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ON THE CHARACTERISTIC POLYNOMIAL AND SPECTRUM OF BASILICA SCHREIER GRAPHS

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Abstract. The Basilica group is one of the most studied automaton groups, and many papers have been devoted to the investigation of the characteristic polynomial and spectrum of the associated Schreier graphs $\{\Gamma_n\}_{n\geq 1}$, even if an explicit description of them has not been given yet.

Our approach to this issue is original, and it is based on the use of the Coefficient Theorem for signed graphs. We introduce a signed version Γ_n^- of the Basilica Schreier graph Γ_n, and we prove that there exist two fundamental relations between the characteristic polynomials of the signed and unsigned versions. The first relation comes from the cover theory of signed graphs. The second relation is obtained by providing a suitable decomposition of Γ_n into three parts, using the self-similarity of Γ_n, via a detailed investigation of its basic figures. By gluing together these relations, we find out a new recursive equation which expresses the characteristic polynomial of Γ_n as a function of the characteristic polynomials of the three previous levels. We are also able to give an explicit description of the eigenspace associated with the eigenvalue 2, and to determine how the eigenvalues are distributed with respect to such eigenvalue.

1. Introduction

Spectral graph theory is a popular branch of mathematics which combines Graph theory and Linear algebra, and has a number of applications in Mathematical chemistry, Computer Science, Probability and Complex networks. Given a graph, the idea is to associate with it a symmetric matrix (typically the adjacency matrix or the matrix of the Laplace operator) representing the geometric structure of the graph and study how the spectrum of such a matrix might reflect the geometric properties of the

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corresponding graph. Remarkable results in this direction involve enumeration theorems, bounds for isoperimetric constants, computation of quantities having probabilistic interpretations and so on. In the context of Group theory, there exists a vary natural way to associate with a finitely generated group a graph by using its Cayley graph. Spectral properties of the Cayley graph can even capture algebraic properties of the corresponding group. Cayley graphs represent the action of a group on itself. More generally, one can consider Schreier graphs, i.e., graphs that graphically encode the action of a finitely generated group on a set. Schreier graphs seem to be the most natural object of study in the context of automaton groups. Finite invertible automata are finite input/output machines that sequentially describe the action of a set of states on a finite alphabet. This action is represented by automorphisms of a rooted regular tree and so it gives rise to bijections that generate a group, called automaton group (for more details about the theory of automaton groups see [4, 21]). The action of an automaton group on a rooted tree preserves the level structure of the tree: considering the restriction of the action on a given level of the tree, one can construct the corresponding Schreier graph. Therefore any automaton group yields infinitely many finite Schreier graphs corresponding to the action on the finite levels of the related rooted regular tree. The class of automaton groups contains interesting examples of groups with wild and exotic properties: among them, there is the so called Basilica group, that is the first example of an amenable but not subexponentially amenable group. The Basilica group was introduced by R. Grigorchuk and A. Żuk in [17] as a group generated by a three state automaton. It appeared later that this group is the iterated monodromy group of the complex map $z^2 - 1$, whose Julia set is the so called Basilica fractal. With this respect the Schreier graphs of the Basilica group converge, in a suitable way, to the Basilica fractal (see [5, 21]). This correspondence drew a special interest in considering geometric and combinatorial properties of the Basilica Schreier graphs. They were studied in the context of enumeration models [10, 14] and their limits in the topology of rooted graphs were classified up to isomorphism in [13]. Recently, some distance-based topological invariants on the Basilica Schreier graphs have been computed in [8]. Once we have an infinite sequence of Schreier graphs we can be interested in their spectra. A natural question arising from spectral graph theory in the context of automaton groups is the explicit computation of the spectrum of the adjacency matrices for the sequence of Schreier graphs. In general, the determination of the spectrum of the Markov operator associated with a graph is a very difficult task, and only few examples are known for families of graphs. The self-similarity of the action of automaton groups can be exploited in the application of the Schur complement method, that allows to compute the characteristic polynomial of the n-th Schreier graph in terms of the characteristic polynomial of the $(n-1)$-th Schreier graph with respect to a new variable. This approach has succeeded in many examples, providing the explicit computation of the spectra of the Schreier graphs of important automaton groups [9, 15, 16]. It is remarkable that the first examples of graphs whose spectrum is a Cantor set of zero Lebesgue measure, or the union of a Cantor set with a countable set of isolated points, have been obtained in the frame of Schreier graphs generated by automaton groups [1]. In the case of the Basilica group, the situation seems to be more complicated and the Schur complement approach does not lead to an explicit description of the spectrum but to a two-dimensional dynamical system describing it [18]. This

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fact has made things very intriguing and many authors have been trying to explicitly determine the spectrum of the Basilica Schreier graphs. By using a different approach, in [22] the authors studied Dirichlet forms and the corresponding Laplace operators on the Basilica Julia set. By considering renormalization of the infinite graph instead finite approximation, in [6] a dynamical system for the spectrum of the Laplace operator of Schreier graphs is described. This allows to prove, among other results, that the spectrum of the Laplace operator has infinitely many gaps and that the support of the KNS spectral measure is a Cantor set

In this paper, we go back to an approximation approach, but by using a new original method that does not use the structure of the adjacency matrix of the Schreier graphs. Our idea is to construct the n-th Basilica Schreier graph as cover graph of a signed version of the $(n-1)$-th Schreier graph. Signed graphs were introduced by Harary in [20]: roughly speaking, they are graphs together with a sign function on every edge. In this way, they generalize the classical notion of graph (interpreted with a signed graph with all positive edges) and permit to develop a consistent spectral theory (see, for instance, the survey [2] for more details, and the periodically updated bibliography [23] by Zaslavsky for a more general panorama). This crucial observation shifts the spectral problem for the Basilica Schreier graphs to the relationship between the characteristic polynomial of the signed and unsigned Schreier graphs. Together with an application of a signed version of the Coefficient Theorem, this allows us to write down a recursive relation for the characteristic polynomial of the Schreier graphs of the Basilica group. By using this approach, we are able to determine, for any n, the exact multiplicity of the eigenvalue 2 for the adjacency matrix of the n-th Schreier graph, and the precise number of eigenvalues smaller than 2 and of eigenvalues greater than 2.

The paper is organized as follows. In Section 2 we introduce basic definitions and preliminaries about automaton group theory and signed graph theory. In Section 3 we recall the basic structural properties of Basilica Schreier graphs. In Section 4 we start the study of the characteristic polynomial of the Basilica Schreier graphs by introducing a suitable signed version of the Basilica Schreier graphs. This permits, by using the theory of signed graphs, to decompose the characteristic polynomial of the n-th Schreier graph as the product of the characteristic polynomial of the $(n-1)$-th signed and unsigned Schreier graphs (see Corollary 4.2). A more sophisticated analysis of the structure of the Basilica Schreier graphs and the application of the Coefficient Theorem leads to a second relation among characteristic polynomials of signed and unsigned version that even involves the $(n-2)$-th Schreier (see Theorem 4.5). The solution of the equations described in the previous results yields to a closed recursive expression for the characteristic polynomial of the Basilica Schreier graphs (see Theorem 4.9). This expression allows us to make an analysis of the structure of the spectrum in relation to its distribution with respect to the eigenvalue 2, its multiplicity, and its associated eigenspace (see Corollary 4.10, Remark 4.11, and Corollary 4.12).

2. Preliminaries

This section is devoted to give some preliminary notions about automaton groups and signed multigraphs.

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2.1. **Automaton groups and the Basilica group.** We start this section by recalling the definition of automaton group.

Definition 2.1. *An automaton is a quadruple $\mathcal{A} = (S, X, \mu, \lambda)$, where:*

(1) S is the set of states;
(2) $X = \{0, 1, \ldots, k-1\}$ is an alphabet;
(3) $\mu : S \times X \to S$ is the transition map;
(4) $\nu : S \times X \to X$ is the output map.

The automaton $\mathcal{A}$ is *finite* if S is finite and it is *invertible* if, for all $s \in S$, the transformation $\nu(s, \dot{}) : X \to X$ is a permutation of X. An automaton $\mathcal{A}$ is usually represented by its *Moore diagram*, which is a directed labeled graph whose vertex set is S. For every $s \in S$ and $x \in X$, the diagram has an arrow from s to $\mu(s, x)$ labeled by $x|\nu(s, x)$. (See in Fig. 1 the Moore diagram of the automaton generating the Basilica group.)

Let $X^n = \{x_1 \cdots x_n : x_i \in X\}$ be the set of words of length n over X, and put $X^* = \bigcup_{n=0}^{\infty} X^n$, where $X^0 = \{\emptyset\}$ and $\emptyset$ denotes the empty word. Then the maps μ and ν can be naturally extended to $S \times X^*$ as:

$$\mu(s, xw) = \mu(\mu(s, x), w) \qquad \nu(s, xw) = \nu(s, x)\nu(\mu(s, x), w), \tag{2.1}$$

by setting $\mu(s, \emptyset) = s$ and $\nu(s, \emptyset) = \emptyset$, for all $s \in S, x \in X$ and $w \in X^*$.

Fix an initial state $s \in S$ and consider the transformation $\nu(s, \dot{})$ on X^* defined by Eq. (2.1). Then the image of the word $x_1 x_2 \cdots x_n$ under the action of $\nu(s, \dot{})$ can be easily found using the Moore diagram. More precisely, one has to consider the directed path starting at the state s with consecutive labels $x_1|y_1$, $x_2|y_2$ and so on: then one gets $\nu(s, x_1 x_2 \cdots x_n) = y_1 \cdots y_n$. Given an invertible automaton $\mathcal{A} = (S, X, \mu, \nu)$, the *automaton group* generated by $\mathcal{A}$ is by definition the group generated by the transformations $\nu(s, \dot{})$, for $s \in S$.

An automaton group can be regarded in a very natural way as a group of automorphisms of the regular rooted tree T_k of degree $|X| = k$, that is, the rooted tree in which each vertex has k children. Then, each vertex of the n-th level of the tree (which consists of k^n vertices) can be identified with an element of X^n, for each $n \geq 1$, whereas the empty word $\emptyset$ can be identified with the root of T_k. In Fig. 2 the rooted binary tree T_2 is represented. Let $Aut(T_k)$ denote the group of all automorphisms of T_k: it is clear that the root and hence the levels of the tree are preserved by each element of $Aut(T_k)$. A group $G \leq Aut(T_k)$ is said to be *spherically transitive* if its action is transitive on each level of the tree.

If $g \in Aut(T_k)$ and $v \in X^*$, we denote by $g|_v \in Aut(T_k)$ the *restriction* of the action of g to the subtree rooted at v, which satisfies

$$g(vw) = g(v)g|_v(w), \qquad \text{for all } v, w \in X^*.$$

Thus, an automorphism $g \in Aut(T_k)$ induces a permutation of the vertices of the first level of T_k, together with k restrictions to all subtrees rooted at those vertices, which are all isomorphic to T_k. Let

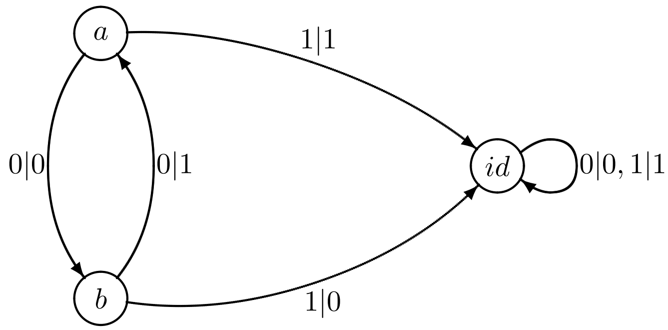

FIGURE 1. The automaton generating the Basilica group.

$Sym(k)$ denote the symmetric group on k elements. It is possible to represent an element $g \in Aut(T_k)$ as

$$g = (g|_0, \ldots, g|_{k-1})\sigma_g,$$

where $\sigma_g \in Sym(k)$ describes the action of g on the first level and $g|_i \in Aut(T_k)$ is the restriction of the action of g on the subtree rooted at the i-th vertex of the first level. A group $G \leq Aut(T_k)$ is said to be *self-similar* if $g|_v \in G$ for all $v \in X^*$ and $g \in G$. In particular, if G is a self-similar group, then any $g \in G$ can be written as

$$g = (g|_0, \ldots, g|_{k-1})\sigma_g,$$

with $\sigma_g \in Sym(k)$ and $g|_i \in G$. In particular, for each $x \in X$ and $w \in X^*$, one has $g(xw) = \sigma_g(x)g|_x(w)$.

