such\ that\ D.N_1 = N_1.P$$

- The generator matrix N_1 of the code C_1 is defined by its 132 column vectors $c_1, c_2, \ldots, c_{132}$ whose transposed vectors form exactly the orbit Orb_2 . As a result we have

$$\begin{aligned} D.N_1 &= D.(c_1, c_2, \ldots, c_{132}) \\ &= (D.c_1, D.c_2, \ldots, D.c_{132}) \\ &= (c_{\pi(1)}, c_{\pi(2)}, \ldots, c_{\pi(132)}) \quad (for\ some\ permutation\ \pi) \\ &= (c_1, c_2, \ldots, c_{132}).P \end{aligned}$$

$D.c_1$, $D.c_2$,...,$D.c_{132}$ is an arrangement of c_1, c_2,...,c_{132} by definition of the orbit Orb_2. Let π be the permutation of indices 1, 2,...,132 corresponding to this arrangement, we clearly have $D.c_i = c_\pi(i)$. Let P the unique permutation matrix associated with the permutation π, we deduce that the application φ is well defined.

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- We consider $D_1, D_2 \in C_2 \times M_{11}$ and $P \in PAut(C_1)$

$$\begin{aligned}\varphi(D_1) = \varphi(D_1)\,(= P) &\Rightarrow D_1.N_1 = N_1.P \quad and \quad D_2.N_1 = N_1.P \\ &\Rightarrow D_1.N_1 = D_2.N_1 \\ &\Rightarrow {D_2}^{-1}.D_1.N_1 = N_1 \\ &\Rightarrow {D_2}^{-1}.D_1 = I_5 \quad (since \quad rank(N_1) = 5) \\ &\Rightarrow D_1 = D_2\end{aligned}$$

  So the application φ is injective.
- We consider $D_1, D_2 \in C_2 \times M_{11}$ and $P_1, P_2 \in PAut(C_1)$ such as $\varphi(D_1) = P_1$ and $\varphi(D_1) = P_2$, then $D_1.N_1 = N_1 \cdot P_1$ and $D_2 \cdot N_1 = N_1 \cdot P_2$. We have

$$\begin{aligned}(D_1.D_2).N_1 &= D_1.(D_2.N_1) \\ &= D_1.(N_1.P_2) \\ &= (D_1.N_1).P_2 \\ &= (N_1.P_1).P_2 \\ &= N_1.(P_1.P_2)\end{aligned}$$

  We deduce that $\varphi(D_1.D_2) = P_1.P_2 = \varphi(D_1).\varphi(D_2)$ and then φ is a homomorphism. We conclude that $C_2 \times M_{11} \cong \varphi\,(C_2 \times M_{11})$ is a subgroup of the full permutation automorphism group $PAut(C_1)$.

We use the same proof for the code $\widetilde{C_2}$

2) Consider the application defined by

$$\begin{aligned}\psi : C_2 \times M_{11} &\to MAut\,(L_1) \\ D &\to \psi\,(D) = P \quad such\ that\ D.S_1 = S_1.P\end{aligned}$$

- The generator matrix S_1 of the code L_1 is defined by its 66 vector columns $w_1, w_2, \ldots, w_{66}$ whose transposed vectors form exactly all projective vectors of the orbit Orb_2 . As a result we have for some permutation σ of cordinates $1, 2, 3, \ldots, 66$ and some $(\alpha_1, \alpha_2, \ldots, \alpha_{66}) \in (GF\,(3) - \{0\})^5$

$$\begin{aligned}D.S_1 &= D.(w_1, w_2, \ldots, w_{66}) \\ &= (D.w_1, D.w_2, \ldots, D.w_{66}) \\ &= (\alpha_1.w_{\sigma(1)}, \alpha_2.w_{\sigma(2)}, \ldots, \alpha_{66}.w_{\sigma(66)}) \\ &= (w_1, w_2, \ldots, w_{66}).Diag(\alpha_1, \alpha_2, \ldots, \alpha_{66}).P_0 \\ &= (w_1, w_2, \ldots, w_{66}).P\end{aligned}$$

  where P_0 is the unique permutation matrix associated with the permutation σ and $Diag(\alpha_1, \alpha_2, \ldots, \alpha_{66})$ is a diagonal matrix. We deduce that the application ψ is well defined

http://dx.doi.org/10.22108/TOC.2022.126254.1786

since

$P = Diag((\alpha)_1, (\alpha)_2, \ldots, (\alpha)_{66}).P_0 \in MAut(L_1)$.

- Using a similar demonstration as 1) in the Theorem 2.1 we prove that the application ψ is an injective homomorphism:

$$\psi(D_1.D_2) = \psi(D_1).\psi(D_2) \text{ and}$$
$$C_2 \times M_{11} \cong \varphi\,(C_2 \times M_{11})$$

$C_2 \times M_{11} \cong \varphi\,(C_2 \times M_{11})$ is a subgroup of the full monomial automorphism group $MAut(L_1)$.

