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ON PRODUCTS OF CONJUGACY CLASSES IN GENERAL LINEAR GROUPS

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ABSTRACT. Let K be a field and $n \geq 3$. Let $\mathrm{E}_n(K) \leq H \leq \mathrm{GL}_n(K)$ be an intermediate group and C a noncentral H-class. Define $m(C)$ as the minimal positive integer m such that $\exists i_1, \dots, i_m \in \{\pm 1\}$ such that the product $C^{i_1} \cdots C^{i_m}$ contains all nontrivial elementary transvections. In this article we obtain a sharp upper bound for $m(C)$. Moreover, we determine $m(C)$ for any noncentral H-class C under the assumption that K is algebraically closed or $n = 3$ or $n = \infty$.

1. Introduction

The investigation of products of conjugacy classes in different types of groups is a popular topic in group theory during the last 30-40 years. Many papers were devoted to this theme, for example [1, 3, 6, 8, 9, 10, 11, 12, 13, 16, 20]. A lot of these works are concerned with the computation of (extended) covering numbers. Recall that if G is a group, then the *covering number* $cn(G)$ is the smallest integer m such that $C^m = G$ for every conjugacy class C of G which is not contained in any proper normal subgroup of G. The *extended covering number* $ecn(G)$ is the smallest integer m such that $C_1 C_2 \cdots C_m = G$ whenever $C_1, C_2, \dots, C_m$ are conjugacy classes of G not contained in any proper normal subgroup of G.

In this paper we address the following problem. Let K be a field, $n \geq 3$ and $\mathrm{E}_n(K) \leq H \leq \mathrm{GL}_n(K)$ an intermediate group. For a noncentral H-class C (see beginning of Section 2) we define

$$m(C) := \min\{m \in \mathbb{N} \mid \exists i_1, \dots, i_m \in \{\pm 1\} : T \subseteq C^{i_1} \cdots C^{i_m}\}$$

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where T denotes the H-class of the elementary transvection $t_{12}(1)$. The Sandwich Classification Theorem (see the next paragraph) implies that the minimum in the definition of $m(C)$ exist. Our goal is to compute $m(C)$, the motivation for this goal is explained below.

In [2, 1964, H. Bass] showed that if R is an (associative, unital) ring then

$$\forall H \leq \mathrm{GL}_n(R) : H^{\mathrm{E}_n(R)} \subseteq H \Leftrightarrow \exists I \lhd R : \mathrm{E}_n(R, I) \leq H \leq \mathrm{C}_n(R, I) \tag{1.1}$$

provided n is large enough with respect to the stable rank of R. Here $\mathrm{E}_n(R)$ denotes the elementary subgroup, $\mathrm{E}_n(R, I)$ the relative elementary subgroup of level I and $\mathrm{C}_n(R, I)$ the full congruence subgroup of level I. Bass's result is known as *Sandwich Classification Theorem*. In the 1970's and 80's, the validity of this theorem was extended by J. Wilson [21], I. Golubchik [7], L. Vaserstein [17, 18, 19] and others. Statement (1.1) holds true for example if R is a commutative ring and $n \geq 3$.

It follows from the Sandwich Classification Theorem that if σ is a matrix in $\mathrm{GL}_n(R)$ where R is a commutative ring and $n \geq 3$, then each of the elementary transvections $t_{kl}(\sigma_{ij}), t_{kl}(\sigma_{ii}-\sigma_{jj})$ $(i \neq j, k \neq l)$ can be expressed as a finite product of $\mathrm{E}_n(R)$-conjugates of σ and σ^{-1}. In [5, 1960, J. Brenner] showed that in the case $R = \mathbb{Z}$ there is a bound for the number of $\mathrm{E}_n(R)$-conjugates needed for such an expression. In 2018, the author [14] proved that indeed for *any* commutative ring there is a bound for the number of $\mathrm{E}_n(R)$-conjugates. In 2020, the author [15] obtained bounds over different classes of noncommutative rings.

In this paper we find the *optimal* bound over the class of fields. Namely, we prove that if K is a field, $n \geq 3$, $\mathrm{E}_n(K) \leq H \leq \mathrm{GL}_n(K)$ and C is a noncentral H-class, then $m(C) \leq 4$ and this bound is sharp (in the sense that there is no better uniform bound over the class of fields). Furthermore, we determine $m(C)$ for any noncentral H-class C under the assumption that K is algebraically closed or $n = 3$ or $n = \infty$.

The rest of the paper is organised as follows. In Section 2 we recall some standard notation which is used throughout the paper. Moreover, we recall the Frobenius form and the generalised Jordan form of a square matrix and some basic definitions regarding general linear groups. In Section 3 we prove that $m(C) \leq 4$ for any noncentral H-class C and that this bound is sharp (see Proposition 3.9 and Example 3.11). We also show that $m(C) \leq 2$ if K is algebraically closed. In Section 4 we compute $m(C)$ in the case $n = 3$, the main result of this section is Theorem 4.5. In Section 5 we consider the case $n = \infty$.

2. Preliminaries

If G is a group and $g, h \in G$, we let $g^h := h^{-1}gh$ and $[g, h] := ghg^{-1}h^{-1}$. Let H be a subgroup of G and $g, g' \in G$. If there is an $h \in H$ such that $g^h = g'$, we write $g \sim_H g'$ (or just $g \sim g'$ if $H = G$). Clearly $\sim_H$ is an equivalence relation on G. We call the $\sim_H$-equivalence class of an element $g \in G$ the *H-class* of g and denote it by g^H. For an H-class $C = g^H$ we define $C^1 := C$ and $C^{-1} := (g^{-1})^H$.

Throughout the paper K denotes a field and K^* the set of all nonzero elements of K. $\mathbb{N}$ denotes the set of positive integers. For any $m, n \in \mathbb{N}$, the set of all $m \times n$ matrices over K is denoted by $\mathrm{M}_{m\times n}(K)$. Instead of $\mathrm{M}_{n\times n}(K)$ we may write $\mathrm{M}_n(K)$. The identity matrix in $\mathrm{M}_n(K)$ is denoted by e or $e_{n\times n}$ and the matrix with a one at position (i, j) and zeros elsewhere is denoted by e^{ij}.

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2.1. **Normal forms for matrices.** In this subsection n denotes a positive integer. We recall the Frobenius form and the generalised Jordan form of a matrix in $\mathrm{M}_n(K)$.

Definition 2.1. *Let $P = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X]$ be a monic polynomial of degree n. The companion matrix of P is the matrix*

$$[P] = \begin{pmatrix} & & & -a_0 \\ 1 & & & -a_1 \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix} \in \mathrm{M}_n(K).$$

Definition 2.2. *Let $\sigma \in \mathrm{M}_m(K)$ and $\tau \in \mathrm{M}_n(K)$. The direct sum of σ and τ is the matrix*

$$\sigma \oplus \tau = \begin{pmatrix} \sigma & \\ & \tau \end{pmatrix} \in \mathrm{M}_{m+n}(K).$$

Definition 2.3. *Let $\sigma, \tau \in \mathrm{M}_n(K)$. If there is an invertible $\rho \in \mathrm{M}_n(K)$ such that $\sigma = \rho\tau\rho^{-1}$, then we write $\sigma \sim \tau$ and call σ and τ similar.*

Theorem 2.4. *Let $\sigma \in \mathrm{M}_n(K)$. Then there are uniquely determined nonconstant, monic polynomials $P_1, \dots, P_r \in K[X]$ such that $P_1|P_2|\cdots|P_r$ and $\sigma \sim [P_1] \oplus \cdots \oplus [P_r]$.*

Proof. See [4, Part V, Chapter 21, Theorem 4.4]. □

The matrix $[P_1] \oplus \cdots \oplus [P_r]$ in Theorem 2.4 is called the *Frobenius form* or *rational canonical form* of σ. We denote it by $F(\sigma)$. The polynomials $P_1, \dots, P_r$ are called the *invariant factors* of σ.

Recall that the *characteristic matrix* of a matrix $\sigma \in \mathrm{M}_n(K)$ is the matrix $Xe - \sigma \in \mathrm{M}_n(K[X])$. The *characteristic polynomial* of σ is the polynomial $\chi_\sigma = \det(Xe - \sigma)$. Note that the characteristic polynomial of a companion matrix $[P]$ is P. The invariant factors of a matrix $\sigma \in \mathrm{M}_n(K)$ are precisely the monic polynomials associated with the nonconstant polynomials in the Smith normal form of the characteristic matrix $Xe - \sigma$. The Smith normal form of $Xe - \sigma$ can be computed using the algorithm described in [4, Part V, Chapter 20, Proof of Theorem 3.2].

