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NORDHAUS-GADDUM TYPE INEQUALITIES FOR TREE COVERING NUMBERS ON UNITARY CAYLEY GRAPHS OF FINITE RINGS

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Abstract. The unitary Cayley graph Γ_n of a finite ring $\mathbb{Z}_n$ is the graph with vertex set $\mathbb{Z}_n$ and two vertices x and y are adjacent if and only if $x-y$ is a unit in $\mathbb{Z}_n$. A family $\mathcal{F}$ of mutually edge disjoint trees in Γ_n is called a tree cover of Γ_n if for each edge $e \in E(\Gamma_n)$, there exists a tree $T \in \mathcal{F}$ in which $e \in E(T)$. The minimum cardinality among tree covers of Γ_n is called a tree covering number and denoted by $\tau(\Gamma_n)$. In this paper, we prove that, for a positive integer $n \geq 3$, the tree covering number of Γ_n is $\frac{\varphi(n)}{2}+1$ and the tree covering number of $\overline{\Gamma}_n$ is at most $n-p$ where p is the least prime divisor of n. Furthermore, we introduce the Nordhaus-Gaddum type inequalities for tree covering numbers on unitary Cayley graphs of rings $\mathbb{Z}_n$.

1. Introduction and Preliminaries

In algebraic graph theory, the structure of algebraic methods are studied and then applied to problems about graphs. An interesting topic is to study properties of graphs in connection to algebraic systems. A well-known connection between graphs and algebraic systems is the construction of graphs from algebras. The unitary Cayley graph is one of several interesting graphs constructed from algebraic systems. For each $n \geq 2$, let Γ_n denote the unitary Cayley graph of a ring $\mathbb{Z}_n$, the ring of integers modulo n, whose vertex set is $\mathbb{Z}_n$ itself and two vertices x and y are joined by edge if $x-y$ is a unit in the ring $\mathbb{Z}_n$. Let us denote all elements in $\mathbb{Z}_n$ by integers $0, 1, 2, \ldots, n-1$. It is well known that all units in the ring $\mathbb{Z}_n$ are the integers a in which $\gcd(a, n) = 1$. Therefore, the edge set of Γ_n can

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be expressed as $E(\Gamma_n) = \{\{x, y\} : x, y \in \mathbb{Z}_n, \gcd(x - y, n) = 1\}$. Clearly, if p is prime, then Γ_p is a complete graph. Moreover, all unitary Cayley graphs of order greater than 2 always contain cycles and their structures are highly symmetric. Moreover, they have some remarkable properties between algebraic graph theory and number theory. Some prominent results of unitary Cayley graphs were studied by several researchers. In 1995, Dejter and Giudici [9] showed that unitary Cayley graphs are unions of disjoint hamiltonian cycles and presented the sufficient condition for being bipartite graphs. In 2007, Klotz and Sander [16] determined some invariant properties of unitary Cayley graphs and studied their perfectness. In 2008, Boggess et al. [6] explored structural properties of unitary Cayley graphs. In 2009, Akhtar et al. [2] computed an automorphism group of a unitary Cayley graph of a finite ring. In 2012, Kiani and Aghaei [15] provided isomorphism theorems for unitary Cayley graphs of rings associated with Jacobson radicals. In 2014, Naghipour [22] considered some properties of induced subgraphs of unitary Cayley graphs of commutative rings.

Moreover, graph covering is one of the most classical topics in graph theory. A survey on covering problems by Beineke [4] appeared in 1969. Graph covering problem asks for the minimum number of graphs with a particular property having a given graph as their union. It is a lively field with deep ramifications over the last decades as well as today. In 1964, Williams [27] provided the minimum number of forests needed to cover a graph in terms of order and size of such the graph. In 1967, Shirakawa et al. [25] presented the minimum number of planar graphs for covering complete graphs. In 1969, Beineke [5] studied the covering of complete graphs by using graphs embeddable in the torus. In 1996, Pyber [24] considered the covering of connected graphs by using paths and cycles. Later in 2010, Hochbaum and Levin [13] used $K_{2,2}$ graphs to cover the edges of bipartite graphs. In 2014, Artes and Dignos [3] investigated the tree cover of graphs consisting of cycles, fans, wheels and cyclic graphs. Further in 2016, Knauer and Ueckerdt [17] purposed several ways to cover a graph by using some special graphs such as forests, stars, and caterpillars. Later in 2019, Abuhijleh et al. [1] determined the covering number of the zero-divisor graph over the rings $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^k q^r}$.