We use the same proof for the code L_2. □

3. Hamming codes defined from the action of the group $GL(k,2)$ on the vector space $GF(2)^k$.

The second point of the following theorem is the well-known theorem stating that the permutation automorphism group of $[2^k-1, 2^k-1-k, 3]$ Hamming code is the general linear group $GL(k,2)$ with dimension 2^k-1-k over the Galois field F_2 [15], but we will prove it differently by using the theory of finite group actions.

Theorem 3.1. *Under the action of $GL(k,2)$, $GF(2)^k$ is partitioned in two orbits, orbit $Orb_a(2)$ which is trivial and orbit $Orb_b(2)$ which is not trivial and we have:*
If H is the transposed matrix of the matrix having the same row vectors as $Orb_b(2)$, then the dual of the code defined by the generator matrix H is the Hamming code $H_k(2)$ (up to an equivalence).
The full automorphism group of the hamming code $H_k(2)$ is the group $GL(k,2)$ (up to an equivalence).

$$PAut\,(H_k\,(2)) \cong GL\,(k,2)$$

Proof. the action of the general linear group $GL(k,2)$ on $GF(2)^k$ is a transitive action. So $GF(2)^k$ is partitioned under this action into two orbits: the trivial orbit $Orb_a(2) = \{000\cdots00\}$ and the orbit $Orb_b(2) = F_q^k - \{000\cdots00\}$ as a list of all transposed vectors of $F_2^k - \{000\cdots00\}$
Let H be the transposed matrix of the matrix having the same row vectors as $Orb_b(2)$, then H is the parity check matrix of the binary Hamming code $H_k(2)$. Let

$$\rho : GL(k,2) \to PAut\,(S_k(2))$$
$$D \to \varphi\,(D) = P \quad \textit{such that } D.H = H.P$$

where $S_k(2)$ is the simplex code which is by definition the dual code of the Hamming code $H_k(2)$. Using a proof similar to that of the theorem 2.1, we prove that the map ρ is well defined and that ρ is an injective homomorphism. It remains to show that ρ is surjective. For this we consider the permutation $P \in PAut\,(S_k(2))$ and $H = (c_1, c_2, \ldots, c_{2^k-1})$ defined by its vector columns c_i for $i = 1, 2, \ldots, 2^k-1$, then we have

$$H.P = (c_1, c_2, \ldots, c_{2^k-1}).P$$
$$= (c_{\pi(1)}, c_{\pi(2)}, \ldots, c_{\pi(2^k-1)})$$

http://dx.doi.org/10.22108/TOC.2022.126254.1786

where the permutation π is associated with the permutation matrix P. We consider the unique matrix D of the group $GL(k,2)$ defined by

$$D.c_i = c_{\pi(i)} \text{ for all } i = 1, 2, \ldots, 2^k - 1.$$

We then have $D.H = H.P$ and consequently we have

$$\begin{aligned} GL(k,2) &\cong PAut(S_k(2)) \\ &\cong PAut(H_k(2)^{\perp}) \quad (from\ 1.3) \\ &\cong PAut(H_k(2)) \end{aligned}$$

□

4. **Conclusion**

Self-orthogonal and two-weight linear codes have important applications in theory and practice, they received a great deal of attention in recent years. A large automorphism group can be used to speed up the decoding process. Decoding algorithms and many other applications are facilitated by the use of error-correcting codes having a large automorphism group. In this article, linear codes with prescribed large automorphism groups are constructed using group action theory. We construct some important ternary linear codes, some of these codes are self-orthogonal and two-weight with a minimum distance achieving the lower bound in the previous records. Then, we define two new codes sharing the same automorphism group isomorphic to $C_2 \times M_{11}$ where M_{11} is the Sporadic Mathieu group and C_2 is a cyclic group of two elements. The following records summarize the important results of this paper, we denote by $\#(w) = s$ the number of codewords w having the weight s.

Linear code	C_1	L_1	$\widetilde{C}_2$	L_2
Length	132	66	110	55
Dimension	5	5	5	5
minimum distance	72	36	72	36
Properties	Self-orthogonal		Self-orthogonal	
	two-weight:	two-weight:	two-weight:	
	$w_1 = 72, \#(w_1) = 22$	$w_1 = 36, \#(w_1) = 22$	$w_1 = 72, \#(w_1) = 220$	
	$w_2 = 90, \#(w_2) = 220$	$w_2 = 45, \#(w_2) = 220$	$w_1 = 90, \#(w_1) = 22$	
			$Lb(110,5) = 72$	$Lb(55,5) = 36$
			$Ub(110,5) = 72$	$Ub(55,5) = 36$
PAut()	$C_2 \times M_{11}$		$C_2 \times M_{11}$	
MAut()		$C_2 \times M_{11}$		$C_2 \times M_{11}$

Acknowledgments

The author would like to thank the referee for his constructive comments and suggestions leading to improvements in the readability and quality of the paper.

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http://dx.doi.org/10.22108/TOC.2022.126254.1786