The Basilica group, which will be denoted by B in this paper, was introduced by R. Grigorchuk and A. Żuk [17] as the group generated by the three-state automaton in Fig. 1. It can be read from the automaton that the Basilica group is the automorphism group of T_2 generated by the two automorphisms a and b of the following self-similar form:

$$a = (b, id)e \qquad\qquad b = (a, id)\varepsilon,$$

where id denotes the trivial automorphism of T_2, and e and ε are the identity and the nontrivial permutation in $Sym(2)$, respectively. It it not difficult to show that the action of B on T_2 is spherically transitive.

2.2. **Signed multigraphs and covers.** A (finite) multigraph G is a triple $G = (V_G, E_G, r_G)$, where the vertex set V_G and the edge set E_G are finite sets, and $r_G \colon E_G \to \{\{u, v\}, u \neq v \in V_G\} \cup \{\{u\}, u \in V_G\}$ is a map assigning to each edge either an unordered pair or a singleton of V_G. If $e \in E_G$ is such that $r_G(e) = \{u, v\}$, the element e is denoted as an edge between the vertices u and v, and the vertices u and v are called endpoints of e. When $r_G(e) = \{u\}$ the edge e is also called a loop with endpoints u. If there exist $e, f \in E_G$ such that $r_G(e) = r_G(f) = \{u, v\}$, then there is a multi-edge connecting the vertices u and v. In this paper we will often use the graph terminology and notation $G = (V_G, E_G)$

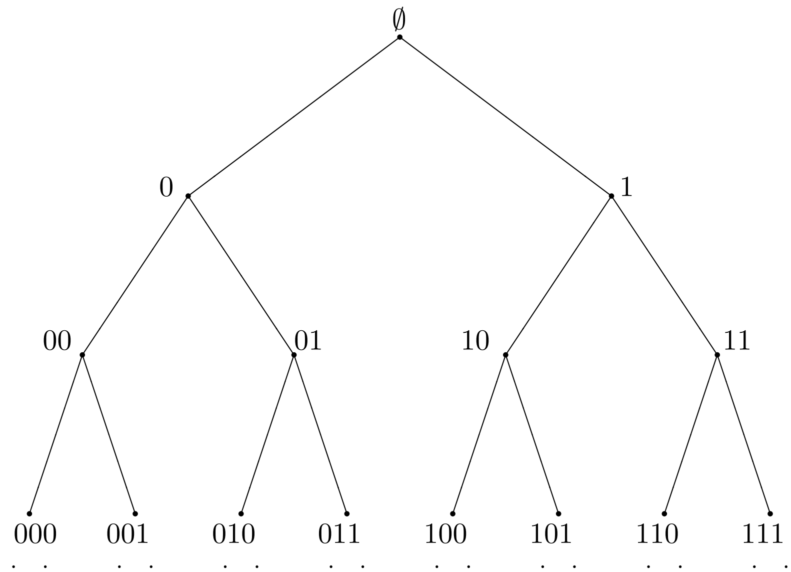

FIGURE 2. The rooted binary tree T_2.

for a multigraph $G = (V_G, E_G, r_G)$ and, when there is no ambiguity, we directly denote by $e = \{u, v\}$ an edge with endpoints $\{u, v\}$.

For two given multigraphs $G_1 = (V_{G_1}, E_{G_1}, r_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2}, r_{G_2})$ a (multigraph) homomorphism $\phi\colon G_1 \to G_2$ is a pair (ϕ_V, ϕ_E) of maps $\phi_V\colon V_{G_1} \to V_{G_2}$ and $\phi_E\colon E_{G_1} \to E_{G_2}$ such that, if $r_{G_1}(e) = \{u, v\}$, then $r_{G_2}(\phi_E(e)) = \{\phi_V(u), \phi_V(v)\}$, for every $e \in E_1$. An isomorphism $\phi = (\phi_V, \phi_E)$ from G_1 to G_2 is an homomorphism such that ϕ_V and ϕ_E are bijections. If such an isomorphism exists, we write $G_1 \cong G_2$.

A *signed multigraph* G^σ is a pair $G^\sigma = (G, \sigma)$ where $G = (V_G, E_G, r_G)$ is a multigraph, called *underlying (multi)graph*, and σ, called *signature*, is a map $\sigma\colon E_G \to \{\pm 1\}$ assigning a sign to every edge. The edges of G^σ can be partitioned into positive and negative edges. In this paper we will often speak about signed graphs instead of signed multigraphs. An unsigned graph can be always equipped with the all-positive signature and then considered as a signed graph. When an ordering of the vertices $V_G = \{v_1, v_2, \dots, v_n\}$ of a signed graph $G^\sigma = (G, \sigma)$ is fixed, one can define its signed adjacency matrix A_{G^σ} by

$$(A_{G^\sigma})_{i,j} = \begin{cases} \displaystyle\sum_{\substack{e \in E_G:\\ r_G(e) = \{v_i, v_j\}}} \sigma(e) & \text{if } i \neq j \\ \displaystyle\sum_{\substack{e \in E_G:\\ r_G(e) = \{v_i\}}} 2\sigma(e) & \text{if } i = j \end{cases} \qquad \text{for each } i, j = 1, \dots, n.$$

We will denote by A_G the classical adjacency matrix of the underlying graph G, which can be also obtained from A_{G^σ}, when σ is the all-positive signature. Notice that different signed graphs may have the same adjacency matrix; in fact deleting or adding a pair consisting of a negative and a positive edge between the same endpoints does not affect the entries of the adjacency matrix. By construction, the matrix A_{G^σ} is symmetric and its characteristic polynomial

$$\Phi(G^\sigma, x) := \det(A_{G^\sigma} - xI)$$

has n real roots, which constitute the spectrum of G^σ.

For a signed graph G^σ its cover graph $L(G^\sigma)$ is defined as the unsigned graph with vertex set $V_G \times \{+1, -1\}$, edge set $E_G \times \{+1, -1\}$, and associated map $r_{L(G^\sigma)}$ defined by

$$r_{L(G^\sigma)}(e, s) = \begin{cases} \{(u, s), (v, s)\} & \text{if } \sigma(e) = +1 \\ \{(u, s), (v, -s)\} & \text{if } \sigma(e) = -1 \end{cases}$$

for each $(e, s) \in E_G \times \{+1, -1\}$ such that $r_G(e) = \{u, v\}$. Notice that each positive edge $\{u, v\}$ of G^σ produces two "parallel" edges

$$\{(u, +1), (v, +1)\} \qquad \text{and} \qquad \{(u, -1), (v, -1)\}$$

in $L(G^\sigma)$; each negative edge $\{u, v\}$ of G^σ produces two "crossed" edges

$$\{(u, +1), (v, -1)\} \qquad \text{and} \qquad \{(u, -1), (v, +1)\}$$

in $L(G^\sigma)$. Observe, in particular, that a negative loop $\{u\}$ gives rise to two edges $\{(u, +1), (u, -1)\}$ and $\{(u, -1), (u, +1)\}$ in $L(G^\sigma)$ that are not loops. Also notice that the cover graph of a graph with the all-positive signature is just the disjoint union of two copies of the graph.

Let $G^\sigma = (G, \sigma)$ be a signed graph. Let us denote by G^σ_+ (resp. G^σ_-) the unsigned graph induced by the positive edges (resp. negative edges) of G^σ. With a suitable ordering of the vertices of $L(G^\sigma)$, we have:

$$A_{G^\sigma} = A_{G^\sigma_+} - A_{G^\sigma_-}, \qquad A_G = A_{G^\sigma_+} + A_{G^\sigma_-},$$

$$A_{L(G^\sigma)} = \left(\begin{array}{c|c} A_{G^\sigma_+} & A_{G^\sigma_-} \\ \hline A_{G^\sigma_-} & A_{G^\sigma_+} \end{array}\right) = \left(\begin{array}{c|c} I_n & I_n \\ \hline I_n & -I_n \end{array}\right)^{-1} \left(\begin{array}{c|c} A_G & 0_n \\ \hline 0_n & A_{G^\sigma} \end{array}\right) \left(\begin{array}{c|c} I_n & I_n \\ \hline I_n & -I_n \end{array}\right),$$

where I_n and 0_n are, respectively, the $n \times n$ identity matrix and the $n \times n$ zero matrix, where $|V_G| = n$. It follows that

$$\Phi(L(G^\sigma), x) = \Phi(G, x)\Phi(G^\sigma, x). \tag{2.2}$$

Actually, Eq. (2.2) can be also seen as a consequence of the more general theory about the spectrum of a *gain graph* (a graph whose edges are labeled by an element of a group) with respect to a unitary representation of the group of labels (see [7, Section 6]).

Let G_1 and G_2 be (unsigned) graphs such that there exists a 2-to-1 homomorphism $\phi\colon G_1 \to G_2$, that is, ϕ_V and ϕ_E are surjective 2-to-1 maps. Then, as a consequence of the theory existing in the setting of voltage graphs [19], there exists a signature σ of G_2 such that $G_1 \cong L(G_2^\sigma)$.

The Coefficient Theorem (also known as Sachs formula) for weighted graphs [11, Eq. (1.35)"] can be applied to signed graphs. This has already been done in the case of a simple underlying graph (see [3, for instance]). The Coefficient Theorem provides a description of the coefficients of the characteristic polynomial of a signed graph through its basic figures. Let K_2 denote the complete graph on 2 vertices and, for each $i \geq 1$, let C_i denote the cyclic graph on i vertices, with the convention that C_1 consists of a unique vertex endowed with a loop. For a signed graph G^σ let us denote by $B(G^\sigma)$ the set of subgraphs of G^σ that are *basic figures*, that is, $f \in B(G^\sigma)$ is a (possibly empty) union of disjoint subgraphs of G^σ whose underlying graphs are isomorphic to K_2 or C_i, with $i \geq 1$. Let us denote by $B_i(G^\sigma)$ the basic figures in G^σ with i vertices, and, for each $f \in B_i(G^\sigma)$, denote by $p(f)$, $c(f)$, and $\sigma(f)$ the number of connected components, the number of cycles, and the product of the sign of the cycles of f, respectively.

Theorem 2.2 (Coefficient Theorem)**.** *Let G^σ be a signed graph on n vertices with characteristic polynomial $\Phi(G^\sigma, x) = \sum_{k=0}^n p_k x^{n-k}$. Then, for each $k = 0, 1, \ldots, n$:*

$$p_k = \sum_{f \in B_k(G^\sigma)} (-1)^{p(f)} 2^{c(f)} \sigma(f).$$

3. Basilica Schreier graphs

This brief section is devoted to the definition of Basilica Schreier graphs and to the description of their main structural properties.

Definition 3.1. *The n-th Schreier graph $\Gamma_n = (V_{\Gamma_n}, E_{\Gamma_n})$ of the action of the Basilica group B on the rooted binary tree T_2, with respect to the symmetric generating set $\Sigma = \{a^{\pm 1}, b^{\pm 1}\}$, is the graph whose vertex set is $\{0,1\}^n$, where two vertices u and v are adjacent if and only if there exists $s \in \Sigma$ such that $s(u) = v$. If this is the case, the edge joining u and v is labeled by s.*

We will refer to the graph Γ_n as the n-th Basilica Schreier graph. Notice that Γ_n is a regular graph of degree 4 on 2^n vertices and it is connected, since the action of Σ is level-transitive. For each $n \geq 1$, let $\pi_{n+1} : \Gamma_{n+1} \longrightarrow \Gamma_n$ be the map defined on $V_{\Gamma_{n+1}}$ as

$$\pi_{n+1}(x_1 \cdots x_n x_{n+1}) = x_1 \cdots x_n.$$

This map induces a surjective homomorphism of Γ_{n+1} onto Γ_n, which is a graph covering of degree 2. In Corollary 4.2, we will exhibit explicitly the signature σ on Γ_n inducing such graph covering.

