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ON INFINITE ANTICOMMUTATIVE GROUPS

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ABSTRACT. We completely describe the structure of locally (soluble-by-finite) groups in which all abelian subgroups are locally cyclic. Moreover, we prove that Engel groups with the above property are locally nilpotent.

1. Introduction

Following [2], a group is termed *anticommutative* if two of its elements commute only when they generate a cyclic subgroup. The class of anticommutative groups is subgroup closed but not quotient closed. It is easy to see that anticommutative groups are precisely those groups in which every abelian subgroup is locally cyclic. In particular, in any anticommutative group an element of finite order cannot commute with an element of infinite order.

The structure of finite anticommutative groups has been fully described by Zassenhaus [11] in the soluble case, and by Suzuki [9] in the non-soluble case (see also [2, Theorem 4.1]). It comes out that a finite group is anticommutative if and only if its Sylow subgroups are either cyclic or a generalized quaternion group

$$Q_{2^n} = \langle a, x \mid a^{2^{n-1}} = 1 \,,\, x^2 = a^{2^{n-2}} \,,\, x^{-1}ax = a^{-1} \rangle$$

of order 2^n with $n > 2$. In particular, a non-abelian finite anticommutative group cannot be simple.

On the other side, there exist infinite simple anticommutative groups. Easy examples are the Tarski p-groups, which are infinite groups all whose proper non-trivial subgroups have prime order p. The structure of locally soluble anticommutative groups was described by Kuzennyi and Maznichenko [5] in the periodic case, and by the authors of the present paper [2] in the non-periodic case. In [2] the structure

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of non-soluble locally finite anticommutative groups is also given. The anonymous referee suggested to consider possible extensions of our results to the class of locally virtually soluble groups. Recall that a group is said to be *locally virtually soluble* if every finitely generated subgroup has a soluble normal subgroup of finite index. In Section 2 we prove that a locally virtually soluble anticommutative group is either metabelian or locally finite, and then its structure comes from the above results.

Given any group G, the commutator of elements $x_1, x_2, \ldots, x_n \in G$ is defined inductively by the rules

$$[x_1, x_2] = {x_1}^{-1}{x_2}^{-1}x_1x_2, \qquad [x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n].$$

It is usual to write $[a, {}_nb]$ instead of $[a, b, b, \ldots, b]$, when the entry b appears n times. A group G is said to be an *Engel group* if for all $a, b \in G$ there exists an integer n, depending on a and b, such that $[a, {}_nb] = 1$. If there exists an integer n such that $[a, {}_nb] = 1$ for all $a, b \in G$, the group G is called an *n-Engel group*. For a comprehensive look at the fundamental results concerning Engel groups and the main open problems, the reader can refer to the beautiful survey by Traustason [10]. Obviously, every locally nilpotent group is an Engel group. On the other side, given any prime number p, the standard wreath product $C_p \wr C_{p^\infty}$ is an example of a $(p+1)$-Engel group which is not nilpotent. Famous examples constructed by Golod show that Engel groups are not locally nilpotent in general. However, Engel groups belonging to many familiar classes are locally nilpotent. For instance, Zorn proved that this is the case for finite groups, Gruenberg for soluble groups, Baer for groups with the maximal condition on subgroups, Peng for groups with the maximal condition on abelian subgroups (see [10]). In Section 3 we prove that anticommutative Engel groups are locally nilpotent. We also give the structure of such groups. It is a longstanding open question whether every n-Engel group is locally nilpotent; up to now, we only know that the answer is positive for $n \leq 4$. By contrast, we show that anticommutative n-Engel groups are nilpotent of class at most n. Again, the structure of these groups is completely determined.

The notation used throughout the paper is standard, and can be found for instance in [8]. In particular, $Z(G)$ and $\mathrm{Fit}(G)$ denote the center and the Fitting subgroup of the group G, respectively. If G is any group, given a subgroup H of G and an element g of G, we denote by $C_H(g)$ the set of all elements of H which commute with g. As usual, $\pi(G)$ is the set of all prime numbers involved in the factorization of the orders of elements of any periodic group G. In the sequel, we set $Q_2 = Q_4 = Q_8$. It is easy to see that the infinite quaternion 2-group

$$Q_{2^\infty} = \bigcup_{n \in \mathbb{N}} Q_{2^n}$$

is anticommutative.

2. Locally virtualy soluble anticommutative groups

Our first result shows that we can restrict our attention to periodic groups.