Let $\sigma \in \mathrm{M}_n(K)$ and $P_1, \dots, P_r$ the invariant factors of σ. For each $1 \leq i \leq r$ we can write $P_i = \prod\limits_{j=1}^{s_i} P_{ij}^{q_{ij}}$ where $P_{i1}, \dots, P_{is_i} \in K[X]$ are pairwise distinct irreducible, monic polynomials and $q_{i1}, \dots, q_{is_i} \geq 1$. This factorisation is unique up to the order of the factors. The polynomials $P_{ij}^{q_{ij}}$ $(1 \leq i \leq r, 1 \leq j \leq s_i)$ form the system of *elementary divisors* of σ. Note that $P_{ij}^{q_{ij}}$ maybe equal to $P_{i'j'}^{q_{i'j'}}$ if $i \neq i'$.

In order to define the generalised Jordan form of a matrix, we need the notion of the generalised Jordan block of a power P^q of an irreducible polynomial P. For polynomials P of degree 1 one gets back the usual Jordan blocks.

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Definition 2.5. *Let $q \in \mathbb{N}$ and $P \in K[X]$ an irreducible, monic polynomial of degree n. The generalised Jordan block corresponding to P^q is the matrix*

$$J(P^q) = \begin{pmatrix} [P] & & & \\ \xi & [P] & & \\ & \ddots & \ddots & \\ & & \xi & [P] \end{pmatrix} \in \mathrm{M}_{qn}(K)$$

where $\xi \in \mathrm{M}_n(K)$ is the matrix that has a 1 at position $(1, n)$ and zeros elsewhere.

Theorem 2.6. *Let $\sigma \in \mathrm{M}_n(K)$ and $(P_i^{q_i})_{i\in\Phi}$ the system of elementary divisors of σ. Then $\sigma \sim \bigoplus_{i\in\Phi} J(P_i^{q_i})$ where the order of direct summands maybe arbitrary.*

Proof. See [4, Part V, Chapter 21, Section 5]. □

The matrix $\bigoplus_{i\in\Phi} J(P_i^{q_i})$ in Theorem 2.6 is called the *generalised Jordan form* or *primary rational canonical form* of σ. It is uniquely determined up to the order of the generalised Jordan blocks. If the characteristic polynomial of σ splits into linear factors, then one gets back the usual Jordan form of σ.

2.2. **The general linear group $\mathrm{GL}_n(K)$.** In this subsection n denotes a positive integer.

Definition 2.7. *The group $\mathrm{GL}_n(K)$ consisting of all invertible elements of $\mathrm{M}_n(K)$ is called the general linear group of degree n over K.*

Definition 2.8. *Let $a \in K$ and $1 \le i \ne j \le n$. Then the matrix $t_{ij}(a) = e + ae^{ij}$ is called an elementary transvection. If $a \ne 0$, then $t_{ij}(a)$ is called nontrivial. The subgroup $\mathrm{E}_n(K)$ of $\mathrm{GL}_n(K)$ generated by the elementary transvections is called the elementary subgroup.*

Lemma 2.9. *For any $\sigma \in \mathrm{GL}_n(K)$ there is an $\epsilon \in \mathrm{E}_n(K)$ such that $\sigma = \epsilon d_n(\det(\sigma))$. It follows that $\mathrm{E}_n(K)$ equals the subgroup $\mathrm{SL}_n(K)$ of $\mathrm{GL}_n(K)$ consisting of all matrices with determinant 1.*

Proof. See for example [2, §5]. □

The lemma below is easy to check.

Lemma 2.10. *The relations*

$$t_{ij}(a)t_{ij}(b) = t_{ij}(a+b), \tag{R1}$$

$$[t_{ij}(a), t_{hk}(b)] = e \text{ and} \tag{R2}$$

$$[t_{ij}(a), t_{jk}(b)] = t_{ik}(ab) \tag{R3}$$

hold where $i \ne k, j \ne h$ in (R2) and $i \ne k$ in (R3).

Definition 2.11. *Let $a \in K^*$ and $1 \le i \ne j \le n$. We define*

$$d_i(a) := e + (a-1)e^{ii} \in \mathrm{GL}_n(K)$$

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and

$$d_{ij}(a) := e + (a-1)e^{ii} + (a^{-1}-1)e^{jj} \in \mathrm{SL}_n(K) = \mathrm{E}_n(K).$$

Moreover, we define

$$p_{ij} := e + e^{ij} + e^{ji} - e^{ii} - e^{jj} \in \mathrm{GL}_n(K)$$

and

$$\hat{p}_{ij} := e + e^{ij} - e^{ji} - e^{ii} - e^{jj} \in \mathrm{SL}_n(K) = \mathrm{E}_n(K).$$

2.3. The stable general linear group $\mathrm{GL}_\infty(K)$.

Definition 2.12. *The direct limit $G_\infty(K) = \varinjlim_n \mathrm{GL}_n(K)$ with respect to the transition homomorphisms $\phi_{n,n+k} : \mathrm{GL}_n(K) \to \mathrm{GL}_{n+k}(K),\ \sigma \mapsto e_{k\times k} \oplus \sigma$ is called the stable general linear group over K.*

Let $\sigma \in \mathrm{GL}_m(K)$ and $\tau \in \mathrm{GL}_n(K)$. If $\phi_{m,\max\{m,n\}}(\sigma) = \phi_{n,\max\{m,n\}}(\tau)$, we write $\sigma \sim_\infty \tau$. We identify $\mathrm{GL}_\infty(K)$ with the set $\bigcup_n \mathrm{GL}_n(K)/\sim_\infty$ of all $\sim_\infty$-equivalence classes made a group by defining $[\sigma]_\infty[\tau]_\infty = [\phi_{m,\max\{m,n\}}(\sigma)\phi_{n,\max\{m,n\}}(\tau)]_\infty$ for any $\sigma \in \mathrm{GL}_m(K)$ and $\tau \in \mathrm{GL}_n(K)$.

Definition 2.13. *The subgroup $\mathrm{E}_\infty(K)$ of $\mathrm{GL}_\infty(K)$ generated by the elements $[t_{n,i,j}(a)]_\infty$ ($n \geq 2,\ 1 \leq i \neq j \leq n,\ a \in K$) is called the elementary subgroup. Here $t_{n,i,j}(a)$ denotes the elementary transvection $t_{ij}(a) \in \mathrm{GL}_n(K)$.*

3. Products of conjugacy classes in $\mathrm{GL}_n(K)$

In this section n denotes an integer greater than 2. We set $G := \mathrm{GL}_n(K)$ and $E := \mathrm{E}_n(K)$. H denotes a subgroup of G containing E, and T denotes the H-class of $t_{12}(1)$. If C is a noncentral H-class, then $m(C)$ is defined as in Section 1.

The lemma below shows that $T = t_{12}(1)^G = t_{12}(1)^E$.

Lemma 3.1. *Let $\sigma \in G$. Then $t_{12}(1) \sim_E \sigma \Leftrightarrow t_{12}(1) \sim_H \sigma \Leftrightarrow t_{12}(1) \sim_G \sigma$.*

Proof. Clearly $t_{12}(1) \sim_E \sigma \Rightarrow t_{12}(1) \sim_H \sigma \Rightarrow t_{12}(1) \sim_G \sigma$. Hence it suffices to show that $t_{12}(1) \sim_G \sigma \Rightarrow t_{12}(1) \sim_E \sigma$. Suppose that $t_{12}(1) \sim_G \sigma$. Then there is a $\rho \in G$ such that $t_{12}(1) = \sigma^\rho$. By Lemma 2.9 there is an $\epsilon \in E$ such that $\rho = \epsilon d_n(\det(\rho))$. Since $n \geq 3$, the matrix $t_{12}(1)$ commutes with $d_n(\det(\rho))$ and hence $t_{12}(1) = \sigma^\epsilon$. Thus $t_{12}(1) \sim_E \sigma$. □

It follows from the next lemma that T contains all nontrivial elementary transvection.

Lemma 3.2. *Let $t_{ij}(a)$ and $t_{kl}(b)$ be nontrivial elementary transvections. Then $t_{ij}(a) \sim_E t_{kl}(b)$.*

Proof. It is an easy exercise to show that $t_{ij}(a)^\epsilon = t_{kl}(a)$ for some $\epsilon \in \mathrm{E}$ which is a product of $\hat{p}_{st}$'s (see Definition 2.11). Hence $t_{ij}(a) \sim_E t_{kl}(a)$. It remains to show that $t_{kl}(a) \sim_E t_{kl}(b)$. But clearly $t_{kl}(a)^{d_{lm}(a^{-1}b)} = t_{kl}(b)$ for any $m \neq k, l$. □

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Let C be an H-class. Since similar matrices have the same characteristic polynomial (resp. determinant, trace, invariant factors, elementary divisors), we can define χ_C (resp. $\det(C)$, $\operatorname{tr}(C)$, the invariant factors of C, the elementary divisors of C, the Frobenius form $F(C)$) in the obvious way. Below we compute $F(T)$. Note that $T = \{g \in G \mid F(\sigma) = F(T)\}$ by Lemma 3.1.

Lemma 3.3. $F(T) = [X-1] \oplus \dots \oplus [X-1] \oplus [(X-1)^2]$.