For studying invariant properties of graphs, Nordhaus-Gaddum type inequalities are remarkable. In 1956, Nordhaus and Gaddum [23] studied the chromatic number for a graph G and its complement $\overline{G}$ together. They presented lower and upper bounds on the sum and the product of the chromatic number of graphs and their complements in terms of the number of vertices in graphs. Consequently, any bound on the sum and the product of an invariant for a graph G and the same invariant for its complement $\overline{G}$ is called the *Nordhaus-Gaddum type inequality*. Many researchers widely investigated such inequalities for several parameters of graphs. In 2005, Furedi et al. [11] studied Nordhaus-Gaddum type theorems for decompositions of graphs. In 2011, Jiang and Kang [14] considered these inequalities for total outer-connected domination in graphs. In the same year, Henning et al. [12] presented more results of Nordhaus-Gaddum type inequalities for total domination numbers. Other results of these inequalities for several parameters of graphs can be found in [8], [18], [19],[20] and [21].

In this paper, we take an interest to measure how difficult it is to cover all edges of a given unitary Cayley graph with special graphs from a given class. This motivates us to study tree covering numbers

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of unitary Cayley graphs of the ring $\mathbb{Z}_n$ where $n \geq 2$. We present the results in the form of Nordhaus-Gaddum type inequalities. Throughout the paper, all sets and graphs are assumed to be finite. Some preliminaries used in this paper are described as follows. We are willing to refer to [7] and [26] for more information about graphs and [10] for others on algebras. Let G be a graph. A family $\mathcal{F}$ of pairwise edge disjoint trees in G is called a *tree cover* of G if for each edge $e \in E(G)$, there exists a tree $T \in \mathcal{F}$ in which $e \in E(T)$. The minimum cardinality among tree covers of G is called a *tree covering number* of G and denoted by $\tau(G)$, that is,

$$\tau(G) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a tree cover of } G\}.$$

Note that if G is an empty graph, then $\tau(G) = 0$ with respect to the corresponding tree cover $\emptyset$ (empty set). We now mention about a decomposition of a graph which plays a crucial role in the following lemmas. A *decomposition* of a graph G is a collection of edge disjoint subgraphs of G such that every edge of G belongs to exactly one subgraph. The following two lemmas are useful for studying the tree cover properties of unitary Cayley graphs Γ_n of rings $\mathbb{Z}_n$. They are concerned with complete graphs of odd orders and even orders, respectively, as follows.

Lemma 1.1. [7] *For every positive integer n, the complete graph K_{2n+1} can be factored into n hamiltonian cycles.*

Lemma 1.2. [7] *For every positive integer n, the complete graph K_{2n} can be factored into n hamiltonian paths.*

For the part of main results, we divide results into three sections. The first one presents the result about the tree covering numbers on unitary Cayley graphs of finite rings. The second one presents more results about the tree covering numbers on complementary unitary Cayley graphs and the last one presents the results of tree covering numbers between unitary Cayley graphs and their complements in the forms of Nordhaus-Gaddum type inequalities.

2. Tree covering numbers on unitary Cayley graphs

Before we state the theorem, certain symbol is prescribed as follows. In number theory, the function called *Euler's totient function*, denoted by $\varphi(n)$, counts the positive integers up to a given positive integer n that are relatively prime to n. That is,

$$\varphi(n) = \big|\{a \in \mathbb{N} : 1 \leq a \leq n \text{ and } \gcd(a, n) = 1\}\big|.$$

For the unitary Cayley graph Γ_n, it is well known that Γ_n is a regular graph of degree $\varphi(n)$ which can be seen in [7]. Consequently, $|E(\Gamma_n)| = \dfrac{n(\varphi(n))}{2}$ and $|E(\overline{\Gamma}_n)| = \dfrac{n((n-1) - \varphi(n))}{2}$ where the graph $\overline{\Gamma}_n$ is the complement of Γ_n.

Theorem 2.1. *For every positive integer $n \geq 3$, $\tau(\Gamma_n) = \dfrac{\varphi(n)}{2} + 1$.*

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Proof. Let $n \in \mathbb{N}$ be such that $n \geq 3$. Assume that $\Phi_n = \{x \in \mathbb{N} : 1 \leq x \leq n \text{ and } \gcd(x, n) = 1\}$. Hence $|\Phi_n| = \varphi(n)$. As the fact that $\varphi(n)$ is even and further, it can be seen that $\gcd(a, n) = 1$ if and only if $\gcd(n - a, n) = 1$, we let

$$A_n = \left\{ a \in \Phi_n : 1 \leq a \leq \left\lfloor \frac{n-1}{2} \right\rfloor \right\}.$$