Lemma 2.1. *Every non-periodic virtually soluble anticommutative group is soluble.*

Proof. Let G be a non-periodic virtually soluble anticommutative group. Then there exists a soluble normal subgroup H having minimal finite index in G. By [2, Lemma 2.2], there exists a maximal torsion-free abelian normal subgroup A of H. Clearly, $A \subseteq \mathrm{Fit}(H)$. On the other hand, by [5, Lemma 3], being a locally nilpotent anticommutative group, $\mathrm{Fit}(H)$ cannot be mixed, so it is torsion-free and therefore abelian. By the maximality of A we get $\mathrm{Fit}(H) = A$. In particular, A is normal in G. Now $A \subseteq C_H(A)$ since A is abelian, and $C_H(A) = C_H(\mathrm{Fit}(H)) \subseteq \mathrm{Fit}(H) = A$ as H is soluble (see, for instance, [8, 5.4.4 (ii)]). Therefore $C_H(A) = A$. The quotient group $C_G(A)H/H$ is a subgroup of G/H, so it is finite. Since $C_G(A) \cap H = A$, this means that $C_G(A)/A$ is finite. Obviously $A \subseteq Z(C_G(A))$, hence the derived subgroup of $C_G(A)$ is finite by a well-known result due to Schur (see, for instance, [8, 10.1.4]). Since G is anticommutative, given any non-trivial element $a \in A$, the subgroup $\langle a, c\rangle$ is cyclic for all $c \in C_G(A)$. This forces $C_G(A)$ to be torsion-free. Thus its derived subgroup is trivial, and $C_G(A)$ is abelian. Hence $C_G(A) \subseteq \mathrm{Fit}(G)$. Now we claim that $\mathrm{Fit}(G) = A$.

Of course, $A \subseteq \mathrm{Fit}(G)$ as A is an abelian normal subgroup of G. Now let K be any nilpotent normal subgroup of G. Then HK is a soluble normal subgroup of G, and $|G : HK| \leq |G : H|$. By our choice of H, we get $K \subseteq H$, and hence $K \subseteq \mathrm{Fit}(H) = A$. This proves our claim.

Since $C_G(A) \subseteq \mathrm{Fit}(G) = A$, we get $C_G(A) = A$. Thus the quotient group G/A is isomorphic to a subgroup of $\mathrm{Aut}(A)$, and therefore it is abelian (see, for instance, [3, Exercise 4, p. 254]). Hence G is soluble, as required. □

Theorem 2.2. *Every infinite locally virtually soluble anticommutative group is either metabelian or locally finite.*

Proof. Let G be an infinite locally virtually soluble anticommutative group, and assume that G is not locally finite. Then G cannot be periodic. Thus every finitely generated subgroup of G is contained in a non-periodic finitely generated subgroup of G, and the latter is soluble by Lemma 2.1. Hence G is locally soluble, and therefore it is metabelian by [2, Corollary 3.3]. □

A finite group is called a $\mathfrak{Z}$-group if its Sylow subgroups are cyclic. Obviously, $\mathfrak{Z}$-groups are anticommutative. Furthermore, they are metacyclic, and their structure is completely determined (see, for instance, [8, 10.1.10]).

Corollary 2.3. *A locally virtually soluble group G is anticommutative if and only if one of the following holds:*

(1) *G is a $\mathfrak{Z}$-group;*
(2) *$G = LQ$, where L is a $\mathfrak{Z}$-group of odd order, and $Q \simeq Q_{2^n}$ for some $n \in \mathbb{N}$;*
(3) *$G = L \times S$, where L is a $\mathfrak{Z}$-group, $S \simeq \mathrm{SL}(2,p)$ for some prime $p > 3$, and $(|L|, |S|) = 1$;*
(4) *$G = ((\langle\sigma\rangle \rtimes \langle\rho\rangle) \times S)\langle x\rangle$, where $S \simeq \mathrm{SL}(2,p)$ for some prime $p > 3$, $|\sigma| = m$, $|\rho| = n$, $|x| = 4$, $\sigma^\rho = \sigma^r$ with $(m,n) = 1$, $(mn, p(p^2-1)) = 1$ and $r^n \equiv 1 \pmod{m}$, $\sigma^x = \sigma^t$ with $t^2 \equiv 1 \pmod{m}$, $\rho^x = \rho$, $x^2 \in Z(S)$, and x acts on S as a non-inner automorphism of order 2;*
(5) *$G = A \rtimes B$, where A and B are periodic locally cyclic groups with $\pi(A) \cap \pi(B) = \emptyset$;*
(6) *$G = A \rtimes (B \times Q)$, where A and B are periodic locally cyclic groups without elements of order 2, $\pi(A) \cap \pi(B) = \emptyset$, $Q \simeq Q_{2^n}$ for some $n \in \mathbb{N} \cup \{\infty\}$, and $[A, Q'] = 1$;*