Proof. By Lemma 3.2, we have $F(T) = F(t_{n,n-1}(1))$. One checks easily that

$$t_{n,n-1}(1)^{t_{n-1,n}(1)} = \left(\begin{array}{c|cc} e_{(n-2)\times(n-2)} & & \\ \hline & & -1 \\ & 1 & 2 \end{array}\right).$$

□

Let C and D be noncentral H-classes. We write $C \sim D$ and call C and D *conjugated* if there is a $\rho \in G$ such that $C^\rho = D$. The lemma below implies that $m(C)$ does only depend on the conjugacy class of C.

Lemma 3.4. *Let $C \sim D$ be conjugated noncentral H-classes and suppose that $T \subseteq C^{i_1} \cdots C^{i_k}$ where $i_1, \dots, i_k \in \{\pm 1\}$. Then $T \subseteq D^{i_1} \cdots D^{i_k}$.*

Proof. Choose a $\sigma \in C$ and a $\tau \in D$. Since $C \sim D$, there is a $\rho \in G$ such that $\sigma = \tau^\rho$. By Lemma 2.9 there is an $\epsilon \in E$ and an $a \in K^*$ such that $\rho = \epsilon d_n(a)$. Hence

$$\sigma = \tau^{\epsilon d_n(a)}. \tag{3.1}$$

Since $T \subseteq C^{i_1} \cdots C^{i_k}$, there are $\rho_1, \dots, \rho_k \in H$ such that

$$t_{12}(1) = (\sigma^{i_1})^{\rho_1} \cdots (\sigma^{i_k})^{\rho_k}.$$

By Lemma 2.9 there are $\epsilon_1, \dots, \epsilon_k \in E$ and $a_1, \dots, a_k \in K$ such that $\rho_i = \epsilon_i d_n(a_i)$ $(1 \le i \le k)$. Note that $d_n(a_1), \dots, d_n(a_k) \in H$ since H contains E. We have

$$t_{12}(1) = (\sigma^{i_1})^{\epsilon_1 d_n(a_1)} \cdots (\sigma^{i_k})^{\epsilon_k d_n(a_k)}. \tag{3.2}$$

It follows from Equations (3.1) and (3.2) that

$$t_{12}(1) = (\tau^{i_1})^{\epsilon d_n(a)\epsilon_1 d_n(a_1)} \cdots (\tau^{i_k})^{\epsilon d_n(a)\epsilon_k d_n(a_k)}.$$

One easily checks that $d_n(a)$ commutes with elementary transvections modulo E. Hence there are $\epsilon_1', \dots, \epsilon_k' \in E$ such that $d_n(a)\epsilon_i = \epsilon_i' d_n(a)$ $(1 \le i \le k)$. We get

$$t_{12}(1) = (\tau^{i_1})^{\epsilon\epsilon_1' d_n(a_1) d_n(a)} \cdots (\tau^{i_k})^{\epsilon\epsilon_k' d_n(a_k) d_n(a)}.$$

Since $n \ge 3$, the matrix $t_{12}(1)$ commutes with $d_n(a)$ and thus

$$t_{12}(1) = (\tau^{i_1})^{\epsilon\epsilon_1' d_n(a_1)} \cdots (\tau^{i_k})^{\epsilon\epsilon_k' d_n(a_k)} \in D^{i_1} \cdots D^{i_k}.$$

□

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Lemma 3.5. *Let C be a noncentral H-class. Then $m(C) = 1$ iff $C = T$.*

Proof. Clear since $T = T^{-1}$ by Lemma 3.2. □

Let C be a noncentral H-class. Theorem 3.6 below shows that if χ_C has a root in K (which is always true if K is algebraically closed), then $m(C) \leq 2$. But we will see later that $m(C)$ can be greater than 2 if χ_C has no root.

Theorem 3.6. *Let C be a noncentral H-class. If χ_C has a root in K, then $T \subseteq CC^{-1}$.*

Proof. Let $a \in K$ be a root of χ_C. First note that $a \neq 0$ since the constant coefficient of χ_C is $(-1)^n \det(C) \neq 0$. Since $(X-a)$ divides χ_C we get that $(X-a)^q$ is an elementary divisor of C for some $q \in \mathbb{N}$ (since χ_C is the product of the elementary divisors of C). Choose a $\sigma \in C$. By Theorem 2.6 we have

$$\sigma \sim \begin{pmatrix} J((X-a)^q) & \\ & \tau \end{pmatrix} =: \rho$$

for some $\tau \in \mathrm{GL}_{n-q}(K)$. Recall that

$$J((X-a)^q) = \begin{pmatrix} a & & & \\ 1 & a & & \\ & \ddots & \ddots & \\ & & 1 & a \end{pmatrix} \in \mathrm{GL}_q(K).$$

By Theorem 2.4 we may assume that τ is in Frobenius form, i.e. there are nonconstant, monic polynomials $P_1, \dots, P_r \in K[X]$ such that $P_1 | P_2 | \cdots | P_r$ and $\tau = [P_1] \oplus \oplus [P_r]$.

Case 1 Suppose that $q = 1$. Then

$$\rho = \begin{pmatrix} a & \\ & \tau \end{pmatrix}.$$

Subcase 1.1 Suppose that each of the P_i's has degree 1. Since $P_1 | P_2 | \cdots | P_r$, it follows that $P_1 = \cdots = P_r = X - b$ for some $b \in K^*$. Hence

$$\rho = \begin{pmatrix} a & & & \\ & b & & \\ & & \ddots & \\ & & & b \end{pmatrix}.$$

One checks easily that $[\rho, t_{12dot}(1)] = t_{12}(ab^{-1} - 1)$. It follows that $t_{12}(ab^{-1} - 1) \in DD^{-1}$ where $D = \rho^H$. Since σ is noncentral, we have $a \neq b$ and hence $ab^{-1} - 1 \neq 0$. Hence $T \subseteq DD^{-1}$ by Lemma 3.2 and thus $T \subseteq CC^{-1}$ by Lemma 3.4.

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Subcase 1.2 Suppose there is an i such that P_i has degree $t \geq 2$. Write $P_i = X^t + b_{t-1}X^{t-1} + \cdots + b_1X + b_0$. Then

$$\sigma \sim \left(\begin{array}{c|ccccc|c} a & & & & & & \\ \hline & & & & & -b_0 & \\ & 1 & & & & -b_1 & \\ & & 1 & & & -b_2 & \\ & & & \ddots & & \vdots & \\ & & & & 1 & -b_{t-1} & \\ \hline & & & & & & * \end{array}\right) =: \xi.$$

It follows that

$$\sigma \sim \xi^{t_{23}(-a)} = \left(\begin{array}{c|ccccc|c} a & & & & & & \\ \hline & a & * & & & * & \\ & 1 & * & & & * & \\ & & 1 & & & -b_2 & \\ & & & \ddots & & \vdots & \\ & & & & 1 & -b_{t-1} & \\ \hline & & & & & & * \end{array}\right) =: \zeta.$$

One checks easily that $[\zeta, t_{21}(1)] = t_{31}(a^{-1})$. It follows as in Subcase 1.1 that $T \subseteq CC^{-1}$.

Case 2 Suppose that $q = 2$. Then

$$\rho = \left(\begin{array}{cc|c} a & & \\ 1 & a & \\ \hline & & \tau \end{array}\right).$$

Subcase 2.1 Suppose that each of the P_i's has degree 1. Since $P_1|P_2|\cdots|P_r$, it follows that $P_1 = \cdots = P_r = X - b$ for some $b \in K^*$. Hence

$$\rho = \left(\begin{array}{cc|ccc} a & & & & \\ 1 & a & & & \\ \hline & & b & & \\ & & & \ddots & \\ & & & & b \end{array}\right).$$

Clearly

$$\sigma \sim \rho^{t_{12}(a-b)} = \left(\begin{array}{cc|ccc} b & * & & & \\ 1 & * & & & \\ \hline & & b & & \\ & & & \ddots & \\ & & & & b \end{array}\right) =: \xi.$$

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One checks easily that $[\xi, t_{13}(1)] = t_{23}(b^{-1})$. It follows as in Subcase 1.1 that $T \subseteq CC^{-1}$.

Subcase 2.2 Suppose there is an i such that P_i has degree $t \geq 2$. Write $P_i = X^t + b_{t-1}X^{t-1} + \cdots + b_1X + b_0$. Then

$$\sigma \sim \left(\begin{array}{cc|ccccc|c} a & & & & & & & \\ 1 & a & & & & & & \\ \hline & & & & & & -b_0 & \\ & & 1 & & & & -b_1 & \\ & & & 1 & & & -b_2 & \\ & & & & \ddots & & \vdots & \\ & & & & & 1 & -b_{t-1} & \\ \hline & & & & & & & * \end{array}\right) =: \xi.$$

It follows that

$$\sigma \sim \xi^{t_{34}(-a)} = \left(\begin{array}{cc|ccccc|c} a & & & & & & & \\ 1 & a & & & & & & \\ \hline & & a & * & & & * & \\ & & 1 & * & & & * & \\ & & & 1 & & & -b_2 & \\ & & & & \ddots & & \vdots & \\ & & & & & 1 & -b_{t-1} & \\ \hline & & & & & & & * \end{array}\right) =: \zeta.$$

One checks easily that $[\zeta, t_{31}(1)] = t_{41}(a^{-1})$. It follows as in Subcase 1.1 that $T \subseteq CC^{-1}$.