For convenience, we write $A_n = \{a_1, a_2, \ldots, a_k\}$ where $k = \dfrac{\varphi(n)}{2}$ and $a_i \langle a_j$ for $1 \leq i \leq k$. Let $a_m \in A_n$. Consider a subgroup $\langle a_m \rangle$ of $\mathbb{Z}_n$ generated by a_m as follows:

$$\langle a_m \rangle = \{0, a_m, 2a_m, 3a_m, \ldots, (|a_m| - 1)a_m\}.$$

Since $\gcd(a_m, n) = 1$, we have that a_m is a generator of $\mathbb{Z}_n$, that is, $|a_m| = n$. From the expression of elements in $\langle a_m \rangle$ above, we can observe that $0, a_m, 2a_m, 3a_m, \ldots, (n-1)a_m$ form a path of length $n - 1$ in Γ_n. Moreover, this path is a tree and will be denoted by T_{a_m}. In addition, we can see that $\{0,\ (n-1)a_m\} \in E(\Gamma_n)$. Now, we observe that $\{0,\ (n-1)a_l\} \in E(\Gamma_n)$ for all $l = 1, 2, \ldots, k$. Let T' be a subgraph of Γ_n induced by such the edges $\{0,\ (n-1)a_l\}$. Hence T' is a star which is also tree. Further, let $\mathcal{T} = \{T_{a_1}, T_{a_2}, T_{a_3}, \ldots, T_{a_k}, T'\}$. We now show that all trees in $\mathcal{T}$ are mutually edge disjoint. Suppose to the contrary that there exists an edge $\{x, y\} \in E(T_{a_r}) \cap E(T_{a_s})$ for some $r, s \in \{1, 2, \ldots, k\}$. Then

$$x = pa_r \text{ and } y = (p+1)a_r \text{ in } V(T_{a_r}) \text{ where } p \in \{0, 1, 2, \ldots, n-2\}, \text{ and}$$

$$x = qa_s \text{ and } y = (q+1)a_s; \text{ or } x = (q+1)a_s \text{ and } y = qa_s \text{ in } V(T_{a_s})$$
$$\text{where } q \in \{0, 1, 2, \ldots, n-2\}.$$

Case 1: $x = pa_r$; and $y = (p+1)a_r$ and $x = qa_s$ and $y = (q+1)a_s$.
Consider $(p+1)a_r = (q+1)a_s$, we have $pa_r + a_r = qa_s + a_s$ which implies that $a_r = a_s$ since $pa_r = qa_s$. This leads to a contradiction.

Case 2: $x = pa_r$ and $y = (p+1)a_r$ and $x = (q+1)a_s$ and $y = qa_s$.
Consider $(p+1)a_r = qa_s$, we have $pa_r + a_r = qa_s$. Thus $(q+1)a_s + a_r = qa_s$ since $(q+1)a_s = x = pa_r$. Hence $qa_s + a_s + a_r = qa_s$ which implies that $a_r = n - a_s$, an inverse element of $a_s \in \mathbb{Z}_n$. It follows that $a_r + a_s \equiv 0 (\bmod n)$, that is, $n | (a_r + a_s)$ which is impossible as $1 \leq a_r, a_s \leq \left\lfloor \dfrac{n-1}{2} \right\rfloor$. Form the above two cases, we obtain that T_{a_i} are mutually edge disjoint for all $i \in \{1, 2, \ldots, k\}$. Next, if there exists an edge $\{x, y\} \in E(T') \cap E(T_{a_j})$ for some $j \in \{1, 2, \ldots, k\}$, then $a_j = (n-1)a_l$ for some $l \in \{1, 2, \ldots, k\}$. This also gives a contradiction as we described above since $a_j = (n-1)a_l = n - a_l$. Therefore, $\mathcal{T}$ is mutually edge disjoint. Next, we show that $\mathcal{T}$ is a tree cover of Γ_n. Let $\{x, y\} \in E(\Gamma_n)$. Without loss of generality, we assume that x. Then $\gcd(x - y, n) = 1$. We have two possibilities to consider. The first one is $x - y \in A_n$, that is, $x - y = a_m$ for some $1 \leq a_m \leq \left\lfloor \dfrac{n-1}{2} \right\rfloor$. Thus $\{x, y\} \in E(T_{a_m})$. The second one is $x - y \in \Phi_n \setminus A_n$, that is, $x - y = n - a_t$ where $\left\lfloor \dfrac{n-1}{2} \right\rfloor \langle n - a_t \langle n - 1$ for some $a_t \in A_n$.

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Since $n - a_t = (n-1)a_t$ in $\mathbb{Z}_n$, we get that $\{x, y\} \in E(T')$. Hence $\mathcal{T}$ is a tree cover of Γ_n. It follows that

$$\tau(\Gamma_n) \leq |\mathcal{T}| = |A_n| + 1 = \frac{\varphi(n)}{2} + 1.$$

We next suppose that there exists a tree cover $\mathcal{F}$ of Γ_n in which $|\mathcal{F}| \leq \frac{\varphi(n)}{2}$. As the fact that, for every tree $T \in \mathcal{F}, |E(T)| \leq n - 1$, so

$$|E(\bigcup_{T \in \mathcal{F}} T)| \leq \frac{(n-1)\varphi(n)}{2} \langle \frac{n\varphi(n)}{2} = |E(\Gamma_n)|.$$

This contradicts the property of the tree cover $\mathcal{F}$. Consequently, $\tau(\Gamma_n) = \frac{\varphi(n)}{2} + 1$. □

3. Tree covering numbers on the complements of unitary Cayley graphs

We begin this section with the theorem related to the decomposition of the complementary unitary Cayley graph $\overline{\Gamma}_{p^k}$ into complete graphs as follows.