Observe that the graph Γ_n can also be described as a multigraph $(V_{\Gamma_n}, E_{\Gamma_n}, r_{\Gamma_n})$, with

$$V_{\Gamma_n} = \{0,1\}^n, \qquad E_{\Gamma_n} = \{0,1\}^n \times \{a, b\}, \qquad r(u,s) = \{u, s(u)\} \text{ for each } u \in \{0,1\}^n, s \in \{a,b\}.$$

In particular, the edges which are incident to the vertex u are precisely (u,a), (u,b), $(a^{-1}(u), a)$, $(b^{-1}(u), b) \in E_{\Gamma_n} = \{0,1\}^n \times \{a,b\}$.

In [13] finite and infinite Schreier graphs of the Basilica group have been studied and classified. In Fig. 3 we list three substitutional rules, consisting in substituting the labeled edges of Γ_n together with their extremal vertices by some labeled subgraphs, allowing to recursively construct Γ_{n+1} starting from Γ_n.

Recall that a connected graph G is *separable* if it can be disconnected by removing only one vertex: such a vertex is called a *cut* vertex. Here below we list some properties of the graphs Γ_n, $n \geq 1$ (see for instance, [10, 13, 14]).

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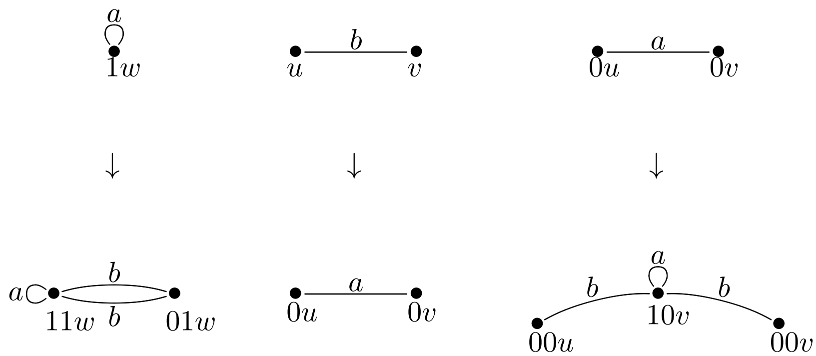


FIGURE 3. Substitutional rules for Γ_n.

(1) For every $n \geq 1$, Γ_n is a separable graph with a cactus structure, that is, it is the union of cycles (each having length a power of 2) arranged in a tree-like way; the central cycle, which is the unique cycle containing both the vertices 0^n and $0^{n-1}1$, has always maximal length, equal to $2^{\lceil \frac{n}{2} \rceil}$.

(2) Every vertex without a loop in Γ_n is a cut vertex. Removing any cut vertex disconnects Γ_n into exactly two connected components.

(3) For each $n \geq 2$, the graph Γ_n contains exactly 2^{n-1} loops rooted at the vertices corresponding to words over the alphabet $\{0, 1\}$ starting by 1, since the action of the generator a is trivial on these words.

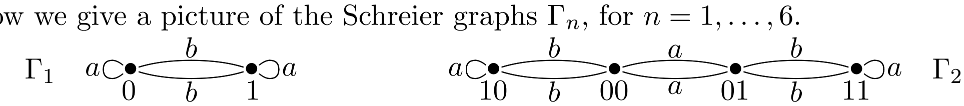
(4) For any $n \geq 4$, the number $c_{n,i}$ of cycles of length 2^i in Γ_n is:

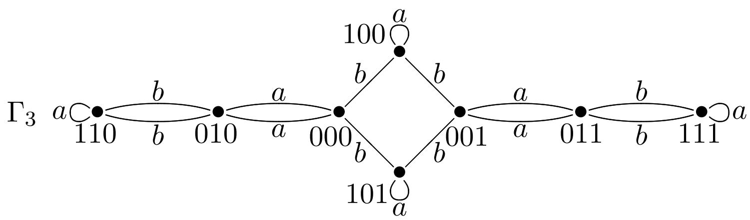
$$c_{n,i} = \begin{cases} 3 \cdot 2^{n-2i-1} & \text{for } 1 \leq i \leq \frac{n}{2} - 1 \\ 3 & \text{for } i = \frac{n}{2}, \end{cases} \qquad \text{for } n \text{ even,}$$

and

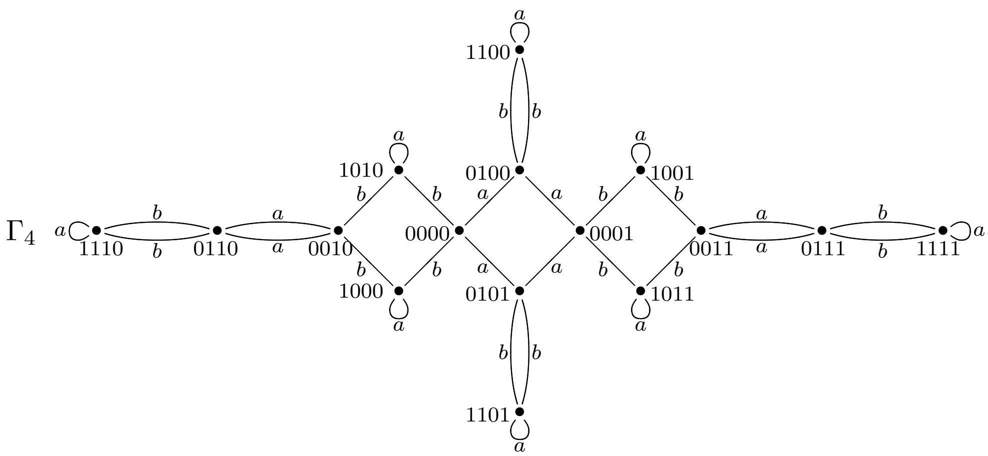
$$c_{n,i} = \begin{cases} 3 \cdot 2^{n-2i-1} & \text{for } 1 \leq i \leq \frac{n-1}{2} - 1 \\ 4 & \text{for } i = \frac{n-1}{2} \\ 1 & \text{for } i = \frac{n+1}{2}, \end{cases} \qquad \text{for } n \text{ odd.}$$

Here below we give a picture of the Schreier graphs Γ_n, for $n = 1, \ldots, 6$.

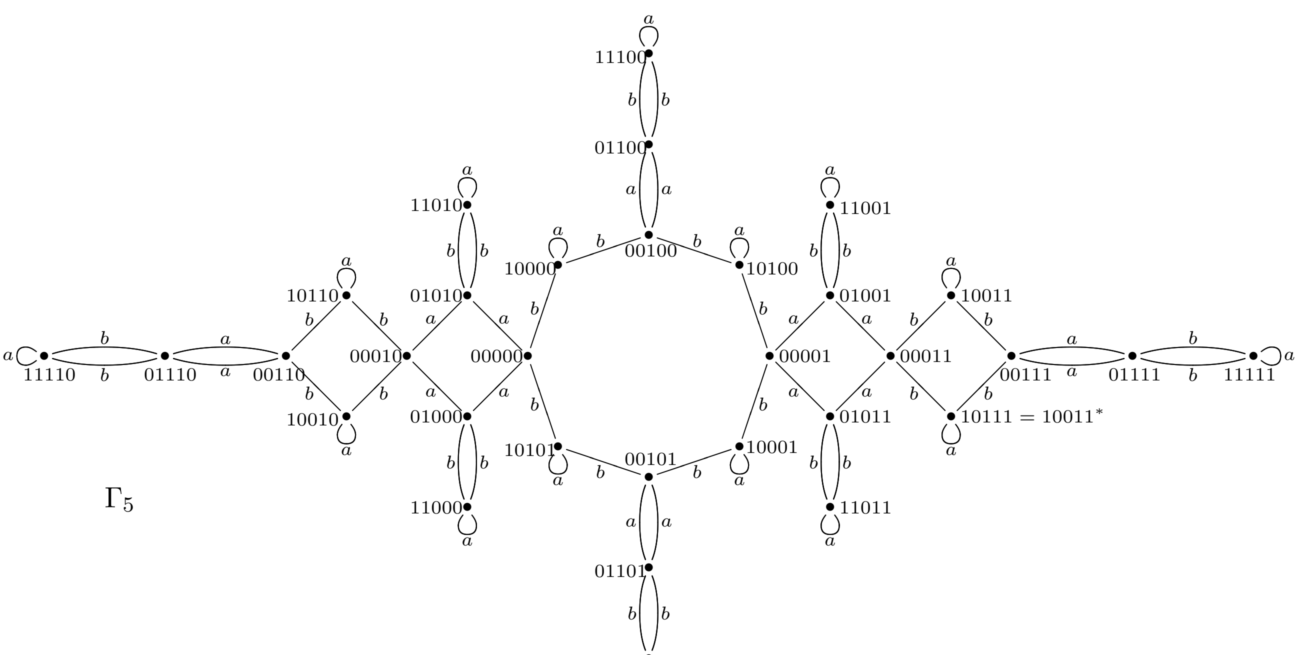




Γ_3
100
110
010
000
001
011
111
101



Γ_4
1100
1010
0100
1001
1110
0110
0010
0000
0001
0011
0111
1111
1000
0101
1011
1101



Γ_5
11100
01100
11010
11001
10000
00100
10100
10110
01010
01001
10011
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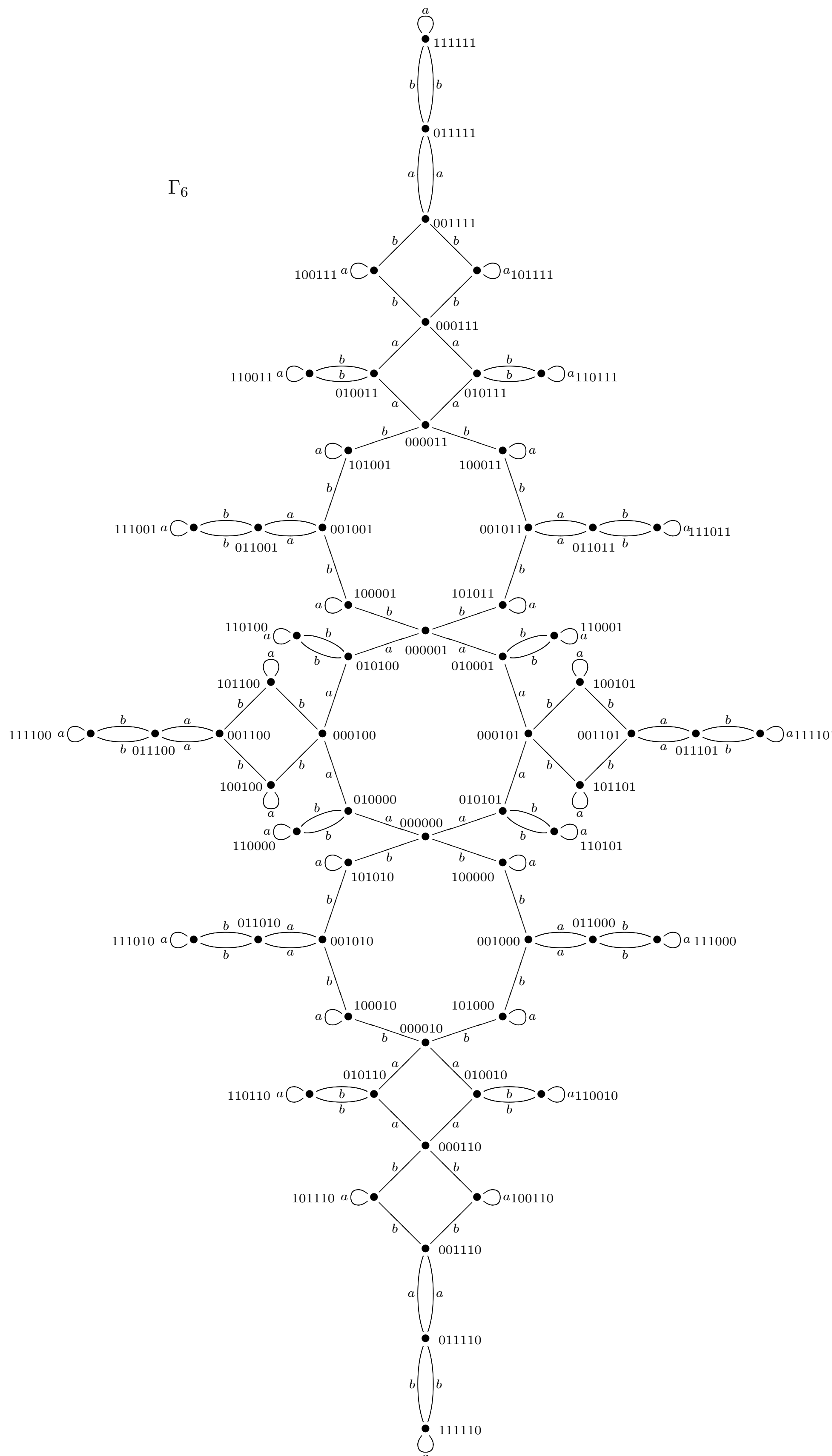

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The remaining part of the paper is devoted to the investigation of the characteristic polynomial $\Phi(\Gamma_n, x)$ of the Basilica Schreier graph Γ_n and its roots. Notice that, although Basilica Schreier graphs are defined as labeled graphs, we will consider them as unlabeled graphs in our spectral investigation. Observe that the spectrum of Γ_n is always contained in the interval $[-4, 4]$, since Γ_n is a regular graph of degree 4.