Case 3 Suppose that $q \geq 3$. Then

$$\rho = \left(\begin{array}{ccc|c} a & & & \\ 1 & a & & \\ & 1 & a & \\ \hline & & & * \end{array}\right).$$

One checks easily that $[\rho, t_{21}(1)] = t_{31}(a^{-1})$. It follows as in Subcase 1.1 that $T \subseteq CC^{-1}$.

□

Lemma 3.5 and Theorem 3.6 directly imply the following theorem.

Theorem 3.7. *Suppose that K is algebraically closed. Let C be a noncentral H-class. Then $m(C) = 1$ if $C = T$ respectively $m(C) = 2$ if $C \neq T$.*

The converse of Theorem 3.6 does not hold as the following example shows. Suppose that $K = \mathbb{F}_2$ and $n = 4$. Let C be the G-class of $[X^4 + X^2 + 1]$. Set

$$\sigma := \begin{pmatrix} & 1 & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \end{pmatrix} \in G \text{ and } \tau := \begin{pmatrix} & & 1 & 1 \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & \end{pmatrix} \in G.$$

Clearly $\sigma = t_{12}(1)\tau$ and hence $\sigma\tau^{-1} = t_{12}(1)$. We leave it to the reader to check that $F(\sigma) = F(\tau) = [X^4 + X^2 + 1]$ which implies that $\sigma, \tau \in C$. It follows that $T \subseteq CC^{-1}$ although $\chi_C = X^4 + X^2 + 1$ has no root in K. However, a "weak converse" of Theorem 3.6 does hold. Namely if C is a noncentral H-class such that $T \subseteq CC^{-1}$, then χ_C is reducible as Theorem 3.8 below shows.

In Theorem 3.8 we use the following notation. If $\sigma \in G$, $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$, then $\sigma^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ denotes the $k \times k$ matrix whose entry at position (s,t) is σ_{i_s,j_t}. Recall that $\det(\sigma^{i_1,\dots,i_k}_{i_1,\dots,i_k})$, where $1 \leq i_1 < \cdots < i_k \leq n$, is called a *principal minor of size* k. It is well-known that

$$\chi_\sigma = \sum_{k=0}^{n} (-1)^k a_k X^{n-k}$$

where a_k is the sum of all principal minors of size k.

Theorem 3.8. *Let C be a noncentral H-class. If $T \subseteq CC^{-1}$, then χ_C is reducible.*

Proof. Suppose there is a noncentral H-class C such that $T \subseteq CC^{-1}$ and χ_C is irreducible. We will show that this assumption leads to a contradiction. Let $\sigma, \tau \in C$ such that $t_{12}(1) = \sigma\tau^{-1}$. It follows that

$$\sigma = t_{12}(1)\tau. \tag{3.3}$$

Clearly we may assume that

$$\tau_{ij} = 0 \text{ for any } i \geq 2 \text{ and } j \geq i + 2 \tag{3.4}$$

(conjugate Equation (3.3) by appropriate elements of E commuting with $t_{12}(1)$). Since $\sigma, \tau \in C$, we have $\operatorname{tr}(\sigma) = \operatorname{tr}(\tau)$. Hence $\operatorname{tr}(\sigma) \overset{(3.3)}{=} \operatorname{tr}(t_{12}(1)\tau) = \operatorname{tr}(\tau) + \tau_{21} = \operatorname{tr}(\sigma) + \tau_{21}$ whence $\tau_{21} = 0$. Suppose that $\tau_{23} = 0$. Then all nondiagonal entries of τ in the second row are 0. Therefore $\chi_\tau = \chi_C$ has a linear factor, which contradicts the assumption that χ_C is irreducible. Hence $\tau_{23} \neq 0$ and therefore we may assume that $\tau_{22} = 0$ (conjugate (3.3) by $t_{32}(-\tau_{23}^{-1}\tau_{22})$) . Hence

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} & \cdots & \cdots & \tau_{1n} \\ 0 & 0 & \tau_{23} & 0 & \cdots & 0 \\ \tau_{31} & \tau_{32} & \tau_{33} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \tau_{n-1,n1} \\ \tau_{n1} & \tau_{n2} & \tau_{n3} & \cdots & \cdots & \tau_{nn} \end{pmatrix}$$

and, because of (3.3),

$$\sigma = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13}+\tau_{23} & \cdots & \cdots & \tau_{1n} \\ 0 & 0 & \tau_{23} & 0 & \cdots & 0 \\ \tau_{31} & \tau_{32} & \tau_{33} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \tau_{n-1,n1} \\ \tau_{n1} & \tau_{n2} & \tau_{n3} & \cdots & \cdots & \tau_{nn} \end{pmatrix}$$

Consider the statement

$$P(k): \ \tau_{k1} = 0.$$

We will show by induction on k that $P(k)$ holds for any $3 \leq k \leq n$.

<u>$k = 3$</u> Since $\sigma, \tau \in C$, we have $\chi_\sigma = \chi_\tau$. We compare the coefficients of X^{n-2} in χ_σ and χ_τ, namely the sums of all principal minors of size 2. Suppose $\det(\sigma^{i_1,i_2}_{i_1,i_2}) \neq \det(\tau^{i_1,i_2}_{i_1,i_2})$. Then clearly $i_1 = 1$ and $i_2 = 3$. Moreover, $\det(\sigma^{1,3}_{1,3}) - \det(\tau^{1,3}_{1,3}) = -\tau_{23}\tau_{31}$. Hence $\tau_{23}\tau_{31} = 0$. Since clearly $\tau_{23} \neq 0$, we obtain $\tau_{31} = 0$. Thus $P(3)$ holds.

<u>$k \to k+1$</u> Suppose that $P(3), \dots, P(k)$ hold for some $k \in \{3, \dots, n-1\}$. We will show that $P(k+1)$ holds. We compare the coefficients of X^{n-k} in χ_σ and χ_τ, namely the sums of all principal minors of size k (multiplied by $(-1)^k$). Suppose $\det(\sigma^{i_1,\dots,i_k}_{i_1,\dots,i_k}) \neq \det(\tau^{i_1,\dots,i_k}_{i_1,\dots,i_k})$. Then clearly $i_1 = 1$ and $i_2 = 3$. Moreover,

$$\det(\sigma^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) - \det(\tau^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) = \tau_{23}\det(\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k})$$

(consider the Laplace expansion along the first row). In view of (3.4) we have

$$\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k} = \begin{pmatrix} \tau_{31} & \tau_{3i_3} & 0 & \dots & 0 \\ \tau_{i_31} & \tau_{i_3i_3} & \tau_{i_3i_4} & \ddots & \vdots \\ \tau_{i_41} & \tau_{i_4i_3} & \tau_{i_4i_4} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \tau_{i_{k-1}i_k} \\ \tau_{i_k1} & \tau_{i_ki_3} & \tau_{i_ki_4} & \cdots & \tau_{i_ki_k} \end{pmatrix}.$$

Consider the statement

$$Q(j): \ i_j = j+1.$$

We will show by induction on j that $Q(j)$ holds for any $3 \leq j \leq k$.

<u>$j = 3$</u> Assume that $i_3 > 4$. Then $\tau_{3i_3} = 0$ by (3.4). But $\tau_{31} = 0$ by $P(3)$, whence $\det(\sigma^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) - \det(\tau^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) = \tau_{23}\det(\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k}) = 0$. But this contradicts the assumption that $\det(\sigma^{i_1,\dots,i_k}_{i_1,\dots,i_k}) \neq \det(\tau^{i_1,\dots,i_k}_{i_1,\dots,i_k})$. Hence $i_3 = 4$ and thus $Q(3)$ holds.