Theorem 3.1. *If $n = p^k$ where p is a prime number and $k \in \mathbb{N}$, then $\overline{\Gamma}_n$ is decomposed into p complete graphs of order p^{k-1}.*

Proof. Let p be a prime number and $k \in \mathbb{N}$. Assume that $n = p^k$. If $k = 1$, then $n = p$ which implies that $\overline{\Gamma}_n$ is an empty graph. The result holds as $\overline{\Gamma}_n$ is decomposed into p copies of K_1. Now, let $k \geq 2$. We first show that $V(\overline{\Gamma}_n)$ can be partitioned into p disjoint subsets. Let $\langle p \rangle$ be a cyclic subgroup of $\mathbb{Z}_n$ generated by p. Consider the set of all left cosets of $\langle p \rangle$ in $\mathbb{Z}_n$ as follows:

$$\mathbb{Z}_n / \langle p \rangle = \{a + \langle p \rangle : a \in \mathbb{Z}_n\}.$$

We take $V_1 = \langle p \rangle, V_2 = 1 + \langle p \rangle, \ldots, V_p = (p-1) + \langle p \rangle$. Clearly, $V_i \cap V_j = \emptyset$ for all $i \neq j$ where $i, j \in \{1, 2, \ldots, p\}$ and $\bigcup_{t=1}^{p} V_t = \mathbb{Z}_n$. Next, we show that every two vertices in the same coset are joined by edge and there is no edge between vertices from different cosets. Let $i, j \in \{1, 2, \ldots, p\}$. Further, let $x, y \in V_i$ and $z \in V_j$ where $i \neq j$. Then $x = (i-1) + cp, y = (i-1) + dp$ and $z = (j-1) + qp$ for some $c, d, q \in \mathbb{Z}$. It follows that $x - y = (c-d)p$ which leads to $\gcd(x - y, n) \neq 1$. That means $\{x, y\} \in E(\overline{\Gamma}_n)$. Consider $x - z = (i - j) + (c - q)p$, since $i, j \in \{1, 2, \ldots, p\}$ and $i \neq j$, we have $p \nmid (i - j)$. Thus $\gcd(i - j, p) = 1$ which implies that $\gcd(i - j, n) = 1$. Hence $\gcd(x - z, n) = 1$, as well. We obtain that $\{x, z\} \notin E(\overline{\Gamma}_n)$. So we can conclude that an induced subgraph $\overline{\Gamma}_n[V_i]$ of $\overline{\Gamma}_n$ is complete for all $i \in \{1, 2, \ldots, p\}$. Therefore, $\overline{\Gamma}_n$ is decomposed into p complete graphs which each of them has order $\frac{n}{p} = \frac{p^k}{p} = p^{k-1}$, as required. This completes the proof of our theorem. □

The following theorems introduce bounds for tree covering numbers on complementary unitary Cayley graphs in term of their orders.

Theorem 3.2. *Let $n = p^k$ be such that p is a prime number and $k \in \mathbb{N} \setminus \{1\}$. Then $\tau(\overline{\Gamma}_n) = 2^{k-1}$ where $p = 2$ and*

$$\left\lceil \frac{p^{2k-1}-p^k}{2p^k-2} \right\rceil \leq \tau(\overline{\Gamma}_n) \leq \frac{p^k+p}{2}$$

for all $p\rangle 2$*.*

Proof. Assume that $n=p^k$ where p is a prime number and $k \in \mathbb{N}\setminus\{1\}$. By Theorem 3.1 , we obtain that $\overline{\Gamma}_n$ is decomposed into p copies of complete graphs $K_{p^{k-1}}$. We first consider $p=2$. By Lemma 1.2 , we have that $K_{2^{k-1}}$ can be decomposed into 2^{k-2} hamiltonian paths. Since those hamiltonian paths are spanning trees, we get that 2^{k-2} is the smallest number of trees required for covering all edges of $K_{2^{k-1}}$. Since we have 2 copies of $K_{2^{k-1}}$, we obtain that $\tau(\overline{\Gamma}_n)=\tau(\overline{\Gamma}_{2^k})=2(2^{k-2})=2^{k-1}$, as required.

