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# CLAW-DECOMPOSITION OF KNESER GRAPHS

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Abstract. A claw is a star with three edges. The Kneser graph $KG_{n,2}$ is the graph whose vertices are the 2-subsets of an $n$-set, in which two vertices are adjacent if and only if their intersection is empty. In this paper, we prove that $KG_{n,2}$ is claw-decomposable, for all $n \geq 6$.

## 1. Introduction

All the graphs considered in this paper are finite. For a graph $G$, $G(\lambda)$ is the graph obtained from $G$ by replacing each of its edges by $\lambda$ parallel edges. If a graph $G$ has no edges, then it is called a *null graph*. Let $K_{m,n}$ denote the *complete bipartite graph* with $m$ and $n$ vertices in the parts. A *star* with $k$ edges is denoted by $S_k$ and $S_k \cong K_{1,k}$. If $k = 3$, then the graph $K_{1,3}$ is called a *claw*. A *path* with $k$ edges is denoted by $P_k$ and a *cycle* with $k$ edges is denoted by $C_k$. A *Hamilton cycle* of $G$ is a cycle that contains every vertex of $G$. A graph $G$ is *Hamiltonian* if it contains a Hamilton cycle. The degree of a vertex $x$ of $G$, denoted by $deg_G x$ is the number of edges incident with $x$ in $G$. Let $k$ be a positive integer. A graph $G$ is said to be *k-regular*, if each vertex in $G$ is of degree $k$. If $H_1, H_2, \ldots, H_l$ are edge disjoint subgraphs of a graph $G$ such that $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_l)$, then we say that $H_1, H_2, \ldots, H_l$ *decompose* $G$ and we denote it by $G = H_1 \oplus H_2 \oplus \cdots \oplus H_l$. If $H_i \cong S_k$ for $i = 1, 2, \ldots, l$, then we say that $G$ is $S_k$-*decomposable* and we denote it by $S_k|G$. For positive integers $l$ and $n$ with $1 \leq l \leq n$, the crown $C_{n,l}$ is the bipartite graph with bipartition $(A, B)$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$, and the edge set $\{a_i b_j | 1 \leq j - i \leq l$ with arithmetic modulo $n\}$. Note that $C_{n,n} \cong K_{n,n}$ and $C_{n,n-1} \cong K_{n,n} - I$, where $I$ is a 1-factor of $K_{n,n}$. The *tensor*

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*product* of $G$ and $H$, denoted by $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. The *line graph* $L(G)$ of a graph $G$ is the graph with $V(L(G)) = E(G)$ and $e_ie_j \in E(L(G))$ if and only if the edges $e_i$ and $e_j$ are incident with a common end vertex in $G$. The complete graph on $n$ vertices is denoted by $K_n$. The line graph of the complete graph $K_n$ is denoted by $L(K_n)$. Let $A = \{1, 2, 3, \ldots, n\}$. Then $\mathcal{P}_k(A)$ denotes the set of all $k$-subsets of $A$. The Kneser graph $KG_{n,2}$ is defined as follows: $V(KG_{n,2})= \mathcal{P}_2(A)$ and $E(KG_{n,2}) = \{XY | X, Y \in \mathcal{P}_2(A) \text{ and } X \cap Y = \emptyset\}$. Note that the graph $KG_{n,2} \cong \overline{L(K_n)}$, where $\overline{L(K_n)}$ denotes the complement of the graph $L(K_n)$. Also, it is interesting to note that $KG_{5,2}$ is the Petersen graph. The Generalized Kneser Graph, $GKG_{n,k,r}$ is the graph whose vertices are the $k$-subsets of $A$, in which two vertices are adjacent if and only if they intersect in precisely $r$ elements.