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$\underline{j \to j+1}$ Suppose that $Q(3), \dots, Q(j)$ hold for some $j \in \{3, \dots, k-1\}$. Write

$$\sigma^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k} = \begin{pmatrix} A & B \\ D & E \end{pmatrix}$$

where $A \in \mathrm{M}_{(j-1)\times(j-1)}(K)$, $B \in \mathrm{M}_{(j-1)\times(n-j+1)}(K)$, $D \in \mathrm{M}_{(n-j+1)\times(j-1)}(K)$ and $E \in \mathrm{M}_{(n-j+1)\times(n-j+1)}(K)$. Then

$$A = \begin{pmatrix} \tau_{31} & \tau_{34} & 0 & \cdots & 0 \\ \tau_{41} & \tau_{44} & \tau_{45} & \ddots & \vdots \\ \tau_{51} & \tau_{54} & \tau_{55} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \tau_{j,j+1} \\ \tau_{j+1,1} & \tau_{j+1,4} & \tau_{j+1,5} & \cdots & \tau_{j+1,j+1} \end{pmatrix}$$

and B is the matrix whose entry at position $(j-1,1)$ is $\tau_{j+1,i_{j+1}}$ and whose other entries are zero. Assume that $i_{j+1} > j+2$. Then $\tau_{j+1,i_{j+1}} = 0$ by (3.4). Hence $\det \sigma^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k} = \det(A)\det(E)$. But $\tau_{31} = \dots = \tau_{j+1,1} = 0$ by $P(3), \dots, P(j+1)$ whence $\det(A) = 0$. It follows that $\det(\sigma^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) - \det(\tau^{1,3,i_3\dots,i_k}_{1,3,i_3\dots,i_k}) = \tau_{23}\det(\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k}) = 0$. But this contradicts the assumption that $\det(\sigma^{i_1,\dots,i_k}_{i_1,\dots,i_k}) \neq \det(\tau^{i_1,\dots,i_k}_{i_1,\dots,i_k})$. Hence $i_{j+1} = j+2$ and thus $Q(j+1)$ holds.

We have shown that $Q(j)$ holds for any $3 \leq j \leq k$. Hence

$$\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k} = \sigma^{3,4\dots,k+1}_{1,4\dots,k+1} = \begin{pmatrix} \tau_{31} & \tau_{34} & 0 & \dots & 0 \\ \tau_{41} & \tau_{44} & \tau_{45} & \ddots & \vdots \\ \tau_{51} & \tau_{54} & \tau_{55} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \tau_{k,k+1} \\ \tau_{k+1,1} & \tau_{k+1,4} & \tau_{k+1,5} & \cdots & \tau_{k+1,k+1} \end{pmatrix}.$$

Since $\tau_{31} = \dots = \tau_{k1} = 0$ by $P(3), \dots, P(k)$, it follows that

$$\tau_{23}\det(\sigma^{3,i_3\dots,i_k}_{1,i_3\dots,i_k}) = (-1)^k \tau_{23}\tau_{k+1,1}\tau_{34}\tau_{45}\cdots\tau_{k,k+1}. \tag{3.5}$$

Suppose that $\tau_{j,j+1} = 0$ for some $3 \leq j \leq k$. Then

$$\sigma = \begin{pmatrix} A & B & C \\ 0 & E & 0 \\ F & G & I \end{pmatrix} \sim \begin{pmatrix} E & 0 \\ * & * \end{pmatrix}$$

where $A \in \mathrm{M}_{1\times1}(K)$, $B \in \mathrm{M}_{1\times(j-1)}(K)$, $C \in \mathrm{M}_{1\times(n-j)}(K)$, $E \in \mathrm{M}_{(j-1)\times(j-1)}(K)$, $F \in \mathrm{M}_{(n-j)\times1}(K)$, $G \in \mathrm{M}_{(n-j)\times(j-1)}(K)$ and $I \in \mathrm{M}_{(n-j)\times(n-j)}(K)$. It follows that $\chi_\sigma = \chi_C$ has a factor of degree $j-1$, which contradicts the assumption that χ_C is irreducible. Hence $\tau_{34}, \dots, \tau_{k,k+1} \neq 0$. It follows from (3.5) that $\tau_{k+1,1} = 0$. Thus $P(k+1)$ holds.

We have shown that $P(k)$ holds for any $3 \leq k \leq n$. Hence all nondiagonal entries of σ in the first

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column are zero and hence $\chi_\sigma = \chi_C$ has a linear factor. But this contradicts the assumption that χ_C is irreducible. □

The next proposition implies that $m(C) \leq 4$ for any noncentral H-class.

Proposition 3.9. *Let C be a noncentral H-class. Then $T \subseteq CC^{-1}CC^{-1}$.*

Proof. Choose a $\sigma \in C$. By Theorem 2.4 and Lemma 3.4 we may assume that σ is in Frobenius form, i.e. there are nonconstant, monic polynomials $P_1, \dots, P_r \in K[X]$ such that $P_1|P_2|\cdots|P_r$ and $\sigma = [P_1] \oplus \cdots \oplus [P_r]$. If one of the P_i's has degree 1, then $T \subseteq CC^{-1} \subseteq CC^{-1}CC^{-1}$ by Theorem 3.6 (since $\chi_C = P_1 \cdots P_r$). Hence we may assume that the degree of each P_i is at least 2.

Case 1 Suppose each P_i has degree 2. It follows that $P_1 = P_2 = \cdots = P_r = X^2 - a_1X - a_0$ for some $a_0, a_1 \in K$ since $P_1|P_2|\cdots|P_r$. Hence

$$\sigma = \left(\begin{array}{cc|cc|c} & a_0 & & & \\ 1 & a_1 & & & \\ \hline & & & a_0 & \\ & & 1 & a_1 & \\ \hline & & & & * \end{array}\right)$$

and

$$\sigma^{-1} = \left(\begin{array}{cc|cc|c} -a_0^{-1}a_1 & 1 & & & \\ a_0^{-1} & & & & \\ \hline & & -a_0^{-1}a_1 & 1 & \\ & & a_0^{-1} & & \\ \hline & & & & * \end{array}\right).$$

One checks easily that $[\sigma, t_{14}(1)] = t_{23}(a_0^{-1})t_{14}(-1)$ which implies that $[[\sigma, t_{14}(1)], t_{12}(1)] = t_{13}(-a_0^{-1})$. It follows from the formula

$$[[a,b],c] = a(a^{-1})^{b^{-1}}a^{b^{-1}c^{-1}}(a^{-1})^{c^{-1}}, \tag{3.6}$$

which holds for elements a, b, c of any group, that $t_{13}(-a_0^{-1}) \in CC^{-1}CC^{-1}$. Thus $T \subseteq CC^{-1}CC^{-1}$ by Lemma 3.2.

Case 2 Suppose that there is an i such that P_i has degree $t \geq 3$. By Lemma 3.4 we may assume that $i = 1$. Write $P_1 = X^t - a_{t-1}X^{t-1} - \cdots - a_1X - a_0$. Then

$$\sigma = \left(\begin{array}{ccccc|c} & & & & a_0 & \\ 1 & & & & a_1 & \\ & 1 & & & a_2 & \\ & & \ddots & & \vdots & \\ & & & 1 & a_{t-1} & \\ \hline & & & & & * \end{array}\right)$$

and

$$\sigma^{-1}=\left(\begin{array}{ccccc|c} -a_0^{-1}a_1 & 1 & & & & \\ -a_0^{-1}a_2 & & 1 & & & \\ \vdots & & & \ddots & & \\ -a_0^{-1}a_{t-1} & & & & 1 & \\ a_0^{-1} & & & & & \\ \hline & & & & & * \end{array}\right)$$

One checks easily that $[\sigma, t_{t-1,t}(1)] = t_{t1}(a_0^{-1})t_{t-1,t}(-1)$ which implies that

$$[[\sigma, t_{t-1,t}(1)], t_{t-1,t}(1)] = t_{t-1,1}(-a_0^{-1})$$

(since $t \geq 3$). It follows from Equation (3.6) that $t_{t-1,1}(-a_0^{-1}) \in CC^{-1}CC^{-1}$. Thus $T \subseteq CC^{-1}CC^{-1}$ by Lemma 3.2.

□

Recall that if $P = a_nX^n + \cdots + a_1X + a_0 \in K[X]$ is a polynomial of degree n, then $P^* = a_0X^n + \cdots + a_{n-1}X + a_n \in K[X]$ is called the *reciprocal polynomial* of P. If $\sigma \in G$, then $\chi_{\sigma^{-1}} = \frac{\chi_\sigma^*}{\det(\sigma)}$. It follows that χ_σ is irreducible iff $\chi_{\sigma^{-1}}$ is irreducible (note that $(P^*)^* = P$ and $(PQ)^* = P^*Q^*$ for any $P, Q \in K[X]$ with nonzero constant coefficient).

Theorem 3.10. *Let C be a noncentral H-class. If $C \neq T$, χ_C is irreducible, $\det(C)^2 \neq 1$ and $\det(C)^3 \neq 1$, then $m(C) = 4$.*

Proof. By Lemma 3.5 we have $m(C) > 1$. By Theorem 3.8 we have $T \not\subseteq CC^{-1}$ since χ_C is irreducible. Similarly $T \not\subseteq C^{-1}C$ since $\chi_{C^{-1}}$ is irreducible by the paragraph right before Theorem 3.10. Clearly $T \not\subseteq CC$ and $T \not\subseteq C^{-1}C^{-1}$ since $\det(C)^2 \neq 1$. Similarly $T \not\subseteq C^{i_1}C^{i_2}C^{i_3}$ for all $i_1, i_2, i_3 \in \{\pm 1\}$ since $\det(C)^3 \neq 1$. Hence $m(C) > 3$. It follows from Proposition 3.9 that $m(C) = 4$. □

Example 3.11. *Suppose $K = \mathbb{Q}$. Let $P = X^n + 2 \in K[X]$. Then P is irreducible by Eisenstein's criterion. Let C denote the H-class of $[P]$. Clearly $C \neq T$ since $F(C) = [P]$ but $F(T) = [X-1] \oplus \cdots \oplus [X-1] \oplus [(X-1)^2]$ by Lemma 3.3. Moreover, $\chi_C = P$ and $\det(C) \in \{\pm 2\}$. It follows from Theorem 3.10 that $m(C) = 4$.*

4. The case $n = 3$

Set $G := GL_3(K)$ and $E := E_3(K)$. H denotes a subgroup of G containing E, and T denotes the H-class of $t_{12}(1)$. We will determine $m(C)$ for any noncentral H-class C.