4. The characteristic polynomial of the Basilica Schreier graphs

Let $\mathbb{F}_2$ denote the free group on two generators. Notice that, for any element $x \in \{0,1\}^n$ and so for any vertex of Γ_n, there exists a word $w \in \mathbb{F}_2$ such that $x = w(a,b)0^n$. Roughly speaking, it is always possible to get x starting from 0^n by letting act finitely many times elements in $\{a^{\pm 1}, b^{\pm 1}\}$. Let us put $x^* = (w(a,b)0^n)^* = w(a^{-1}, b^{-1})0^n$. The involution $*$ on V_{Γ_n} extends to an automorphism of order two of Γ_n, that can be easily interpreted, by choosing a suitable plan embedding, as the symmetry with respect to the horizontal axis passing through the most external vertices 1^n and $1^{n-1}0$, and dividing the graph Γ_n into two isomorphic halves (top and bottom). As an example, if we look at the graph Γ_5, then for $x = 10011$ one has $x^* = 10111$.

We denote by $d_1(\Gamma_n)$ and $d_2(\Gamma_n)$ two special elements of E_{Γ_n}, always contained in the central cycle of Γ_n, and which are opposite to each other, defined as:

$$d_1(\Gamma_n) := \begin{cases} \{0^n, a(0^n) = (01)^{\frac{n}{2}}\} & \text{if } n \text{ is even} \\ \{0^n, b(0^n) = (10)^{\frac{n-1}{2}}1\} & \text{if } n \text{ is odd} \end{cases}$$

$$d_2(\Gamma_n) := \begin{cases} \{0^{n-1}1, a(0^{n-1}1) = (01)^{\frac{n}{2}-1}0^2\} & \text{if } n \text{ is even} \\ \{0^{n-1}1, b(0^{n-1}1) = (10)^{\frac{n-1}{2}}0\} & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, we denote by $m_1(\Gamma_n)$ and $m_2(\Gamma_n)$ two special elements of E_{Γ_n}, always contained in the cycle in the left part of Γ_n containing the vertex 0^n, and which are opposite to each other, defined as:

$$m_1(\Gamma_n) := \begin{cases} \{0^n, b(0^n) = (10)^{\frac{n}{2}}\} & \text{if } n \text{ is even} \\ \{0^n, a(0^n) = (01)^{\frac{n-1}{2}}0\} & \text{if } n \text{ is odd} \end{cases}$$

$$m_2(\Gamma_n) = \begin{cases} \{0^{n-2}10, b(0^{n-2}10) = (10)^{\frac{n}{2}-1}0^2\} & \text{if } n \text{ is even} \\ \{0^{n-2}10, a(0^{n-2}10) = (01)^{\frac{n-3}{2}}0^3\} & \text{if } n \text{ is odd.} \end{cases}$$

We denote by $C_1(\Gamma_n)$ the central cycle of Γ_n, which contains $d_1(\Gamma_n)$ and $d_2(\Gamma_n)$, and has length $2^{\lceil \frac{n}{2} \rceil}$; and by $C_2(\Gamma_n)$ the cycle containing $m_1(\Gamma_n)$ and $m_2(\Gamma_n)$, which has length $2^{\lfloor \frac{n}{2} \rfloor}$. See Fig. 4 and Fig. 5 for the graph Γ_5 and Γ_6, respectively.

For an edge $e = \{x, y\} \in E_{\Gamma_n}$ and a word $w \in \{0,1\}^k$ we put $ew = \{xw, yw\}$ where, in general, ew is not necessarily an edge in Γ_{n+k}. For a subset $F \subset E_{\Gamma_n}$ we set $Fw = \{ew : e \in F\}$. Then the following lemma holds.

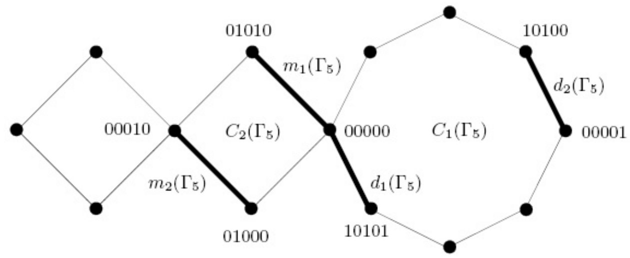

FIGURE 4. A portion of Γ_5 containing the edges $d_1(\Gamma_5), d_2(\Gamma_5), m_1(\Gamma_5), m_2(\Gamma_5)$.

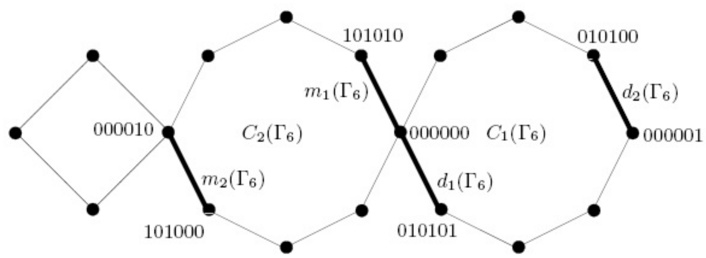

FIGURE 5. A portion of Γ_6 containing the edges $d_1(\Gamma_6), d_2(\Gamma_6), m_1(\Gamma_6), m_2(\Gamma_6)$.

Lemma 4.1. *Let $\Gamma_{n-1}^0 = (V_{n-1}^0, E_{n-1}^0)$ and $\Gamma_{n-1}^1 = (V_{n-1}^1, E_{n-1}^1)$ be two isomorphic copies of the graph Γ_{n-1}. Then:*

$$E_{\Gamma_n} = \left(E_{n-1}^0 \setminus \{m_1\left(\Gamma_{n-1}^0\right)\}\right) 0 \bigcup \left(E_{n-1}^1 \setminus \{m_1\left(\Gamma_{n-1}^1\right)\}\right) 1 \bigcup \{d_1\left(\Gamma_n\right), d_2\left(\Gamma_n\right)\}.$$

Proof. An edge in Γ_{n-1} can be written as $\{u, s(u)\}$ with $u \in \{0,1\}^{n-1}$ and $s \in \{a, b\}$. If the associated directed path in the Moore diagram (starting from the state s and reading the word u) ends in the state a or id, we have that $s(u0) = s(u)0$ and $s(u1) = s(u)1$ and then $\{u0, s(u)0\}$ and $\{u1, s(u)1\}$ are edges of Γ_n. On the other hand, the only directed path of length $n-1$ in the Moore diagram ending in the state b is that alternating the state a and b, associated with the edge $m_1(\Gamma_{n-1})$. It follows that

$$\left(E_{n-1}^0 \setminus \{m_1\left(\Gamma_{n-1}^0\right)\}\right) 0 \subset E_{\Gamma_n} \quad \text{and} \quad \left(E_{n-1}^1 \setminus \{m_1\left(\Gamma_{n-1}^1\right)\}\right) 1 \subset E_{\Gamma_n}.$$

Finally, for cardinality reasons we know that two edges of E_{Γ_n} are missing, and $d_1\left(\Gamma_n\right)$ and $d_2\left(\Gamma_n\right)$ are the only edges whose endpoints end with different letters. This completes the proof. □

In Fig. 6 the construction of Lemma 4.1 is depicted for Γ_5.
We denote by $\Gamma_n^- = (\Gamma_n, \sigma)$ the signed graph with underlying graph Γ_n and such that

$$\sigma(e) = \begin{cases} +1 & \text{if } e \neq m_1(\Gamma_n) \\ -1 & \text{if } e = m_1(\Gamma_n) \end{cases} \qquad \text{for each } e \in E_{\Gamma_n}.$$

Also, recall that the unsigned graph Γ_n can be regarded as a signed graph endowed with the all-positive signature.

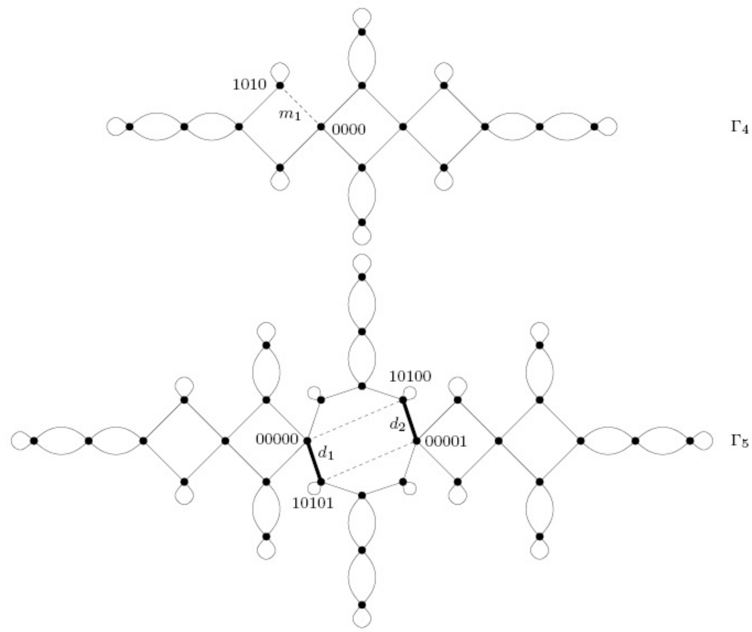

FIGURE 6. The construction of Lemma 4.1 for Γ_5.

Corollary 4.2. *We have*

$$\Gamma_n \cong L(\Gamma_{n-1}^-).$$

In particular $\Phi(\Gamma_n, x) = \Phi(\Gamma_{n-1}, x) \cdot \Phi(\Gamma_{n-1}^-, x)$.

Proof. We are going to define an isomorphism $\phi = (\phi_V, \phi_E)$ between $L(\Gamma_{n-1}^-)$ and Γ_n. We set

$$\phi_V : V_{L(\Gamma_{n-1}^-)} \to V_{\Gamma_n}$$

such that $\phi_V((u, +1)) = u0$ and $\phi_V((u, -1)) = u1$, for all $u \in \{0,1\}^{n-1}$. We set

$$\phi_E : E_{L(\Gamma_{n-1}^-)} \to E_{\Gamma_n}$$

such that $\phi_E((e, +1)) = e0$, and $\phi_E((e, -1)) = e1$ for all $e \in E_{\Gamma_{n-1}^-} \setminus \{m_1(\Gamma_{n-1}^-)\}$; and such that $\phi_E((m_1(\Gamma_{n-1}^-), +1)) = d_1(\Gamma_n)$ and $\phi_E((m_1(\Gamma_{n-1}^-), -1)) = d_2(\Gamma_n)$. By using Lemma 4.1 it is possible to prove that ϕ is a (multigraph) isomorphism. The claim about the decomposition of the characteristic polynomial is a consequence of Eq. (2.2). □

It follows that the spectrum of Γ_n is the union of the spectra of the graphs Γ_{n-1} and Γ_{n-1}^- and that

$$\Phi(\Gamma_n, x) = \Phi(\Gamma_1, x) \cdot \prod_{i=1}^{n-1} \Phi(\Gamma_i^-, x).$$

Now we are going to decompose Γ_n in a different way, in order to obtain one more independent relation between the characteristic polynomials of signed and unsigned Basilica graphs.