Next, assume that $p\rangle 2$. We will prove the upper bound of $\tau(\overline{\Gamma}_n)$. We now consider $K_{p^{k-1}}$ which is decomposed from $\overline{\Gamma}_n$. Since p^{k-1} is odd, we have by Lemma 1.1 that $K_{p^{k-1}}$ can be decomposed into $\frac{p^{k-1}-1}{2}$ hamiltonian cycles. Let v be a fixed vertex of $\overline{\Gamma}_n$. It follows that v belongs to every hamiltonian cycle. For each $i \in \{1,2,\ldots,\frac{p^{k-1}-1}{2}\}$. Let $C^{(i)}$ be a hamiltonian cycle decomposed from $\overline{\Gamma}_n$. Further, let T_{a_i} be a graph with vertex set $\{v,a_i\}$ and edge set $\{\{v,a_i\}\}$ where v and a_i are adjacent in $C^{(i)}$. Hence $C^{(i)}\setminus\{\{v,a_i\}\}$ is a hamiltonian path and T_{a_i} is a path of length 1. That means every $C^{(i)}$ is separated into $C^{(i)}\setminus\{\{v,a_i\}\}$ and T_{a_i}. Moreover, we can see that all edges $\{v,a_i\}$ are distinct. Thus $\bigcup_{i=1}^{l} T_{a_i}$ is a tree with vertex set $\{v,a_1,a_2,\ldots,a_l\}$ and edge set $\{\{v,a_1\},\{v,a_2\},\ldots,\{v,a_l\}\}$ where $l=\frac{p^{k-1}-1}{2}$. We conclude that $K_{p^{k-1}}$ is decomposed into l hamiltonian paths and one tree $\bigcup_{i=1}^{l} T_{a_i}$. Therefore, the set of such l hamiltonian paths and the tree $\bigcup_{i=1}^{l} T_{a_i}$ is a tree cover of $K_{p^{k-1}}$. Consequently, $\tau(\overline{\Gamma}_n) \leq p(l+1) = p(\frac{p^{k-1}-1}{2}+1) = \frac{p^k+p}{2}$.

For proving the left inequality, consider the following fact. If $\overline{\Gamma}_n$ can be decomposed into q spanning trees, then such the number q should be the smallest number of trees used to cover $\overline{\Gamma}_n$. It is known that the number of edges in each spanning tree is p^k-1. So we have that the number of trees required to cover $\overline{\Gamma}_n$ must be greater than or equal to $\frac{|E(\overline{\Gamma}_n)|}{p^k-1}$. That is,

$$\begin{aligned}
\tau(\overline{\Gamma}_n) \geq \frac{|E(\overline{\Gamma}_n)|}{p^k-1} &= \frac{p^k[(p^k-1)-\varphi(p^k)]}{2(p^k-1)} \\
&= \frac{p^k[(p^k-1)-(p^k-p^{k-1})]}{2(p^k-1)} \\
&= \frac{p^k(p^{k-1}-1)}{2(p^k-1)} \\
&= \frac{p^{2k-1}-p^k}{2p^k-2}.
\end{aligned}$$

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Since $\tau(\overline{\Gamma}_n)$ is an integer, we have $\tau(\overline{\Gamma}_n) \geq \left\lceil \frac{p^{2k-1}-p^k}{2p^k-2} \right\rceil$. □

In general, the lower bound mentioned in the previous theorem always holds for $n \in \mathbb{N}$. That is, if we can find a tree cover of $\overline{\Gamma}_n$ and all graphs in such the tree cover are precisely spanning trees, then the cardinality of the tree cover must be a tree covering number of $\overline{\Gamma}_n$. In fact, the number of edges in each spanning tree is $n-1$. Consequently, $\tau(\overline{\Gamma}_n) \geq \frac{|E(\overline{\Gamma}_n)|}{n-1} = \frac{n((n-1)-\varphi(n))}{2(n-1)}$. Since $\tau(\overline{\Gamma}_n)$ is an integer, we have $\tau(\overline{\Gamma}_n) \geq \left\lceil \frac{n((n-1)-\varphi(n))}{2(n-1)} \right\rceil$.

Theorem 3.3. *If n is an even positive integers and not a power of 2, then*

$$\tau(\overline{\Gamma}_n) \leq \begin{cases} \frac{3n}{4} & \text{if } 4|n; \\ \left\lfloor \frac{n}{4} \right\rfloor + \frac{n}{2} + 1 & \text{if } 4 \nmid n. \end{cases}$$

Proof. Let n be an even positive integer and not a power of 2. Assume that $4|n$. Consider the right cosets of a subgroup $\langle 2 \rangle$ in a group $(\mathbb{Z}_n, +)$ as follows:

$$A := \langle 2 \rangle = \{0, 2, 4, \ldots, n-2\} \text{ and}$$

$$B := \langle 2 \rangle + 1 = \{1, 3, 5, \ldots, n-1\}.$$

Let $\overline{\Gamma}_n[A]$ and $\overline{\Gamma}_n[B]$ be subgraphs of $\overline{\Gamma}_n$ induced from A and B, respectively. We can see that $\overline{\Gamma}_n[A]$ and $\overline{\Gamma}_n[B]$ are complete graphs of order $\frac{n}{2}$ which is even since $4|n$. By Lemma 1.2, both of them can be decomposed into $\frac{n}{4}$ hamiltonian paths, respectively. For convenience, let P_i and T_i be hamiltonian paths decomposed from $\overline{\Gamma}_n[A]$ and $\overline{\Gamma}_n[B]$, respectively, where $i \in \{1, 2, \ldots, \frac{n}{4}\}$. Since n is not a power of 2, there exists an odd prime number p such that $p|n$. Moreover, we can see that $2i \in A$ is adjacent to $2i + p \in B$ for all $i \in \{1, 2, \ldots, \frac{n}{4}\}$. Connect the path P_i to the path T_i by edge $\{(2i), (2i+p)\}$ for all $i \in \{1, 2, \ldots, \frac{n}{4}\}$, then we have $\frac{n}{4}$ spanning trees of $\overline{\Gamma}_n$. In addition, for each $a \in A$, we can construct a tree $S(a)$ to be a graph with vertex set $\{a\} \cup \{b \in B : \{a, b\} \in E(\overline{\Gamma}_n)\}$ and edge set $\{\{a, b\} \in E(\overline{\Gamma}_n) : b \in B\}$. Note that all edges of $S(a)$ must exclude edges $\{(2i), (2i+p)\}$ in the construction of spanning trees above. Hence we have $\frac{n}{2}$ trees from the construction $S(a)$ for each $a \in A$. Furthermore, those $\frac{n}{4}$ spanning trees and such $\frac{n}{2}$ trees form a tree cover of $\overline{\Gamma}_n$. Consequently, we get that

$$\tau(\overline{\Gamma}_n) \leq \frac{n}{4} + \frac{n}{2} = \frac{3n}{4} \text{ where } 4|n.$$

For the case $4 \nmid n$, similar to the above case, we have that $\overline{\Gamma}_n[A]$ and $\overline{\Gamma}_n[B]$ are complete subgraphs of order $\frac{n}{2}$. However, in this case, $\frac{n}{2}$ is odd since $4 \nmid n$. By Lemma 1.1, both of $\overline{\Gamma}_n[A]$ and $\overline{\Gamma}_n[B]$ can be decomposed into $\left\lfloor \frac{n}{4} \right\rfloor$ hamiltonian cycles, respectively. For each $i \in \{1, 2, \ldots, \left\lfloor \frac{n}{4} \right\rfloor\}$, the i^{th} hamiltonian cycle in $\overline{\Gamma}_n[A]$ can be decomposed into a hamiltonian path and a single edge $\{0, v_i\}$.

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Moreover, the union (as graph) of edges $\{0, v_i\}$, say M, forms a tree in $\overline{\Gamma}_n$ where $i \in \{1, 2, \ldots, \left\lfloor \frac{n}{4} \right\rfloor\}$. We now obtain that $\overline{\Gamma}_n[A]$ is decomposed into $\left\lfloor \frac{n}{4} \right\rfloor$ hamiltonian paths and one tree. Similarly, the i^{th} hamiltonian cycle in $\overline{\Gamma}_n[B]$ can be decomposed into a hamiltonian path and a single edge $\{p, w_i\}$ where p is an odd prime divisor of n. Thus the union of edge $\{p, w_i\}$, say N, also forms a tree in $\overline{\Gamma}_n$ where $i \in \{1, 2, \ldots, \left\lfloor \frac{n}{4} \right\rfloor\}$. Fortunately, the trees M and N can be connected by edge $\{0, p\}$ and we get a new tree, say Q. Further, $\left\lfloor \frac{n}{4} \right\rfloor$ hamiltonian paths in $\overline{\Gamma}_n[A]$ can be connected to $\left\lfloor \frac{n}{4} \right\rfloor$ hamiltonian paths in $\overline{\Gamma}_n[B]$ and we also obtain the trees $S(a)$ for every $a \in A$ by applying the proof in the case $4|n$. Finally, we can cover $\overline{\Gamma}_n$ by $\left\lfloor \frac{n}{4} \right\rfloor$ trees constructed by joining those hamiltonian paths, $\frac{n}{2}$ trees constructed from the trees $S(a)$, and the tree Q. So we can conclude that if $4 \nmid n$, then

$$\tau(\overline{\Gamma}_n) \leq \left\lfloor \frac{n}{4} \right\rfloor + \frac{n}{2} + 1.$$

□

In order to complete the proof of the following theorem, we need to denote by notation $N(u)$ the set $\{v \in \mathbb{Z}_n : \{u, v\} \in E(\overline{\Gamma}_n)\}$ for every $u \in \mathbb{Z}_n$.