Proposition 4.1. *Let C be a noncentral H-class. Then the following are equivalent.*

(i) χ_C *has a root in* K.
(ii) $T \subseteq CC^{-1}$.
(iii) $T \subseteq C^{-1}C$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Theorem 3.6 and the implication $(ii) \Rightarrow (i)$ from Theorem 3.8 (note that a polynomial $P \in K[X]$ of degree 3 is reducible iff it has a root in K). Applying the equivalence $(i) \Leftrightarrow (ii)$ to C^{-1} we obtain

$$\chi_{C^{-1}} \text{ has a root in } K \Leftrightarrow T \subseteq C^{-1}C.$$

It follows from the paragraph right before Theorem 3.10 that

$$\chi_{C^{-1}} \text{ has a root in } K \Leftrightarrow \chi_C \text{ has a root in } K.$$

Thus we have shown $(i) \Leftrightarrow (iii)$. □

Next we will consider H-classes C such that χ_C does not have a root in K.

Proposition 4.2. *Let C be an H-class such that χ_C does not have a root in K. Then the following are equivalent.*

(i) $\det(C)^2 = 1$.
(ii) $T \subseteq CC$.
(iii) $T \subseteq C^{-1}C^{-1}$.

Proof. We will first show $(i) \Leftrightarrow (ii)$ and then $(i) \Leftrightarrow (iii)$.
The implication $(ii) \Rightarrow (i)$ is obvious since $\det(T) = 1$. Suppose now that $\det(C)^2 = 1$. Choose a $\sigma \in C$. By Lemma 3.4 we may assume that σ is in Frobenius form. Since χ_C does not have a root in K, it is irreducible (because $n = 3$). Hence the matrix σ has only one invariant factor and therefore

$$\sigma = \begin{pmatrix} & & a \\ 1 & & b \\ & 1 & c \end{pmatrix}$$

for some $a, b, c \in K$. Note that $a = \det(\sigma)$ and hence $a^2 = 1$. Set

$$\sigma_0 = \begin{pmatrix} & & a \\ 1 & & 1 \\ & 1 & -a \end{pmatrix}.$$

Then

$$\sigma_0^2 = \begin{pmatrix} & a & -1 \\ & 1 & \\ 1 & -a & 2 \end{pmatrix}.$$

One checks easily that $(\sigma_0^2)^{p_{12}t_{21}(a)} = [X-1] \oplus [(X-1)^2] = F(t_{12}(1)) \sim t_{12}(1)$. Hence $\sigma_0^2 \sim t_{12}(1)$ and therefore $\sigma_0^2 \sim_E t_{12}(1)$ by Lemma 3.1. It follows that there is an $\epsilon \in E$ such that

$$(\sigma_0^2)^{\epsilon} = t_{12}(1). \tag{4.1}$$

Set

$$\xi = \begin{pmatrix} -1 & & -b-1 \\ & -1 & a+c \\ & & 1 \end{pmatrix}.$$

Clearly $\xi = t_{13}(-b-1)t_{23}(a+c)d_{12}(-1) = d_{12}(-1)t_{13}(b+1)t_{23}(-a-c) \in E$ and

$$\xi^2 = e. \tag{4.2}$$

One checks easily that

$$\sigma_0\xi = \sigma^{d_{13}(-1)}. \tag{4.3}$$

Clearly

$$\begin{aligned} &\sigma^{d_{13}(-1)\epsilon}\sigma^{d_{13}(-1)\epsilon\xi^\epsilon} \\ \overset{(4.3)}{=}\ & (\sigma_0\xi)^\epsilon(\sigma_0\xi)^{\epsilon\xi^\epsilon} \\ =\ & \sigma_0^\epsilon\xi^\epsilon\xi^{-\epsilon}\sigma_0^\epsilon\xi^\epsilon\xi^\epsilon \\ \overset{(4.2)}{=}\ & \sigma_0^\epsilon\sigma_0^\epsilon \\ \overset{(4.1)}{=}\ & t_{12}(1). \end{aligned}$$

Hence $t_{12}(1) \in CC$ and therefore $T \subseteq CC$. Thus we have shown $(i) \Leftrightarrow (ii)$.
Applying the equivalence $(i) \Leftrightarrow (ii)$ to C^{-1} we obtain

$$\det(C^{-1})^2 = 1 \Leftrightarrow T \subseteq C^{-1}C^{-1}.$$

But clearly

$$\det(C^{-1})^2 = 1 \Leftrightarrow \det(C)^2 = 1.$$

Thus we have shown $(i) \Leftrightarrow (iii)$. □

Lemma 4.3. *Let $a, d, f, x \in K^*$ and $b, c \in K$. Then*

$$\begin{pmatrix} & & a \\ d & & b \\ & f & c \end{pmatrix} \sim_E \begin{pmatrix} & & d \\ f & & a^{-1}bf \\ & a & c \end{pmatrix} \sim_E \begin{pmatrix} & & ax^2 \\ dx^{-1} & & bx \\ & fx^{-1} & c \end{pmatrix}.$$

Proof. A straightforward computation shows that

$$\begin{pmatrix} & & a \\ d & & b \\ & f & c \end{pmatrix}^\epsilon = \begin{pmatrix} & & d \\ f & & a^{-1}bf \\ & a & c \end{pmatrix}$$

where $\epsilon = \hat{p}_{32}\hat{p}_{31}t_{23}(-a^{-1}c)t_{13}(a^{-1}b)d_{13}(-1) \in E$. Moreover,

$$\begin{pmatrix} & & a \\ d & & b \\ & f & c \end{pmatrix}^{d_{31}(x)} = \begin{pmatrix} & & ax^2 \\ dx^{-1} & & bx \\ & fx^{-1} & c \end{pmatrix}.$$

□

Proposition 4.4. *Let C be an H-class such that χ_C does not have a root in K. Then the following are equivalent.*

(i) $\det(C)^3 = 1$.
(ii) $T \subseteq CCC$.
(iii) $T \subseteq C^{-1}C^{-1}C^{-1}$.

Proof. We will first show $(i) \Leftrightarrow (ii)$ and then $(i) \Leftrightarrow (iii)$.
The implication $(ii) \Rightarrow (i)$ is obvious since $\det(T) = 1$. Suppose now that $\det(C)^3 = 1$. Choose a $\sigma \in C$. By Lemma 3.4 we may assume that σ is in Frobenius form. Since χ_C does not have a root in K, it is irreducible (because $n = 3$). Hence the matrix σ has only one invariant factor and therefore

$$\sigma = \begin{pmatrix} & & a \\ 1 & & b \\ & 1 & c \end{pmatrix}$$

for some $a, b, c \in K$. Note that $a = \det(\sigma)$ and hence $a^3 = 1$.

Case 1 Suppose that $b \neq 0$. Set

$$\sigma_0 = \begin{pmatrix} & & a \\ 1 & & \\ & 1 & \end{pmatrix}.$$

Then

$$(\sigma_0^2)^{\hat{p}_{32}} = \begin{pmatrix} & & -a \\ 1 & & \\ & -a & \end{pmatrix}.$$

It follows from Lemma 4.3 that there is an $\epsilon \in E$ such that

$$(\sigma_0^2)^{\epsilon} = \begin{pmatrix} & & a^2 \\ 1 & & \\ & 1 & \end{pmatrix}. \tag{4.4}$$

Set

$$\xi = \begin{pmatrix} -1 & & -b \\ & -1 & c \\ & & 1 \end{pmatrix}.$$

Clearly $\xi = t_{13}(-b)t_{23}(c)d_{12}(-1) = d_{12}(-1)t_{13}(b)t_{23}(-c) \in E$ and

$$\xi^2 = e. \tag{4.5}$$

One checks easily that

$$\sigma_0\xi = \sigma^{d_{13}(-1)}. \tag{4.6}$$

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Clearly

$$\begin{aligned}
&\sigma^{d_{13}(-1)\epsilon}\sigma^{d_{13}(-1)\epsilon\xi^{\epsilon}}\\
\overset{(4.6)}{=}\ &(\sigma_0\xi)^{\epsilon}(\sigma_0\xi)^{\epsilon\xi^{\epsilon}}\\
=\ &\sigma_0^{\epsilon}\xi^{\epsilon}\xi^{-\epsilon}\sigma_0^{\epsilon}\xi^{\epsilon}\xi^{\epsilon}\\
\overset{(4.5)}{=}\ &\sigma_0^{\epsilon}\sigma_0^{\epsilon}\\
\overset{(4.4)}{=}\ &\begin{pmatrix} & & a^2\\ 1 & & \\ & 1 & \end{pmatrix}.
\end{aligned} \tag{4.7}$$