Definition 4.3. *Let* $\Gamma^{10}_{n-2} = (V^{10}_{n-2}, E^{10}_{n-2})$ *and* $\Gamma^{00}_{n-2} = (V^{00}_{n-2}, E^{00}_{n-2})$ *be two isomorphic copies of* Γ_{n-2}*, and let* $\Gamma^{1}_{n-1} = (V^{1}_{n-1}, E^{1}_{n-1})$ *be a copy of* Γ_{n-1}*. We define a bijection*

$$\rho\colon V^{10}_{n-2} \cup V^{00}_{n-2} \cup V^{1}_{n-1} \longrightarrow V_n \quad s.t. \quad \rho(x) = \begin{cases} x10 & \text{if } x \in V^{10}_{n-2} \\ x00 & \text{if } x \in V^{00}_{n-2} \\ x1 & \text{if } x \in V^{1}_{n-1}. \end{cases}$$

Moreover, we put:

$$\overline{E^{10}_{n-2}} := E^{10}_{n-2} \setminus \{m_1(\Gamma^{10}_{n-2})\}, \qquad \overline{\Gamma^{10}_{n-2}} := (V^{10}_{n-2}, \overline{E^{10}_{n-2}});$$
$$\overline{E^{00}_{n-2}} := E^{00}_{n-2} \setminus \{m_1(\Gamma^{00}_{n-2}), d_1(\Gamma^{00}_{n-2})\}, \qquad \overline{\Gamma^{00}_{n-2}} := (V^{00}_{n-2}, \overline{E^{00}_{n-2}});$$
$$\overline{E^{1}_{n-1}} := E^{1}_{n-1} \setminus \{m_1(\Gamma^{1}_{n-1})\}, \qquad \overline{\Gamma^{1}_{n-1}} := (V^{1}_{n-1}, \overline{E^{1}_{n-1}});$$

and we denote by $\overline{\Gamma_n}$ *the graph obtained from* Γ_n *by removing the edges* $m_1(\Gamma_n), m_2(\Gamma_n), d_1(\Gamma_n)$ *and* $d_2(\Gamma_n)$*.*

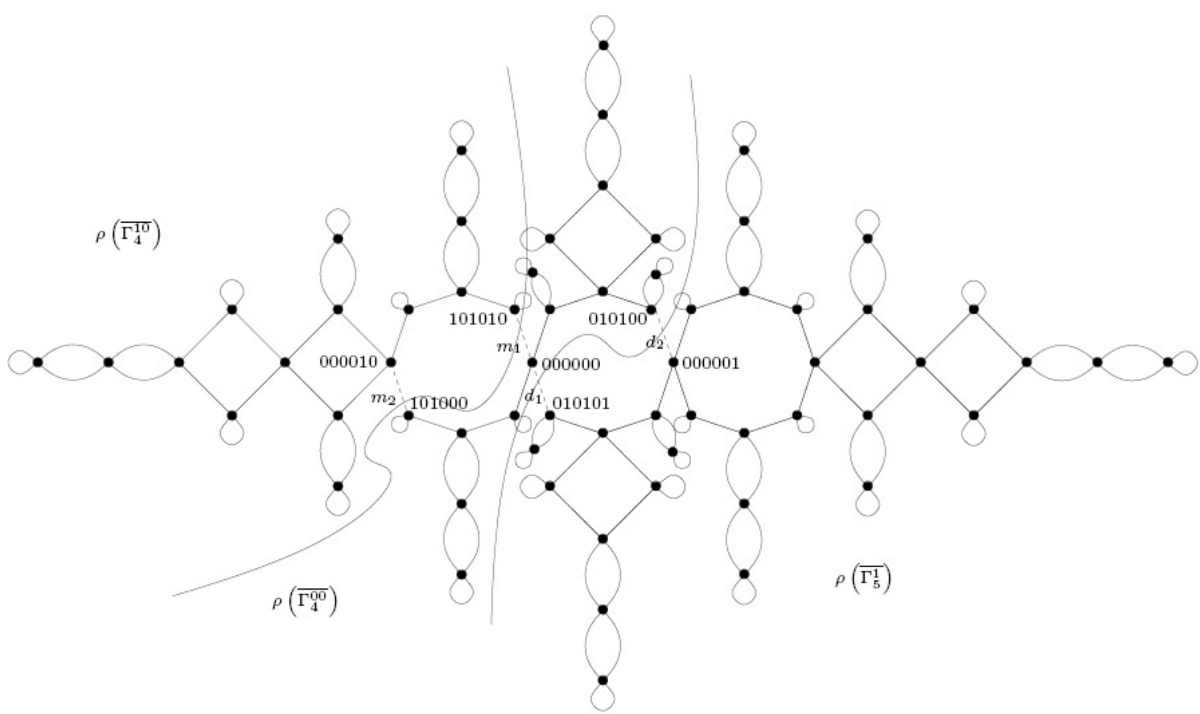

FIGURE 7. The construction of Proposition 4.4 for Γ_6.

Proposition 4.4. *(i) The map* ρ *induces a graph isomorphism between* $\overline{\Gamma_n}$ *and the disjoint union of the graphs* $\overline{\Gamma^{10}_{n-2}}$, $\overline{\Gamma^{00}_{n-2}}$ *and* $\overline{\Gamma^{1}_{n-1}}$*. In particular, one has:*

$$E_{\Gamma_n} = \overline{E^{10}_{n-2}}10 \cup \overline{E^{00}_{n-2}}00 \cup \overline{E^{1}_{n-1}}1 \cup \{m_1(\Gamma_n), m_2(\Gamma_n), d_1(\Gamma_n), d_2(\Gamma_n)\}.$$

(ii) The map ρ *induces a bijection between* $B(\overline{\Gamma^{10}_{n-2}}) \times B(\overline{\Gamma^{00}_{n-2}}) \times B(\overline{\Gamma^{1}_{n-1}})$ *and* $B(\overline{\Gamma_n})$*.*

(iii) *The image under* ρ *of the union of the vertices of* $C_2(\Gamma^{10}_{n-2})$ *and* $C_2(\Gamma^{00}_{n-2})$ *is the set of vertices of* $C_2(\Gamma_n)$.

(iv) *The image under* ρ *of the union of the vertices of* $C_1(\Gamma^{00}_{n-2})$ *and* $C_2(\Gamma^{1}_{n-1})$ *is the set of vertices of* $C_1(\Gamma_n)$.

Proof. (i) As in the proof of Lemma 4.1, if the directed path in the Moore diagram starting from the state $s \in \{a,b\}$ with a word $u \in \{0,1\}^{n-1}$ ends in the state a or id, we have that $s(u1) = s(u)1$. The only edge of Γ_{n-1} associated with a directed path ending in the state b is $m_1(\Gamma_{n-1})$. As a consequence, if $\{u, s(u)\} \in \overline{E^1_{n-1}}$ then $\{u1, s(u)1\} \in E_{\Gamma_n}$.

Moreover, if the directed path in the Moore diagram starting from the state $s \in \{a,b\}$ with a word $u \in \{0,1\}^{n-2}$ ends in the state a or id, we have that $s(u10) = s(u)10$. The only edge of Γ_{n-2} associated with a directed path ending in the state b is $m_1(\Gamma_{n-2})$. As a consequence, if $\{u, s(u)\} \in \overline{E^{10}_{n-2}}$ then $\{u10, s(u)10\} \in E_{\Gamma_n}$. On the other hand, if the directed path ends in the state id, also $s(u00) = s(u)00$. The only edges of Γ_{n-2} associated with directed paths not ending in id are $m_1(\Gamma_{n-2})$ and $d_1(\Gamma_{n-2})$. As a consequence, if $\{u, s(u)\} \in \overline{E^{00}_{n-2}}$ then $\{u00, s(u)00\} \in E_{\Gamma_n}$. Finally, the only edges of Γ_n not having both endpoints ending with 00, or 10, or 1, are exactly $m_1(\Gamma_n)$, $m_2(\Gamma_n)$, $d_1(\Gamma_n)$ and $d_2(\Gamma_n)$. For cardinality reasons the assertion (i) follows.

(ii) It is a direct consequence of (i). (iii) Suppose that $n = 2k$. The set $V(C_2(\Gamma_{2k}))$ of vertices of $C_2(\Gamma_{2k})$ is such that $V(C_2(\Gamma_{2k})) = \{b^i(0^n),\, i \geq 0\}$, and $|V(C_2(\Gamma_{2k}))| = 2^k$. We are going to prove that

$$V(C_2(\Gamma_{2k})) = \{00, 10\}^k. \tag{4.1}$$

For cardinality reasons, it is enough to prove, by induction on i, that $b^i(0^n) \in \{00,10\}^k$. For $i = 0$ this is trivial, since $0^{2k} \in \{00,10\}^k$. Suppose $b^{i-1}(0^n) \in \{00,10\}^k$, so that we have to prove that $b(b^{i-1}(0^n)) \in \{00,10\}^k$. This is true since, by Eq. (2.1),

$$\nu(b,00) = 10,\ \mu(b,00) = b \quad \text{and} \quad \nu(b,10) = 00,\ \mu(b,10) = id,$$

and then b maps $\{00,10\}^k$ to itself. Being $\{00,10\}^k = \{00,10\}^{k-1}00 \cup \{00,10\}^{k-1}10$, we have:

$$V(C_2(\Gamma_{2k})) = V(C_2(\Gamma_{2k-2}))00 \cup V(C_2(\Gamma_{2k-2}))10.$$

When $n = 2k+1$ we have that $V(C_2(\Gamma_{2k+1})) = \{a^i(0^n),\, i \geq 0\}$. Since $a^i(0^{2k+1}) = 0b^i(0^{2k})$, it follows that

$$\begin{aligned} V(C_2(\Gamma_{2k+1})) &= 0V(C_2(\Gamma_{2k})) = 0\,(V(C_2(\Gamma_{2k-2}))00 \cup V(C_2(\Gamma_{2k-2}))10) \\ &= 0V(C_2(\Gamma_{2k-2}))00 \cup 0V(C_2(\Gamma_{2k-2}))10 = V(C_2(\Gamma_{2k-1}))00 \cup V(C_2(\Gamma_{2k-1}))10. \end{aligned}$$

(iv) Suppose $n = 2k+1$. Then $V(C_1(\Gamma_{2k+1})) = \{b^i(0^n),\, i \geq 0\}$. We have

$$V(C_1(\Gamma_{2k+1})) = \{00,10\}^k 0 \cup \{00,10\}^k 1. \tag{4.2}$$

In fact, $0^n \in \{00,10\}^k 0$ and, if $u \in \{00,10\}^k 0 \cup \{00,10\}^k 1$, then $b(u) \in \{00,10\}^k 0 \cup \{00,10\}^k 1$, so that one can prove Eq. (4.2) by induction as in the even case of (iii). Moreover, it is not difficult to check that

$$\{00,10\}^k 0 = (\{00,10\}^{k-1}0 \cup \{00,10\}^{k-1}1)00 = V(C_1(\Gamma_{n-2}))00;$$

then combining with Eq. (4.1) and Eq. (4.2) we obtain

$$V(C_1(\Gamma_{2k+1})) = V(C_1(\Gamma_{2k-1}))00 \cup V(C_2(\Gamma_{2k}))1 \tag{4.3}$$

and the thesis follows.