(7) $G = A \rtimes (B \times (Q \rtimes T))$, *where* A *and* B *are periodic locally cyclic groups without elements of order* 2 *and* 3, $\pi(A) \cap \pi(B) = \emptyset$, $Q \simeq Q_8$ *is a Sylow* 2*-subgroup of* G, T *is a cyclic Sylow* 3*-subgroup of* G, $[A, B \times T] = A$, $[A, Q] = 1$, *and* $[Q, T] = Q$;

(8) $G = A \rtimes (B \times ((Q \rtimes T)\langle x \rangle))$, *where* A *and* B *are periodic locally cyclic groups without elements of order* 2 *and* 3, $\pi(A) \cap \pi(B) = \emptyset$, $Q \simeq Q_8$, T *is a cyclic Sylow* 3*-subgroup of* G, $|x| = 4$, $[A, B \times \langle x \rangle] = A$, $[A, Q \rtimes T] = 1$, $[Q, T] = Q$, $[T, \langle x \rangle] = T$, *and* $Q\langle x \rangle \simeq Q_{16}$;

(9) $G = R \times S$, *where* $S \simeq \mathrm{SL}(2, p)$ *for some prime* $p > 3$, $R = A \rtimes B$ *for some periodic locally cyclic groups* A *and* B *with* $\pi(A) \cap \pi(B) = \emptyset$, *and* $\pi(R) \cap \pi(\mathrm{SL}(2, p)) = \emptyset$;

(10) $G = K\langle x \rangle$, *where* $K = (A \rtimes B) \times S$ *is as in* (9), $|x| = 4$, $x^2 \in Z(S)$, $b^x = b$ *for every* $b \in B$, x *acts on* A *as a power automorphism of order at most* 2, *and on* S *as a non-inner automorphism of order* 2;

(11) G *is isomorphic to a semidirect product* $A \rtimes \langle x \rangle$, *where* A *is a rational group,* x *has order* 2 *and* $a^x = a^{-1}$ *for all* $a \in A$;

(12) G *is isomorphic to a semidirect product* $A \rtimes \langle x \rangle$, *where* A *is a rational group and either* $x = 1$ *or* x *acts on* A *as an automorphism having infinite order.*

Proof. Let G be a locally virtually soluble anticommutative group.

First suppose that G is finite. Then the structure of G is that described in (1) or in (2) if G is soluble [11], and that described in (3) or in (4) if G is non-soluble [9] (see also [2, Theorem 4.1]).

Now suppose that G is infinite. Then G is soluble or locally finite by Theorem 2.2. If G is locally finite, then its structure is that described in (5) or in (6) when G is soluble and locally supersoluble, that described in (7) or in (8) when G is soluble but not locally supersoluble ([5, Theorem 1 and Theorem 2], see also [2, Theorem 3.1]), and that described in (9) or in (10) when G is non-soluble [2, Theorem C].

Finally, suppose that G is not locally finite, and hence non-periodic. Now, if G is mixed, then its structure is as in (11) by [2, Theorem B]. Otherwise G is torsion-free, and its structure is as in (12) by [2, Theorem A].

Conversely, all groups listed under (1) – (12) are anticommutative (see [2]). □

3. Anticommutative Engel groups

A result due to Plotkin ([6], see also [7, page 56]) states that an Engel group whose abelian subgroups have finite rank and finite torsion subgroup is nilpotent of finite rank. In particular, this implies that torsion-free anticommutative Engel groups are abelian. Now we show that anticommutative Engel groups cannot be mixed, and hence they are either abelian or periodic.

Proposition 3.1. *Every anticommutative Engel group having an element of infinite order is torsion-free and abelian.*

Proof. Let G be an anticommutative Engel group, and let $b \in G$ be an element of infinite order. We claim that if a is an element of G such that $[a, b, b] = 1$ then $[a, b] = 1$.

From $[a, b, b] = 1$ it follows that $[b^a, b] = 1$, and so the subgroup $\langle b^a, b \rangle$ is cyclic. Hence there exists an element $c \in G$ and non-zero integers α and β such that $b^a = c^\alpha$ and $b = c^\beta$, so we get $b^\alpha = c^{\beta\alpha} = (c^\alpha)^\beta = (b^a)^\beta$. Let B be any abelian maximal subgroup of G containing b. Thus $b^\alpha \in Z(\langle B, B^a \rangle)$. Since in an anticommutative group an element of infinite order cannot commute with an element of finite order, this implies that the subgroup $\langle B, B^a \rangle$ is torsion-free and therefore abelian by the Plotkin result. Hence $B = B^a$ by the maximality of B. Thus B is normal in $\langle B, a \rangle$, and the subgroup $B\langle a \rangle$ is soluble. As a finitely generated soluble Engel group is nilpotent, and a nilpotent anticommutative group cannot be mixed, $B\langle a \rangle$ is torsion-free and hence abelian. This means that $[a, b] = 1$, as claimed.