Our next to goal is to show that there are $\epsilon',\epsilon''\in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''}=\begin{pmatrix} & & a^2\\ 1 & & \\ & 1 & \end{pmatrix},$$

which will finish Case 1 in view of (4.7). Clearly

$$\sigma^{-1}=\begin{pmatrix} -ba^{-1} & 1 & \\ -ca^{-1} & & 1\\ a^{-1} & & \end{pmatrix}.$$

A straightforward computation shows that

$$(\sigma^{-1})^{\epsilon'}=\begin{pmatrix} -ba^{-1}+b^{-1}c & & -b\\ ba^{-1} & & \\ b^{-3}c^2 & -b^{-2} & -b^{-1}c \end{pmatrix}$$

where $\epsilon'=\hat{p}_{23}t_{31}(b^{-1}c)d_{32}(-b)\in E$. It follows that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{t_{31}(b^{-2}c)}=\begin{pmatrix} & & -b\\ ba^{-1} & & \\ & -b^{-2} & \end{pmatrix}$$

whence, by Lemma 4.3, there is an $\epsilon''\in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''}=\begin{pmatrix} & & a^2\\ 1 & & \\ & 1 & \end{pmatrix}. \tag{4.8}$$

By (4.7) and (4.8) we have

$$\sigma^{d_{13}(-1)\epsilon(\epsilon'')^{-1}}\sigma^{d_{13}(-1)\epsilon\xi^{\epsilon}(\epsilon'')^{-1}}\sigma^{\epsilon'}=t_{12}(1).$$

Thus $T\subseteq CCC$.

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Case 2 Suppose that $b = 0$ and $c \neq 0$. Let σ_0, ϵ and ξ be as in Case 1. We will show that there are $\epsilon', \epsilon'' \in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''} = \begin{pmatrix} & & a^2 \\ 1 & & \\ & 1 & \end{pmatrix},$$

which will finish Case 2 in view of (4.7). Clearly

$$\sigma^{-1} = \begin{pmatrix} & 1 & \\ -ca^{-1} & & 1 \\ a^{-1} & & \end{pmatrix}$$

and hence

$$(\sigma^{-1})^{\epsilon'} = \begin{pmatrix} & c^{-1} & \\ & & c^2 \\ c^{-1}a^{-1} & -c^{-1}a^{-1} & \end{pmatrix}$$

where $\epsilon' = t_{31}(ca^{-1})d_{32}(c) \in E$. It follows that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{t_{12}(1)\hat{p}_{23}} = \begin{pmatrix} & & c^{-1} \\ -c^{-1}a^{-1} & & \\ & -c^2 & \end{pmatrix}$$

whence, by Lemma 4.3, there is an $\epsilon'' \in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''} = \begin{pmatrix} & & a^2 \\ 1 & & \\ & 1 & \end{pmatrix}. \tag{4.9}$$

By (4.7) and (4.9) we have

$$\sigma^{d_{13}(-1)\epsilon(\epsilon'')^{-1}}\sigma^{d_{13}(-1)\epsilon\xi^{\epsilon}(\epsilon'')^{-1}}\sigma^{\epsilon'} = t_{12}(1).$$

Thus $T \subseteq CCC$.

Case 3 Suppose that $b = c = 0$. Set

$$\sigma_0 = \begin{pmatrix} & & a \\ 1 & & 1 \\ & 1 & \end{pmatrix}.$$

Then

$$(\sigma_0^2)^{\hat{p}_{32}t_{23}(-a^{-1})} = \begin{pmatrix} & & -a \\ 1 & & a^{-1} \\ & -a & 2 \end{pmatrix}.$$

It follows from Lemma 4.3 that there is an $\epsilon \in E$ such that

$$(\sigma_0^2)^{\epsilon} = \begin{pmatrix} & & a^2 \\ 1 & & -1 \\ & 1 & 2 \end{pmatrix}. \tag{4.10}$$

Set

$$\xi=\begin{pmatrix}-1 & & -1\\ & -1 & \\ & & 1\end{pmatrix}.$$

Clearly $\xi=t_{13}(-1)d_{12}(-1)=d_{12}(-1)t_{13}(1)\in E$ and

$$\xi^2=e. \tag{4.11}$$

One checks easily that

$$\sigma_0\xi=\sigma^{d_{13}(-1)}. \tag{4.12}$$

Clearly

$$\begin{aligned}&\sigma^{d_{13}(-1)\epsilon}\sigma^{d_{13}(-1)\epsilon\xi^{\epsilon}}\\ \overset{(4.12)}{=}\ &(\sigma_0\xi)^{\epsilon}(\sigma_0\xi)^{\epsilon\xi^{\epsilon}}\\ =\ &\sigma_0^{\epsilon}\xi^{\epsilon}\xi^{-\epsilon}\sigma_0^{\epsilon}\xi^{\epsilon}\xi^{\epsilon}\\ \overset{(4.11)}{=}\ &\sigma_0^{\epsilon}\sigma_0^{\epsilon}\\ \overset{(4.10)}{=}\ &\begin{pmatrix} & & a^2\\ 1 & & -1\\ & 1 & 2\end{pmatrix}.\end{aligned} \tag{4.13}$$

Our next to goal is to show that there are $\epsilon',\epsilon''\in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''}=\begin{pmatrix} & & a^2\\ 1 & & -1\\ & 1 & 2\end{pmatrix},$$

which will finish Case 3 in view of (4.13). Clearly

$$\sigma^{-1}=\begin{pmatrix} & 1 & \\ & & 1\\ a^{-1} & & \end{pmatrix}$$

and hence

$$(\sigma^{-1})^{\epsilon'}=\begin{pmatrix} & -a^{-1} & 2a^{-2}\\ 2 & & -a^{-1}\\ a & & \end{pmatrix}$$

where $\epsilon'=d_{13}(-a)t_{23}(-2a^{-1})\in E$. It follows that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{t_{23}(2a^{-1})t_{13}(2a^{-1})\hat{p}_{21}}=\begin{pmatrix} & & -a^{-1}\\ a^{-1} & & a^{-1}\\ & -a & 2\end{pmatrix}$$

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whence, by Lemma 4.3, there is an $\epsilon'' \in E$ such that

$$(t_{12}(1)(\sigma^{-1})^{\epsilon'})^{\epsilon''} = \begin{pmatrix} & & a^2 \\ 1 & & -1 \\ & 1 & 2 \end{pmatrix}. \tag{4.14}$$

By (4.13) and (4.14) we have

$$\sigma^{d_{13}(-1)\epsilon(\epsilon'')^{-1}} \sigma^{d_{13}(-1)\epsilon\xi^{\epsilon}(\epsilon'')^{-1}} \sigma^{\epsilon'} = t_{12}(1).$$

Thus $T \subseteq CCC$.

We have shown $(i) \Leftrightarrow (ii)$. Applying the equivalence $(i) \Leftrightarrow (ii)$ to C^{-1} we obtain

$$\det(C^{-1})^3 = 1 \Leftrightarrow T \subseteq C^{-1}C^{-1}C^{-1}.$$

But clearly

$$\det(C^{-1})^3 = 1 \Leftrightarrow \det(C)^3 = 1.$$

Thus we have shown $(i) \Leftrightarrow (iii)$. □

The theorem below follows from Lemma 3.5 and Propositions 3.9, 4.1, 4.2, 4.4.

Theorem 4.5. *Let C be a noncentral H-class. Then the following holds.*

(i) If $C = T$, then $m(C) = 1$.
(ii) If $C \neq T$ and χ_C has a root in K, then $m(C) = 2$. In this case $T \subseteq CC^{-1}$.
(iii) If $C \neq T$, χ_C has no root in K and $\det(C)^2 = 1$, then $m(C) = 2$. In this case $T \subseteq CC$.
(iv) If $C \neq T$, χ_C has no root in K, $\det(C)^2 \neq 1$ and $\det(C)^3 = 1$, then $m(C) = 3$. In this case $T \subseteq CCC$.
(v) If $C \neq T$, χ_C has no root in K, $\det(C)^2 \neq 1$ and $\det(C)^3 \neq 1$, then $m(C) = 4$. In this case $T \subseteq CC^{-1}CC^{-1}$.