Finally, suppose $n = 2k$. Since $a^i(0^{2k}) = 0b^i(0^{2k-1})$, we have $V(C_1(\Gamma_{2k})) = 0V(C_1(\Gamma_{2k-1}))$. Combining with Eq. (4.3) one has

$$\begin{aligned} V(C_1(\Gamma_{2k})) = 0V(C_1(\Gamma_{2k-1})) &= 0V(C_1(\Gamma_{2k-3}))00 \cup 0V(C_2(\Gamma_{2k-2}))1 \\ &= V(C_1(\Gamma_{2k-2}))00 \cup V(C_2(\Gamma_{2k-1}))1, \end{aligned}$$

and the thesis follows. □

In the example of Fig. 7, it is possible to see the decomposition of the edge set of Γ_6 as

$$E_{\Gamma_6} = \rho\left(\overline{E_4^{10}}\right) \cup \rho\left(\overline{E_4^{00}}\right) \cup \rho\left(\overline{E_5^{1}}\right) \cup \{m_1(\Gamma_6), m_2(\Gamma_6), d_1(\Gamma_6), d_2(\Gamma_6)\}.$$

For a subset $F \subseteq B(\Gamma_n)$ of basic figures, and for a basic figure f, we put

$$F^f := \{g \in F : f \text{ is a subgraph of } g\}.$$

Now we define:

$$B^-(\Gamma_n) := B(\Gamma_n)^{C_2(\Gamma_n)} = \{f \in B(\Gamma_n) : C_2(\Gamma_n) \subseteq f\}; \qquad B^+(\Gamma_n) := B(\Gamma_n) \setminus B^-(\Gamma_n).$$

In other words, $B^-(\Gamma_n)$ is the set of basic figures in Γ_n which contain the cycle $C_2(\Gamma_n)$ as a subgraph, whereas $B^+(\Gamma_n)$ is the set of basic figures in Γ_n whose intersection with $C_2(\Gamma_n)$ is either empty, or a proper subgraph of $C_2(\Gamma_n)$. Moreover, we set:

$$B_i^-(\Gamma_n) := B_i(\Gamma_n) \cap B^-(\Gamma_n); \qquad B_i^+(\Gamma_n) := B_i(\Gamma_n) \setminus B_i^-(\Gamma_n).$$

For a basic figure $f \in B(\Gamma_n)$, we put

$$W(f) := (-1)^{p(f)} 2^{c(f)}.$$

Theorem 4.5. *There exists a map* $\varphi\colon B^-(\Gamma_{n-2}^{10}) \times B^-(\Gamma_{n-2}^{00}) \times B^+(\Gamma_{n-1}^{1}) \to B^-(\Gamma_n)$ *such that:*

(i) φ *is a bijection;*

(ii) for each $k = 0, 1, \ldots, 2^n$*:*

$$B_k^-(\Gamma_n) = \bigcup_{\substack{l \in \{0,\ldots,2^{n-1}\} \\ i,j \in \{0,\ldots,2^{n-2}\}: \\ i+j+l=k}} \varphi(B_i^-(\Gamma_{n-2}^{10}) \times B_j^-(\Gamma_{n-2}^{00}) \times B_l^+(\Gamma_{n-1}^{1}));$$

(iii) for each $(f_1, f_2, f_3) \in B^-(\Gamma_{n-2}^{10}) \times B^-(\Gamma_{n-2}^{00}) \times B^+(\Gamma_{n-1}^{1})$*:*

$$W(\varphi(f_1, f_2, f_3)) = -\frac{1}{2} W(f_1) W(f_2) W(f_3).$$

Proof. Let us start by proving Claim (i). As a first step, we construct a bijection

$$\varphi_0 \colon B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times \left(B^+(\Gamma^1_{n-1}) \setminus \left(B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}\right)\right) \to B^-(\Gamma_n) \setminus \left(B^-(\Gamma_n)^{d_2(\Gamma_n)}\right)$$

in the following way:

$$\varphi_0(f_1, f_2, f_3) := C_2(\Gamma_n) \cup \rho\left(f_1 \setminus C_2(\Gamma^{10}_{n-2})\right) \cup \rho\left(f_2 \setminus C_2(\Gamma^{00}_{n-2})\right) \cup \rho(f_3).$$

Notice that the edges of $f_1 \setminus C_2(\Gamma^{10}_{n-2})$ are in $\overline{E^{10}_{n-2}}$, the edges of $f_2 \setminus C_2(\Gamma^{00}_{n-2})$ are in $\overline{E^{00}_{n-2}}$, since f_2, containing $C_2(\Gamma^{00}_{n-2})$, cannot contain $d_1(\Gamma^{00}_{n-2})$. By construction, the edges of f_3 are in $\overline{E^1_{n-1}}$. As a consequence of Proposition 4.4, $\rho\left(f_1 \setminus C_2(\Gamma^{10}_{n-2})\right) \cup \rho\left(f_2 \setminus C_2(\Gamma^{00}_{n-2})\right) \cup \rho(f_3)$ is a basic figure.

Now in order to prove that $\varphi_0(f_1, f_2, f_3)$ is a basic figure, it is enough to notice that, by Proposition 4.4, the set of vertices of $\rho\left(f_1 \setminus C_2(\Gamma^{10}_{n-2})\right) \cup \rho\left(f_2 \setminus C_2(\Gamma^{00}_{n-2})\right) \cup \rho(f_3)$ is disjoint from the set of vertices of $C_2(\Gamma_n)$. As a consequence, $\varphi_0(f_1, f_2, f_3)$ is a basic figure in $B^-(\Gamma_n)$. Moreover, by construction, the edge $d_2(\Gamma_n)$ cannot appear as an edge of the basic figure $\varphi_0(f_1, f_2, f_3)$.
Notice that, if $f_3 \in B^+(\Gamma^1_{n-1})$ contains $m_1(\Gamma^1_{n-1})$, then the reflected figure f_3^* does not and it is in the domain of φ_0. We are now in position to define the required map φ as:

$$\varphi(f_1, f_2, f_3) := \begin{cases} \varphi_0(f_1, f_2, f_3), & \text{if } m_1(\Gamma^1_{n-1}) \nsubseteq f_3, \\ \varphi_0(f_1^*, f_2^*, f_3^*)^* & \text{if } m_1(\Gamma^1_{n-1}) \subseteq f_3. \end{cases}$$

Notice that if $m_1(\Gamma^1_{n-1}) \subseteq f_3$, the basic figure $\varphi(f_1, f_2, f_3) = \varphi_0(f_1^*, f_2^*, f_3^*)^*$ certainly contains $d_2(\Gamma_n)$. By construction and by virtue of Proposition 4.4, the restrictions of the map φ to $B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times \left(B^+(\Gamma^1_{n-1}) \setminus \left(B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}\right)\right)$ and to $B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}$ are injective. Since their images are disjoint (the first one does not contain $d_2(\Gamma_n)$ but the second one does), the map φ is injective.

Now we have to prove that φ is surjective. If $f \in B^-(\Gamma_n) \setminus \left(B^-(\Gamma_n)^{d_2(\Gamma_n)}\right)$, we have that $f \setminus C_2(\Gamma_n)$ does not contain any edge among $m_1(\Gamma_n), m_2(\Gamma_n), d_1(\Gamma_n)$ and $d_2(\Gamma_n)$, and then, by Proposition 4.4, the element $(g_1, g_2, g_3) := \rho^{-1}(f \setminus C_2(\Gamma_n))$ is well defined, where:

- g_1 is a basic figure of $\overline{\Gamma^{10}_{n-2}}$, with no vertex in $C_2(\Gamma^{10}_{n-2})$,
- g_2 is a basic figure of $\overline{\Gamma^{00}_{n-2}}$, with no vertex in $C_2(\Gamma^{00}_{n-2})$,
- g_3 is a basic figure of $\overline{\Gamma^1_{n-1}}$, and in particular $g_3 \in \left(B^+(\Gamma^1_{n-1}) \setminus \left(B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}\right)\right)$.

Defining $(f_1, f_2, f_3) := (g_1 \cup C_2(\Gamma^{10}_{n-2}), g_2 \cup C_2(\Gamma^{00}_{n-2}), g_3)$, we have that $(f_1, f_2, f_3) \in B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times \left(B^+(\Gamma^1_{n-1}) \setminus \left(B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}\right)\right)$, and then

$$\varphi(f_1, f_2, f_3) = \varphi_0(f_1, f_2, f_3) = f.$$

On the other hand, if $f \in B^-(\Gamma_n)$ and f contains the edge $d_2(\Gamma_n)$, clearly f^* does not and so, by applying the previous argument, we can find

$$(f_1, f_2, f_3) \in B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times \left(B^+(\Gamma^1_{n-1}) \setminus \left(B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}\right)\right)$$

such that $\varphi_0(f_1, f_2, f_3) = f^*$. By noticing that $d_2(\Gamma_n)^* = \rho(m_1(\Gamma^1_{n-1})^*)$, we have the following chain of implications:

$$d_2(\Gamma_n) \subseteq f \implies d_2(\Gamma_n)^* \subseteq f^* \implies m_1(\Gamma^1_{n-1})^* \subseteq f_3 \implies m_1(\Gamma^1_{n-1}) \subseteq f_3^*,$$

that is, $f_3^* \in B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}$. Moreover $f_1 \in B^-(\Gamma^{10}_{n-2})$ implies $f_1^* \in B^-(\Gamma^{10}_{n-2})$, and similarly $f_1 \in B^-(\Gamma^{00}_{n-2})$ implies $f_1^* \in B^-(\Gamma^{00}_{n-2})$. It follows that $(f_1^*, f_2^*, f_3^*) \in B^-(\Gamma^{10}_{n-2}) \times B^-(\Gamma^{00}_{n-2}) \times B^+(\Gamma^1_{n-1})^{m_1(\Gamma^1_{n-1})}$, that implies

$$\varphi(f_1^*, f_2^*, f_3^*) = \varphi_0(f_1, f_2, f_3)^* = f.$$

The Claims (ii) and (iii) follow from the fact that the number of vertices of $\varphi(f_1, f_2, f_3)$ is exactly the sum of the number of vertices of f_1,f_2 and f_3, and the number of cycles (resp. connected components) of $\varphi(f_1, f_2, f_3)$ is the sum of the numbers of cycles (resp. connected components) of f_1, f_2 and f_3 minus 1. In fact, the two starting cycles $C_2(\Gamma^{10}_{n-2})$ and $C_2(\Gamma^{00}_{n-2})$ glue together to constitute the cycle $C_2(\Gamma_n)$. □

For a polynomial $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_0$, we introduce the notation $[p(x)]_i := p_i$.

Lemma 4.6. *For each $k = 0, 1, \ldots, 2^n$, one has:*

$$[\Phi(\Gamma_n, x)]_{2^n-k} = \sum_{f\in B_k^+(\Gamma_n)} W(f) + \sum_{f\in B_k^-(\Gamma_n)} W(f)$$

$$[\Phi(\Gamma_n^-, x)]_{2^n-k} = \sum_{f\in B_k^+(\Gamma_n)} W(f) - \sum_{f\in B_k^-(\Gamma_n)} W(f).$$

Proof. Being $B_k(\Gamma_n) = B_k^+(\Gamma_n) \cup B_k^-(\Gamma_n)$, by Theorem 2.2 we have

$$\begin{aligned}[\Phi(\Gamma_n, x)]_{2^n-k} &= \sum_{f\in B_k(\Gamma_n)} (-1)^{p(f)}2^{c(f)}\sigma(f)\\ &= \sum_{f\in B_k^+(\Gamma_n)} (-1)^{p(f)}2^{c(f)}\sigma(f) + \sum_{f\in B_k^-(\Gamma_n)} (-1)^{p(f)}2^{c(f)}\sigma(f)\\ &= \sum_{f\in B_k^+(\Gamma_n)} W(f) + \sum_{f\in B_k^-(\Gamma_n)} W(f),\end{aligned}$$

since $\sigma(f) = +1$ for every $f \in B_k(\Gamma_n)$. On the other hand, we also get

$$[\Phi(\Gamma_n^-, x)]_{2^n-k} = \sum_{f\in B_k(\Gamma_n^-)} (-1)^{p(f)}2^{c(f)}\sigma(f) = \sum_{f\in B_k^+(\Gamma_n)} W(f) - \sum_{f\in B_k^-(\Gamma_n)} W(f),$$

since, for a given $f \in B_k(\Gamma_n^-)$, we have $\sigma(f) = 1$ if the corresponding figure in the underlying graph Γ_n belongs to $B_k^+(\Gamma_n)$, while we have $\sigma(f) = -1$ if the corresponding figure in the underlying graph Γ_n belongs to $B_k^-(\Gamma_n)$. □

For each $n \geq 1$, let us introduce the auxiliary polynomials

$$A_n(x) := \Phi(\Gamma_n, x) + \Phi(\Gamma_n^-, x) \qquad\qquad B_n(x) := \Phi(\Gamma_n^-, x) - \Phi(\Gamma_n, x)$$

and put

$$A_n(x) = \sum_{i=0}^{2^n} a_{n,i} x^{2^n - i} \qquad B_n(x) = \sum_{i=0}^{2^n} b_{n,i} x^{2^n - i},$$

so that $[A_n(x)]_{2^n-i} = a_{n,i}$ and $[B_n(x)]_{2^n-i} = b_{n,i}$, for each $i = 0, \ldots, 2^n$. The following lemma holds.