For all elements $a \in G$ there exists an integer n, depending on a and b, such that $[a,\, _n b] = 1$. Our previous claim ensures that $[a,\, _{n-1} b] = 1$, and using backward induction we easily get $[a, b] = 1$. Hence a has infinite order, as required. □

Lemma 3.2. *Every anticommutive 2-group is locally nilpotent.*

Proof. Let G be an anticommutative 2-group, and suppose that $a, b \in G$ are involutions. If a and b do not commute, then they generate a finite dihedral group, which is not anticommutative. Hence $[a, b] = 1$, and so a and b generate a cyclic subgroup. Thus $a = b$. By [1, Theorem 8.1], a 2-group has an unique element of order 2 if and only if it is isomorphic either to C_{2^n} or to Q_{2^n} for some $n \in \mathbb{N} \cup \{\infty\}$. Hence the result follows. □

Theorem 3.3. *Every anticommutative Engel group is locally nilpotent. Moreover it is either locally cyclic or isomorphic to the direct product of a group isomorphic to Q_{2^n} for some $n \in \mathbb{N} \cup \{\infty\}$ and a periodic locally cyclic group without elements of order 2.*

Proof. Let G be an anticommutative Engel group. By Proposition 3.1 and the above mentioned Plotkin result, we can assume that G is periodic. Moreover, by Lemma 3.2 we can assume that G has elements of odd order. First we claim that every element of odd order is central in G.

Let $b \in G$ an element of odd order, and consider an element $a \in G$ such that $[a, b, b] = 1$. Then $[b^a, b] = 1$, and so the subgroup $\langle b^a, b \rangle$ is finite cyclic. Hence $\langle b \rangle = \langle b^a \rangle$ since a finite cyclic group cannot have two subgroups of the same order. Thus $\langle b \rangle$ is normal in $\langle a, b \rangle$, and the subgroup $\langle a, b \rangle$ is soluble and hence nilpotent. Since in any anticommutative nilpotent group the elements of odd order are central, this means that $[a, b] = 1$. Now for all elements $a \in G$ there exists an integer n, depending on a and b, such that $[a,\, _n b] = 1$. Our previous argument yields that $[a,\, _{n-1} b] = 1$, and using backward induction we obtain that $[a, b] = 1$. Hence $b \in Z(G)$, as claimed.

In particular, the elements of odd order in G form a subgroup $D \subseteq Z(G)$ and the quotient G/D is a 2-group. Now we will prove that G/D is anticommutative.

Let S/D be any abelian subgroup of G/D. Then S is nilpotent as $D \subseteq Z(G)$. Since G is anticommutative, this implies that $S = U \times D$ where U is a 2-subgroup of S. Thus $S/D \simeq U$ is an anticommutative abelian group, and so it is locally cyclic. Therefore G/D is anticommutative, as required.

Hence, by Lemma 3.2, the quotient G/D is locally nilpotent. Now let H be any finitely generated subgroup of G. Then HD/D is nilpotent, so H is nilpotent since $D \leq Z(G)$. Therefore G is locally nilpotent.

It follows by [5, Lemma 3] that G is either locally cyclic or isomorphic to the direct product of a group isomorphic to Q_{2^n} for some $n \in \mathbb{N} \cup \{\infty\}$ and a periodic locally cyclic group without elements of order 2. □

Corollary 3.4. *Every anticommutative n-Engel group is nilpotent of class $\leq n$. Moreover it is either locally cyclic or isomorphic to the direct product of a group isomorphic to Q_{2^t} for some $t \leq n+1$ and a periodic locally cyclic group without elements of order 2.*

Proof. Let G be an anticommutative n-Engel group. By Theorem 3.3 we know that G is either locally cyclic or isomorphic to the direct product of a group isomorphic to Q_{2^n} for some $n \in \mathbb{N} \cup \{\infty\}$ and a periodic locally cyclic group without elements of order 2. Since the infinite quaternion group Q_{2^∞} is not n-Engel for any positive integer n, and the generalized quaternion group Q_{2^t} is n-Engel if and only if $t \leq n+1$, if G is non-abelian then its 2-component is a generalized quaternion group of order 2^t with $t \leq n+1$. Thus the result follows. □

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