Theorem 3.4. *Let n be a positive integer greater than 3. If p is the least prime divisor of n, then $\tau(\overline{\Gamma}_n) \leq n - p$.*

Proof. Let $n \in \mathbb{N}$ be such that $n \rangle 3$. Assume that p is the least prime divisor of n. Let S_0 be a graph defined as follows:

$$V(S_0) = \{0\} \cup N(0) \text{ and } E(S_0) = \{\{0, v\} \in E(\overline{\Gamma}_n) : v \in N(0)\}.$$

Moreover, for each $m \in \{1, 2, \ldots, (n-1) - p\}$, we define S_m to be a graph with

$$V(S_m) = \{m\} \cup [N(m) \setminus \{0, 1, 2, \ldots, m-1\}] \text{ and}$$

$$E(S_m) = \{\{m, v\} \in E(\overline{\Gamma}_n) : v \in N(m) \setminus \{0, 1, 2, \ldots, m-1\}\}.$$

It is clear that those subgraphs S_i of $\overline{\Gamma}_n$ are trees for all $i \in \{0, 1, 2, \ldots, (n-1) - p\}$. From the construction, we can see that such trees are mutually edge disjoint. Next, let $\{u, v\} \in E(\overline{\Gamma}_n)$ be an arbitrary edge. Without loss of generality, we assume that u. We show that $u \leq (n-1) - p$. Suppose that $u \rangle (n-1) - p$. From $\{u, v\} \in E(\overline{\Gamma}_n)$, we have $\gcd(v - u, n) \neq 1$. Then there exists a prime divisor q of n such that $q|(v-u)$. Thus $v - u = qk$ for some $k \in \mathbb{N}$. Consequently, $v = u + qk \rangle (n-1) - p + qk \geq n - 1 - p + pk \geq n - 1$ which is a contradiction since $v \in V(\overline{\Gamma}_n)$, that is, $v \leq n - 1$. Hence we now have $u \leq (n-1) - p$ which implies that $\{u, v\} \in E(S_u)$. Therefore, $\{S_0, S_1, S_2, \ldots, S_{(n-1)-p}\}$ is a tree cover of $\overline{\Gamma}_n$. So we conclude that $\tau(\overline{\Gamma}_n) \leq n - p$, as desired. □

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4. **Nordhaus-Gaddum type inequalities**

This section provides the Nordhaus-Gaddum type inequalities expressed as lower and upper bounds of the sum and the product between $\tau(\Gamma_n)$ and $\tau(\overline{\Gamma}_n)$. Recall that for any an integer $n \geq 3$, we have $\tau(\overline{\Gamma}_n) \geq \left\lceil \frac{n((n-1)-\varphi(n))}{2(n-1)} \right\rceil$. As consequences of results for the tree covering numbers on unitary Cayley graphs and their complements we presented before, the following theorems are obtained as follows.

Theorem 4.1. *Let $n = p^k$ be such that p is a prime number and $k \in \mathbb{N} \setminus \{1\}$. Then $\tau(\Gamma_n) + \tau(\overline{\Gamma}_n) = 3(2^{k-2}) + 1$ where $p = 2$ and*

$$\frac{p^{2k} + p^{k-1} - 2}{2(p^k - 1)} \leq \tau(\Gamma_n) + \tau(\overline{\Gamma}_n) \leq \frac{2(p^k + 1) + p - p^{k-1}}{2}$$

for all $p \rangle 2$.

Proof. Let $n = p^k$ be such that p is a prime number and $k \in \mathbb{N} \setminus \{1\}$. By Theorem 2.1, we have

$$\tau(\Gamma_n) = \frac{\varphi(n)}{2} + 1 = \frac{\varphi(p^k)}{2} + 1 = \frac{p^k - p^{k-1}}{2} + 1.$$

Moreover, by Theorem 3.2, we obtain that $\tau(\overline{\Gamma}_n) = 2^{k-1}$ where $p = 2$, and for $p \rangle 2$, we have

$$\left\lceil \frac{p^{2k-1} - p^k}{2p^k - 2} \right\rceil \leq \tau(\overline{\Gamma}_n) \leq \frac{p^k + p}{2}.$$

Therefore, the lower and upper bounds of $\tau(\Gamma_n) + \tau(\overline{\Gamma}_n)$ are easily computed and we then have

$$\tau(\Gamma_n) + \tau(\overline{\Gamma}_n) = 3(2^{k-2}) + 1$$

where $p = 2$ and

$$\frac{p^{2k} + p^{k-1} - 2}{2(p^k - 1)} \leq \tau(\Gamma_n) + \tau(\overline{\Gamma}_n) \leq \frac{2(p^k + 1) + p - p^{k-1}}{2}$$

for all $p \rangle 2$. □

Theorem 4.2. *Let n be an even positive integer and not a power of 2. Then*

$$\frac{(n+2)(n+1) - \varphi(n)}{2(n-1)} \leq \tau(\Gamma_n) + \tau(\overline{\Gamma}_n) \leq \begin{cases} \frac{3n + 2\varphi(n) + 4}{4} & \text{if } 4|n; \\ \frac{3n + 2\varphi(n) + 8}{4} & \text{if } 4 \nmid n. \end{cases}$$