Example 4.6. *Suppose that $K = \mathbb{F}_2$ or $K = \mathbb{F}_3$. Then $m(C) \leq 2$ for any noncentral H-class C since $a^2 = 1$ for any $a \in K^*$.*

Example 4.7. *Suppose that $K = \mathbb{F}_5$. Let C be the H-class of $[X^3 - X + 2]$. Then $C \neq T$ by Lemma 3.3, $\chi_C = X^3 - X + 2$ has no root in K, $\det(C)^2 = -1 \neq 1$ and $\det(C)^3 = 2 \neq 1$. Hence $m(C) = 4$.*

Example 4.8. *Suppose that $K = \mathbb{Q}$. Let C be the H-class of $[X^3 - 2]$. Then $C \neq T$ by Lemma 3.3, $\chi_C = X^3 - 2$ has no root in K, $\det(C)^2 = 4 \neq 1$ and $\det(C)^3 = 8 \neq 1$. Hence $m(C) = 4$.*

Example 4.9. *Suppose that $K = \mathbb{R}$. Then $m(C) \leq 2$ for any noncentral H-class C since any cubic polynomial with real coefficients has at least one real root.*

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5. Products of conjugacy classes in $\mathrm{GL}_\infty(K)$

Set $G := \mathrm{GL}_\infty(K)$ and $E := \mathrm{E}_\infty(K)$. H denotes a subgroup of G containing E, and T denotes the H-class of $[t_{2,1,2}(1)]_\infty$. We will determine $m(C)$ for any noncentral H-class C ($m(C)$ is defined as in Section 1). Note that the center of G consists only of $[e]_\infty$. Hence there is only one central H-class.

Our first goal is to define the Frobenius form of an element of G.

Lemma 5.1. *Let $n \in \mathbb{N}$ and $\sigma \in \mathrm{GL}_n(K)$. If $F(\sigma) = [P_1] \oplus \dots \oplus [P_r]$, then*

$$F(1 \oplus \sigma) = [P_1] \oplus \dots \oplus [P_{i-1}] \oplus [P_i(X-1)] \oplus [P_{i+1}] \oplus \dots \oplus [P_r]$$

where $i = \min\{j \in \{0,\dots,r\} \mid P_j(X-1) \text{ divides } P_{j+1}\}$. Here we set $P_0 := 1$ and $P_{r+1} := P_r(X-1)$.

Proof. Clearly the characteristic matrix of $1 \oplus \sigma$ is the matrix $A = (X-1) \oplus (Xe_{n\times n} - \sigma)$. Since the invariant factors of σ are $P_1, \dots, P_r$, the matrix A is equivalent to the matrix $B = (X-1) \oplus 1 \oplus \dots \oplus 1 \oplus P_1 \oplus \dots \oplus P_r$. By applying the algorithm described in [4, Part V, Chapter 20, Proof of Theorem 3.2] we get that B is equivalent to the matrix $C = 1 \oplus \dots \oplus 1 \oplus P_1 \oplus \dots \oplus P_{i-1} \oplus P_i(X-1) \oplus P_{i+1} \dots \oplus P_r$. Clearly C is the Smith normal form of A and hence $P_1, \dots, P_{i-1}, P_i(X-1), P_{i+1}, \dots P_r$ are the invariant factors of $1 \oplus \sigma$. □

For any $\sigma \in G$ there is a minimal $n_\sigma \in \mathbb{N}$ such that σ has a representative in $\mathrm{GL}_{n_\sigma}(K)$. For any $n \geq n_\sigma$ we write $\sigma^{(n)}$ for the unique representative of σ in $\mathrm{GL}_n(K)$. Proposition 5.2 below shows that the sequence

$$F(\sigma^{(n_\sigma)}), F(\sigma^{(n_\sigma+1)}), F(\sigma^{(n_\sigma+2)}) \cdots$$

eventually stabilises (up to the equivalence relation $\sim_\infty$).

Proposition 5.2. *Let $\sigma \in G$. Then there is an $s \geq n_\sigma$ such that $[F(\sigma^{(s)})]_\infty = [F(\sigma^{(t)})]_\infty$ for any $t \geq s$.*

Proof. If $F(\sigma^{(n_\sigma)}) = [P_1] \oplus \dots \oplus [P_r]$, then by Lemma 5.1,

$$F(\sigma^{(n_\sigma+1)}) = [P_1] \oplus \dots \oplus [P_{i-1}] \oplus [P_i(X-1)] \oplus [P_{i+1}] \oplus \dots \oplus [P_r]$$

where $i = \min\{j \in \{0,\dots,r\} \mid P_j(X-1) \text{ divides } P_{j+1}\}$. Since $P_{i-1}(X-1)$ divides $P_i(X-1)$ but $P_j(X-1)$ does not divide P_{j+1} for any $j \in \{0,\dots,i-2\}$, we get

$$F(\sigma^{(n_\sigma+2)}) = [P_1] \oplus \dots \oplus [P_{i-2}] \oplus [P_{i-1}(X-1)] \oplus [P_i(X-1)] \oplus [P_{i+1}] \oplus \dots \oplus [P_r],$$

again by Lemma 5.1. After repeating this step a finite number of times we arrive at

$$F(\sigma^{(n_\sigma+i)}) = [P_1(X-1)] \oplus \dots \oplus [P_i(X-1)] \oplus [P_{i+1}] \oplus \dots \oplus [P_r].$$

Hence, again by Lemma 5.1,

$$F(\sigma^{(n_\sigma+t)}) = [X-1] \oplus \dots \oplus [X-1] \oplus [P_1(X-1)] \oplus \dots \oplus [P_i(X-1)] \oplus [P_{i+1}] \oplus \dots \oplus [P_r].$$

for any $t \geq i$. Since $[X-1]$ is the 1×1 matrix (1), the assertion of the proposition follows. □

Let $\sigma \in G$. By Proposition 5.2 there is an $s \geq n_\sigma$ such that $[F(\sigma^{(s)})]_\infty = [F(\sigma^{(t)})]_\infty$ for any $t \geq s$. We define the *Frobenius form* $F(\sigma)$ of σ by $F(\sigma) = [F(\sigma^{(s)})]_\infty$. Clearly $F(\sigma)$ is well-defined.

Proposition 5.3. *Let $\sigma, \tau \in G$. Then $\sigma \sim \tau$ iff $F(\sigma) = F(\tau)$.*

Proof. Choose an $s \geq n_\sigma, n_\tau$ such that $[F(\sigma^{(s)})]_\infty = [F(\sigma^{(t)})]_\infty$ and $[F(\tau^{(s)})]_\infty = [F(\tau^{(t)})]_\infty$ for any $t \geq s$. Then $F(\sigma) = [F(\sigma^{(s)})]_\infty$ and $F(\tau) = [F(\tau^{(s)})]_\infty$. One checks easily that

$$\begin{aligned} & \sigma \sim \tau \\ \Leftrightarrow & \exists t \geq s : \sigma^{(t)} \sim \tau^{(t)} \text{ in } \mathrm{GL}_t(K) \\ \Leftrightarrow & \exists t \geq s : F(\sigma^{(t)}) = F(\tau^{(t)}) \\ \Leftrightarrow & F(\sigma^{(s)}) = F(\tau^{(s)}) \\ \Leftrightarrow & F(\sigma) = F(\tau). \end{aligned}$$

□

By Proposition 5.3 we can define the Frobenius form $F(C)$ of an H-class C in the obvious way. Below we compute $F(T)$. Note that $T = \{g \in G \mid F(\sigma) = F(T)\}$ by Lemma 3.1.

Lemma 5.4. $F(T) = [[(X-1)^2]]_\infty$.

Proof. It is an easy exercise to show that

$$F(t_{2,1,2}(1)) = \begin{pmatrix} & -1 \\ 1 & 2 \end{pmatrix} = [(X-1)^2].$$

Hence $F(e_{n\times n} \oplus t_{2,1,2}(1)) = [X-1] \oplus \cdots \oplus [X-1] \oplus [(X-1)^2]$ for any $n \in \mathbb{N}$, by Lemma 5.1. The assertion of the lemma follows. □

Lemma 5.5. *Let C be a noncentral H-class. Then $m(C) = 1$ iff $C = T$.*

Proof. Clear since $T = T^{-1}$ by Lemma 3.2. □

Lemma 5.6. *Let C be a noncentral H-class. Then $T \subseteq CC^{-1}$.*

Proof. Choose a $\sigma \in C$ and an $n > n_\sigma$. Then clearly $\sigma^{(n)}$ is noncentral and $\chi_{\sigma^{(n)}}$ has a linear factor. By Theorem 3.6 we get $t_{n,n-1,n}(1) \in C^{(n)}(C^{(n)})^{-1}$ where $C^{(n)}$ is the $E_n(K)$-class of $\sigma^{(n)}$. It follows that $[t_{2,1,2}(1)]_\infty \in CC^{-1}$ and thus $T \subseteq CC^{-1}$. □

Theorem 5.7 below follows directly from Lemmas 5.5 and 5.6.

Theorem 5.7. *Let C be a noncentral H-class. Then $m(C) = 1$ if $C = T$ respectively $m(C) = 2$ if $C \neq T$.*

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