Lemma 4.7. *For each $k = 0, \ldots, 2^n$, one has:*

$$a_{n,k} = 2 \sum_{f \in B_k^+(\Gamma_n)} W(f) \qquad b_{n,k} = -2 \sum_{f \in B_k^-(\Gamma_n)} W(f).$$

Proof. By definition, and by using Lemma 4.6, one has:

$$\begin{aligned} a_{n,k} &= [A_n(x)]_{2^n-k} = [\Phi(\Gamma_n, x) + \Phi(\Gamma_n^-, x)]_{2^n-k} \\ &= [\Phi(\Gamma_n, x)]_{2^n-k} + [\Phi(\Gamma_n^-, x)]_{2^n-k} \\ &= \sum_{f \in B_k^+(\Gamma_n)} W(f) + \sum_{f \in B_k^-(\Gamma_n)} W(f) + \sum_{f \in B_k^+(\Gamma_n)} W(f) - \sum_{f \in B_k^-(\Gamma_n)} W(f) \\ &= 2 \sum_{f \in B_k^+(\Gamma_n)} W(f) \end{aligned}$$

and similarly

$$\begin{aligned} b_{n,k} &= [B_n(x)]_{2^n-k} = [\Phi(\Gamma_n^-, x) - \Phi(\Gamma_n, x)]_{2^n-k} \\ &= [\Phi(\Gamma_n^-, x)]_{2^n-k} - [\Phi(\Gamma_n, x)]_{2^n-k} \\ &= \sum_{f \in B_k^+(\Gamma_n)} W(f) - \sum_{f \in B_k^-(\Gamma_n)} W(f) - \sum_{f \in B_k^+(\Gamma_n)} W(f) - \sum_{f \in B_k^-(\Gamma_n)} W(f) \\ &= -2 \sum_{f \in B_k^-(\Gamma_n)} W(f). \end{aligned}$$

□

Lemma 4.8. *For each $n \geq 3$, one has:*

$$B_n(x) = \frac{1}{8} A_{n-1}(x) B_{n-2}(x)^2.$$

Proof. Fix $k \in \{0, 1, \ldots, 2^n\}$. Then a direct computation, together with Theorem 4.5, gives:

$$\begin{aligned}
\left[A_{n-1}(x)B_{n-2}(x)^2\right]_{2^n-k} &= \left[\left(\sum_{i=0}^{2^{n-1}} a_{n-1,i}x^{2^{n-1}-i}\right)\cdot\left(\sum_{i=0}^{2^{n-2}} b_{n-2,i}x^{2^{n-2}-i}\right)^2\right]_{2^n-k} \\
&= \left[\sum_{l=0}^{2^{n-1}}\sum_{i=0}^{2^{n-2}}\sum_{j=0}^{2^{n-2}} a_{n-1,l}b_{n-2,i}b_{n-2,j}x^{2^n-(l+i+j)}\right]_{2^n-k} \\
&= \sum_{\substack{l\in\{0,\ldots,2^{n-1}\} \\ i,j\in\{0,\ldots,2^{n-2}\}: \\ l+i+j=k}} a_{n-1,l}b_{n-2,i}b_{n-2,j} \\
&= 8\sum_{\substack{l\in\{0,\ldots,2^{n-1}\} \\ i,j\in\{0,\ldots,2^{n-2}\}: \\ l+i+j=k}}\ \sum_{f_1\in B_l^+(\Gamma_{n-1})}\ \sum_{f_2\in B_i^-(\Gamma_{n-2})}\ \sum_{f_3\in B_j^-(\Gamma_{n-2})} W(f_1)W(f_2)W(f_3) \\
&= 8\sum_{\substack{l\in\{0,\ldots,2^{n-1}\} \\ i,j\in\{0,\ldots,2^{n-2}\}: \\ l+i+j=k}}\ \sum_{f_1\in B_l^+(\Gamma_{n-1})}\ \sum_{f_2\in B_i^-(\Gamma_{n-2})}\ \sum_{f_3\in B_j^-(\Gamma_{n-2})} -2W(\varphi(f_1,f_2,f_3)) \\
&= 8\left(-2\sum_{f\in B_k^-(\Gamma_n)} W(f)\right) = 8b_{n,k} = 8[B_n(x)]_{2^n-k}.
\end{aligned}$$

□

Corollary 4.2 and Lemma 4.8 provide the following two relations for the characteristic polynomials of the signed and unsigned Basilica graphs:

$$\begin{cases} \Phi(\Gamma_n,x) = \Phi(\Gamma_{n-1},x)\Phi(\Gamma_{n-1}^-,x) \\ \Phi(\Gamma_n^-,x) - \Phi(\Gamma_n,x) = \frac{1}{8}(\Phi(\Gamma_{n-1}^-,x)+\Phi(\Gamma_{n-1},x))(\Phi(\Gamma_{n-2}^-,x)-\Phi(\Gamma_{n-2},x))^2. \end{cases} \tag{4.4}$$

These two relations are independent and allow to express, in the next theorem, the characteristic polynomial of Γ_n as a function of the characteristic polynomials of Γ_{n-1} and Γ_{n-2}.

Theorem 4.9. *For each $n \geq 3$, one has:*

$$\Phi(\Gamma_{n+1},x) = \Phi(\Gamma_n,x)\left(\Phi(\Gamma_n,x)+\frac{1}{8}\frac{(\Phi(\Gamma_n,x)+\Phi(\Gamma_{n-1},x)^2)(\Phi(\Gamma_{n-1},x)-\Phi(\Gamma_{n-2},x)^2)^2}{\Phi(\Gamma_{n-1},x)\Phi(\Gamma_{n-2},x)^2}\right).$$

Proof. By combining the two relations in Eq. (4.4), we obtain:

$$\begin{aligned}
\Phi(\Gamma_{n+1},x) &= \Phi(\Gamma_n,x)\Phi(\Gamma_n^-,x) \\
&= \Phi(\Gamma_n,x)\left(\Phi(\Gamma_n,x)+\frac{1}{8}(\Phi(\Gamma_{n-1}^-,x)+\Phi(\Gamma_{n-1},x))(\Phi(\Gamma_{n-2}^-,x)-\Phi(\Gamma_{n-2},x))^2\right) \\
&= \Phi(\Gamma_n,x)\left(\Phi(\Gamma_n,x)+\frac{1}{8}\left(\frac{\Phi(\Gamma_n,x)}{\Phi(\Gamma_{n-1},x)}+\Phi(\Gamma_{n-1},x)\right)\left(\frac{\Phi(\Gamma_{n-1},x)}{\Phi(\Gamma_{n-2},x)}-\Phi(\Gamma_{n-2},x)\right)^2\right) \\
&= \Phi(\Gamma_n,x)\left(\Phi(\Gamma_n,x)+\frac{1}{8}\frac{(\Phi(\Gamma_n,x)+\Phi(\Gamma_{n-1},x)^2)(\Phi(\Gamma_{n-1},x)-\Phi(\Gamma_{n-2},x)^2)^2}{\Phi(\Gamma_{n-1},x)\Phi(\Gamma_{n-2},x)^2}\right)
\end{aligned}$$

and the claim is proved. □

TABLE 1. The polynomials $\Phi(\Gamma_1, x)$ and $\Phi(\Gamma_i^-, x)$, for $i = 1, \ldots, 5$.

$\Phi(\Gamma_1, x)$	$x(x-4)$
$\Phi(\Gamma_1^-, x)$	$x^2 - 8$
$\Phi(\Gamma_2^-, x)$	$(x-2)(x^3 - 2x^2 - 8x + 8)$
$\Phi(\Gamma_3^-, x)$	$(x-2)(x^2 - 2x - 4)(x^5 - 4x^4 - 8x^3 + 32x^2 + 8x - 16)$
$\Phi(\Gamma_4^-, x)$	$(x-2)(x^2-2x-4)(x^4-4x^3-6x^2+28x-8)\cdot$ $\cdot(x^9-8x^8+2x^7+116x^6-192x^5-432x^4+968x^3+96x^2-512x+64)$
$\Phi(\Gamma_5^-, x)$	$(x-2)^3(x^2-2x-4)(x^4-4x^3-6x^2+28x-8)\cdot$ $\cdot(x^7-6x^6-6x^5+76x^4-28x^3-240x^2+128x+32)\cdot$ $\cdot(x^{16}-14x^{15}+44x^{14}+228x^{13}-1536x^{12}+160x^{11}+15520x^{10}-22432x^9-62200x^8+$ $+149296x^7+65488x^6-336064x^5+108416x^4+162048x^3-79872x^2-4096x+1024)$

Corollary 4.10. *The multiplicity of the eigenvalue* 2 *in the spectrum of* Γ_n *is* $\left\lceil \frac{2^{n-2}}{3} \right\rceil$*, for each* $n \geq 3$.

Proof. For every $n \geq 3$, put $\Phi(\Gamma_n, x) = (x-2)^{a_n} p_n(x)$ for some polynomial p_n such that $p_n(2) \neq 0$. Notice that $a_n \geq 1$ since 2 is a root of $\Phi(\Gamma_2^-, x)$, see Table 1. As a first step, we prove by induction on $k \geq 2$ that

$$a_{2k} = 2a_{2k-1}, \qquad a_{2k+1} = 2a_{2k} - 1, \qquad p_{2k}(2) > 0, \qquad p_{2k+1}(2) < 0. \tag{4.5}$$

One can explicitly check that $a_4 = 2 = 2a_3$, $a_5 = 3 = 2a_4 - 1$ and $p_4(2) > 0$, $p_5(2) < 0$ and then Eq. (4.5) is true for $k = 2$.

Now let $k > 2$. Then, by virtue of Theorem 4.9, we have:

$$\Phi(\Gamma_{2k}, x) = (x-2)^{a_{2k}} p_{2k}(x) = (x-2)^{a_{2k-1}} p_{2k-1}(x) \Big[(x-2)^{a_{2k-1}} p_{2k-1}(x) + \frac{1}{8} \frac{((x-2)^{a_{2k-1}} p_{2k-1}(x) + (x-2)^{2a_{2k-2}} p_{2k-2}(x)^2)((x-2)^{a_{2k-2}} p_{2k-2}(x) - (x-2)^{2a_{2k-3}} p_{2k-3}(x)^2)^2}{(x-2)^{a_{2k-2}} p_{2k-2}(x)(x-2)^{2a_{2k-3}} p_{2k-3}(x)^2} \Big].$$

By using inductive hypothesis we obtain:

$$\Phi(\Gamma_{2k}, x) = (x-2)^{a_{2k}} p_{2k}(x) = (x-2)^{a_{2k-1}} p_{2k-1}(x) \Big[(x-2)^{a_{2k-1}} p_{2k-1}(x) + \frac{1}{8} \frac{((x-2)^{a_{2k-1}} p_{2k-1}(x) + (x-2)^{a_{2k-1}+1} p_{2k-2}(x)^2)((x-2)^{a_{2k-2}} p_{2k-2}(x) - (x-2)^{a_{2k-2}} p_{2k-3}(x)^2)^2}{(x-2)^{a_{2k-2}} p_{2k-2}(x)(x-2)^{a_{2k-2}} p_{2k-3}(x)^2} \Big] = (x-2)^{2a_{2k-1}} q(x),$$

with

$$q(x) = p_{2k-1}(x) \left(p_{2k-1}(x) + \frac{1}{8} \frac{(p_{2k-1}(x) + (x-2) p_{2k-2}(x)^2)(p_{2k-2}(x) - p_{2k-3}(x)^2)^2}{p_{2k-2}(x) p_{2k-3}(x)^2} \right).$$