Proof. Let n be an even positive integer and not a power of 2. By Theorem 2.1, we have

$$\tau(\Gamma_n) = \frac{\varphi(n)}{2} + 1$$

and by Theorem 3.3, we obtain that

$$\tau(\overline{\Gamma}_n) \leq \begin{cases} \frac{3n}{4} & \text{if } 4|n; \\ \left\lfloor \frac{n}{4} \right\rfloor + \frac{n}{2} + 1 & \text{if } 4 \nmid n. \end{cases}$$

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In addition, by the lower bound of $\tau(\overline{\Gamma}_n)$ we recalled above, that is,

$$\tau(\overline{\Gamma}_n) \geq \left\lceil \frac{n((n-1)-\varphi(n))}{2(n-1)} \right\rceil,$$

the lower and upper bounds of $\tau(\Gamma_n)+\tau(\overline{\Gamma}_n)$ are directly obtained as follows:

$$\frac{(n+2)(n+1)-\varphi(n)}{2(n-1)} \leq \tau(\Gamma_n)+\tau(\overline{\Gamma}_n) \leq \begin{cases} \dfrac{3n+2\varphi(n)+4}{4} & \text{if } 4|n; \\ \dfrac{3n+2\varphi(n)+8}{4} & \text{if } 4\nmid n. \end{cases}$$

□

Theorem 4.3. *Let n be a positive integer greater than 3. If p is the least prime divisor of n, then*

$$\frac{(n+2)(n+1)-\varphi(n)}{2(n-1)} \leq \tau(\Gamma_n)+\tau(\overline{\Gamma}_n) \leq \frac{2(n-p+1)+\varphi(n)}{2}.$$

Proof. Let n be a positive integer greater than 3 and let p be the least prime divisor of n. For the tree covering number of Γ_n, we have by Theorem 2.1 that

$$\tau(\Gamma_n) = \frac{\varphi(n)}{2}+1.$$

For the complement $\overline{\Gamma}_n$, the lower bound of $\tau(\overline{\Gamma}_n)$ is known as

$$\tau(\overline{\Gamma}_n) \geq \left\lceil \frac{n((n-1)-\varphi(n))}{2(n-1)} \right\rceil$$

and the upper bound of $\tau(\overline{\Gamma}_n)$ is obtained by Theorem 3.4 that $\tau(\overline{\Gamma}_n) \leq n-p$. Therefore, the lower and upper bounds of $\tau(\Gamma_n)+\tau(\overline{\Gamma}_n)$ are shown as follows:

$$\frac{(n+2)(n+1)-\varphi(n)}{2(n-1)} \leq \tau(\Gamma_n)+\tau(\overline{\Gamma}_n) \leq \frac{2(n-p+1)+\varphi(n)}{2}.$$

□

Theorem 4.4. *Let $n = p^k$ be such that p is a prime number and $k \in \mathbb{N}\setminus\{1\}$. Then $\tau(\Gamma_n)\tau(\overline{\Gamma}_n) = 2^{2k-2}-2^{2k-3}+2^{k-1}$ where $p = 2$ and*

$$\frac{p^{3k-2}(p-1)-p^{2k-1}(p-3)-2p^k}{4(p^k-1)} \leq \tau(\Gamma_n)\tau(\overline{\Gamma}_n) \leq \frac{p^{2k}(p-1)+p^k(p+1)+2p}{4}$$

for all $p\rangle 2$.

Proof. For proving this theorem, the process is similar to the proof of Theorem 4.1 by using Theorems 2.1 and 3.2. Hence the result follows, directly. □

Theorem 4.5. *Let n be an even positive integer and not a power of 2. Then*

$$\frac{(n(n-1)-n\varphi(n))(\varphi(n)+2)}{4(n-1)} \leq \tau(\Gamma_n)\tau(\overline{\Gamma}_n) \leq \begin{cases} \dfrac{3n\varphi(n)+6n}{8} & \textit{if } 4|n; \\ \dfrac{(3n+4)(\varphi(n))+6n+8}{8} & \textit{if } 4\nmid n. \end{cases}$$

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Proof. To prove this theorem, the process is similar to the proof of Theorem 4.2 by using Theorems 2.1 and 3.3. Thus the result holds, as required. □

Theorem 4.6. *Let n be a positive integer greater than 3. If p is the least prime divisor of n, then*

$$\frac{(n(n-1) - n\varphi(n))(\varphi(n)+2)}{4(n-1)} \leq \tau(\Gamma_n)\tau(\overline{\Gamma}_n) \leq \frac{(n-p)(\varphi(n)+2)}{2}.$$

Proof. To prove this theorem, the process is similar to the proof of Theorem 4.3 by using Theorems 2.1 and 3.4 and we can obtain the bounds for $\tau(\Gamma_n)\tau(\overline{\Gamma}_n)$, as desired. □

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