Now we have:

$$q(2) = p_{2k-1}(2)^2 \left(1 + \frac{1}{8} \frac{(p_{2k-2}(2) - p_{2k-3}(2)^2)^2}{p_{2k-2}(2) p_{2k-3}(2)^2} \right) > 0$$

since $p_{2k-1}(2) \neq 0$ by definition and $p_{2k-2}(2) > 0$ by inductive hypothesis. As a consequence, $a_{2k} = 2a_{2k-1}$ and $p_{2k}(2) = q(2) > 0$. On the other hand, from Theorem 4.9 we also have:

$$\Phi(\Gamma_{2k+1}, x) = (x-2)^{a_{2k+1}} p_{2k+1}(x) = (x-2)^{a_{2k}} p_{2k}(x) \Big[(x-2)^{a_{2k}} p_{2k}(x) +$$
$$\frac{1}{8} \frac{((x-2)^{a_{2k}} p_{2k}(x) + (x-2)^{2a_{2k-1}} p_{2k-1}(x)^2)((x-2)^{a_{2k-1}} p_{2k-1}(x) - (x-2)^{2a_{2k-2}} p_{2k-2}(x)^2)^2}{(x-2)^{a_{2k-1}} p_{2k-1}(x)(x-2)^{2a_{2k-2}} p_{2k-2}(x)^2} \Big].$$

By using inductive hypothesis we obtain:

$$\Phi(\Gamma_{2k+1}, x) = (x-2)^{a_{2k+1}} p_{2k+1}(x) = (x-2)^{a_{2k}} p_{2k}(x) \Big[(x-2)^{a_{2k}} p_{2k}(x) +$$
$$\frac{1}{8} \frac{((x-2)^{a_{2k}} p_{2k}(x) + (x-2)^{a_{2k}} p_{2k-1}(x)^2)((x-2)^{a_{2k-1}} p_{2k-1}(x) - (x-2)^{a_{2k-1}+1} p_{2k-2}(x)^2)^2}{(x-2)^{a_{2k-1}} p_{2k-1}(x)(x-2)^{a_{2k-1}+1} p_{2k-2}(x)^2} \Big] =$$
$$(x-2)^{2a_{2k}-1} q(x),$$

with

$$q(x) = p_{2k}(x) \left((x-2) p_{2k}(x) + \frac{1}{8} \frac{(p_{2k}(x) + p_{2k-1}(x)^2)(p_{2k-1}(x) - (x-2) p_{2k-2}(x)^2)^2}{p_{2k-1}(x) p_{2k-2}(x)^2} \right).$$

Now we have:

$$q(2) = p_{2k}(2) \frac{1}{8} \frac{(p_{2k}(2) + p_{2k-1}(2)^2) p_{2k-1}(2)}{p_{2k-2}(2)^2} < 0$$

since we proved that $p_{2k}(2) > 0$ and by inductive hypothesis $p_{2k-1}(2) < 0$. As a consequence $p_{2k+1}(2) = q(2) < 0$ and $a_{2k+1} = 2a_{2k} - 1$.

Finally, the Arima sequence $\{b_n\}_{n\geq 1} = \{\lceil \frac{2^n}{3} \rceil\}_{n\geq 1}$ (sequence A005578 in the OEIS) is also defined recursively:

$$b_{n+1} = \begin{cases} 2b_n - 1 & \text{if } n \text{ is even} \\ 2b_n & \text{if } n \text{ is odd,} \end{cases} \qquad \text{and} \quad b_1 = 1.$$

Combining Eq. (4.5) with the fact that $a_3 = 1$, we conclude $a_n = b_{n-2} = \left\lceil \frac{2^{n-2}}{3} \right\rceil$ for every $n \geq 3$. □

Remark 4.11. For $n \geq 3$ the graph Γ_n contains at least one cycle of length greater than or equal to 4, half of whose vertices have a loop. For example, in Γ_3 the cycle $C_1(\Gamma_3)$, whose vertex set is $\{000, 100, 001, 101\}$, is the unique cycle with this property. Let us denote by a_n the number of such cycles in Γ_n. Having in mind the construction of Γ_{n+1} from two copies of Γ_n, we have that $a_{n+1} = 2a_n$ when $C_2(\Gamma_n)$ has no loops, and $a_{n+1} = 2a_n - 1$ when $C_2(\Gamma_n)$ is one of those cycles with loops. Then:

$$a_{n+1} = \begin{cases} 2a_n - 1 & \text{if } n \text{ is even} \\ 2a_n & \text{if } n \text{ is odd} \end{cases}$$

with $a_3 = 1$; therefore, for $n \geq 3$ one has $a_n = \left\lceil \frac{2^{n-2}}{3} \right\rceil$.

Suppose $v_1, v_2, \ldots, v_{2^k}$ are the vertices of such a cycle in Γ_n, with $k > 1$, and in particular let $v_2, v_4, \ldots, v_{2^k}$ be the vertices with a loop. Consider the vector of $\mathbb{C}^{2^n}$ whose entries are 0, except for those in correspondence of the vertices $v_2, v_6, \ldots, v_{2^k-2}$, that are 1, and those in correspondence of the vertices $v_4, v_8, \ldots, v_{2^k}$, that are -1. It is straightforward to check that such a vector is an eigenvector of eigenvalue 2 for the adjacency matrix A_{Γ_n}. Since vectors associated with distinct cycles with loops are pairwise orthogonal, we have constructed in this way a basis for the eigenspace of the eigenvalue 2, whose dimension is in fact $\left\lceil \frac{2^{n-2}}{3} \right\rceil$.

Corollary 4.12. *For $n \geq 3$ the spectrum of the graph Γ_n consists exactly of 2^{n-1} eigenvalues less than 2, $\left\lceil \frac{2^{n-2}}{3} \right\rceil$ eigenvalues equal to 2 and $\left\lfloor \frac{5\cdot 2^{n-2}}{3} \right\rfloor$ eigenvalues greater than 2.*

Proof. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2^n}$ be the 2^n eigenvalues of Γ_n. Consider the subgraph G of Γ_n induced by the 2^{n-1} vertices with loops (those vertices whose associated words in $\{0,1\}^n$ start with 1). The graph G clearly has eigenvalue 2 with multiplicity 2^{n-1}. As a consequence of the Interlacing Theorem (see, for example, [12, Theorem 1.3.11]) we obtain

$$\lambda_i \leq 2 \leq \lambda_{2^{n-1}+i}, \qquad \text{for each } i = 1, \ldots, 2^{n-1}.$$

In particular Γ_n has at least 2^{n-1} eigenvalues greater than or equal to 2.

In the next step we prove that Γ_n has at least $2^{n-1} + \left\lceil \frac{2^{n-2}}{3} \right\rceil$ eigenvalues less than or equal to 2. In order to do this, we consider the subgraph G_1 induced by the vertices of Γ_n with loops, together with the vertices having two distinct neighbors with loops. In other words, the connected components of the graph G_1 are:

(1) cycles C with vertices $V_C = \{v_1, \ldots, v_{2^k}\}$, with $k \geq 2$, where each vertex $v_2, v_4, \ldots, v_{2^k}$ has a loop;
(2) isolated vertices with loops.

The spectrum of G_1 is clearly given by the union of the spectra of its connected components. First observe that an isolated vertex with loop has eigenvalue 2. On the other hand, the adjacency matrix A_C of a cycle C of type (1) is conjugated with the matrix $2 - A_C$ via the anti-diagonal matrix P alternating $+1$ and -1 at its anti-diagonal entries:

$$P_{i,j} = \begin{cases} (-1)^{i+1} & \text{if } j = 2^k - i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, in the language of signed graphs, P is the matrix of a *switching isomorphism* between C and the signed graph whose adjacency matrix is $2 - A_C$. It follows that, if λ is an eigenvalue of C, then also $2 - \lambda$ is an eigenvalue of C. In particular, for each eigenvalue λ greater than 2, there exists an eigenvalue less than 0 (and so less than 2). Moreover, we have seen in Remark 4.11 that 2 is always an eigenvalue of C. This implies that the eigenvalues less than or equal to 2 of C are at least $\frac{1}{2}|V_C| + 1$. Notice that $\frac{1}{2}|V_C|$ is also the number of vertices with loops in C. It follows that the eigenvalues less than or equal to 2 of G_1 are at least $2^{n-1} + \left\lceil \frac{2^{n-2}}{3} \right\rceil$, since 2^{n-1} is the number of all the vertices with loops in Γ_n and by Remark 4.11 the number of cycles of type (1) is exactly $\left\lceil \frac{2^{n-2}}{3} \right\rceil$. By using again the Interlacing Theorem, we have that also Γ_n must have at least $2^{n-1} + \left\lceil \frac{2^{n-2}}{3} \right\rceil$ eigenvalues less than or equal to 2. Combining with Corollary 4.10 and with the fact that Γ_n has at least 2^{n-1} eigenvalues greater then or equal to 2, we can conclude that Γ_n has exactly 2^{n-1} eigenvalues less than 2.

The last thing we have to prove is the following equality:

$$\left\lceil \frac{2^{n-2}}{3} \right\rceil + \left\lfloor \frac{5 \cdot 2^{n-2}}{3} \right\rfloor = 2^{n-1},$$

(that is, $A005578(n-1) + A081254(n) = 2^n$ in the OEIS). Indeed:

$$\left\lceil \frac{2^{n-2}}{3} \right\rceil + \left\lfloor \frac{5 \cdot 2^{n-2}}{3} \right\rfloor = \left\lceil \frac{2^{n-2}}{3} \right\rceil + \left\lfloor \frac{(6-1) \cdot 2^{n-2}}{3} \right\rfloor = \left\lceil \frac{2^{n-2}}{3} \right\rceil + \left\lfloor 2^{n-1} - \frac{2^{n-2}}{3} \right\rfloor$$
$$= \left\lceil \frac{2^{n-2}}{3} \right\rceil + \left\lfloor -\frac{2^{n-2}}{3} \right\rfloor + 2^{n-1} = 2^{n-1}.$$

□

Example 4.13. In Fig. 8 the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{128}$ of Γ_7 are represented. As expected by Corollary 4.12, one has:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{64} < 2, \qquad \lambda_{65} = \lambda_{66} = \cdots = \lambda_{75} = 2, \qquad 2 < \lambda_{76} \leq \lambda_{77} \leq \cdots \leq \lambda_{128}.$$

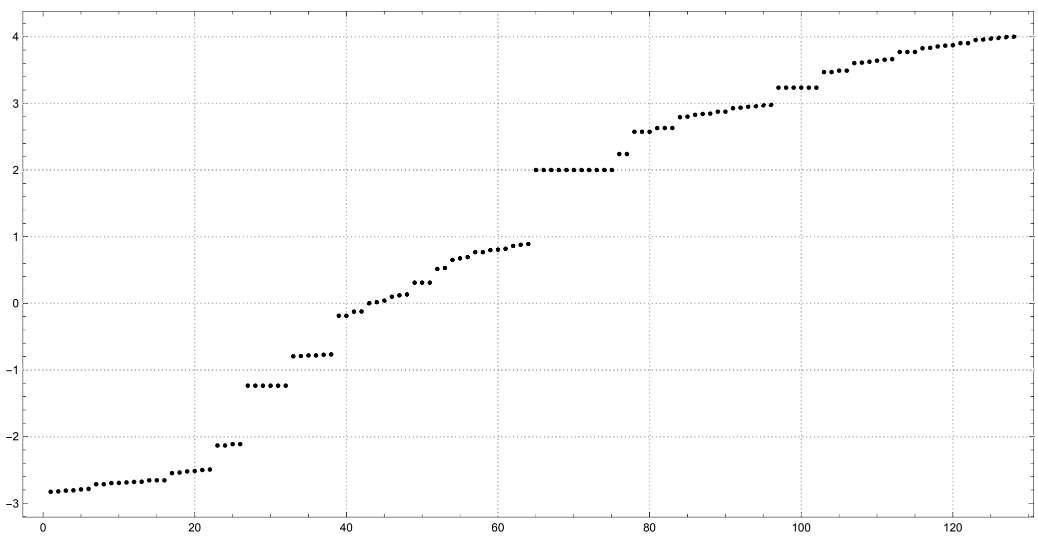

FIGURE 8. Spectrum of Γ_7.

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