In 1955, M. Kneser [3] introduced the Kneser graph. In 2000, Chen [1] proved that $KG_{n,2}$ is Hamiltonian, when $n \geq 3k$, $k \geq 1$. In 2004, Shields and Savage [6] proved that all connected Kneser graphs (except $KG_{5,2}$) have Hamilton cycles, when $n \leq 27$ and the problem that all graphs $KG_{n,2}$(except $KG_{5,2}$) is Hamiltonian is still open. In 2015, Rodger and Whitt [5] established the necessary and sufficient conditions for a $P_3$-decomposition of the Kneser graph $KG_{n,2}$ and the Generalized Kneser Graph $GKG_{n,3,1}$. In 2015, Whitt and Rodger [7] proved that the Kneser graph $KG_{n,2}$ is $P_4$-decomposable if and only if $n \equiv 0, 1, 2, 3 (\text{mod } 16)$. In 2018, Ganesamurthy and Paulraja [2] proved that if $n \equiv 0, 1, 2, 3 (\text{mod } 8k), k \geq 2$, then the Kneser graph $KG_{n,2}$ can be decomposed into paths of length $2k$. In the same paper they also proved that, for $k = 2^l$, $l \geq 1$, $KG_{n,2}$ has a $P_{2k+1}$-decomposition if and only if $n \equiv 0, 1, 2, 3 (\text{mod } 2^{l+3})$. In this paper, we discuss claw-decomposition of the Kneser graphs. We will prove $KG_{n,2}$ is claw-decomposable, for all $n \geq 6$.

## 2. Preliminaries

Let $G$ be a graph on $n$ vertices and $\{1, 2, 3, \ldots, k\} \subset V(G)$. The notation $(1; 2, 3, \ldots, k)$ denotes a star with a center vertex 1 and $k-1$ pendent edges $12, 13, \ldots, 1k$. Let $X$ and $Y$ be two disjoint subsets of $V(G)$. Then $E(X, Y)$ denotes the set of edges in $G$, whose one end vertex is in $X$ and the other end vertex is in $Y$. The notation $\langle E(X, Y)\rangle$ denotes the graph induced by the edges of $E(X, Y)$. To prove our results we use the following:

**Theorem 2.1.** *(Lin et al.[4]) Let $\lambda, k, l$ and $n$ be positive integers. The graph $C_{n,l}(\lambda)$ is $S_k$-decomposable if and only if $k \leq l$ and $\lambda nl \equiv 0 (\text{mod } k)$.*

**Theorem 2.2.** *(Yamamoto et al.[8]) Let $k, m$ and $n \in \mathbb{Z}_+$ with $m \leq n$. There exists an $S_k$-decomposition of $K_{m,n}$ if and only if one of the following holds:*

*(i) $k \leq m$ and $mn \equiv 0 (\text{mod } k)$;*

*(ii) $m < k \leq n$ and $n \equiv 0 (\text{mod } k)$.*

Note that the graphs $KG_{2,2}$ and $KG_{3,2}$ are null graphs. For $n \geq 4$, $|E(KG_{n,2})| = \frac{n(n-1)(n-2)(n-3)}{8}$, which is divisible by 3. Therefore for all $n \geq 4$, the graph $KG_{n,2}$ satisfies the obvious edge-divisibility condition to have a claw-decomposition. As the graph $KG_{4,2}$ is 1-regular, it does not admit a claw-decomposition. We know that the graph $KG_{5,2}$ (Petersen graph) doesn't admit a claw-decomposition.

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For, $KG_{5,2}$ is a simple, 3-regular and non-bipartite graph. It is easy to observe that a simple 3-regular graph is claw-decomposable if and only if it is bipartite. Therefore, we look for a claw-decomposition of $KG_{n,2}$, when $n \geq 6$.

Let $n \geq 6$ and $n_1, n_2 \geq 3$ be positive integers such that $n = n_1 + n_2$. We define $V_1 = \{1, 2, 3, \ldots, n_1\}$, $V_2 = \{n_1 + 1, n_1 + 2, \cdots, n\}$ and $V(KG_{n,2}) = A_1 \cup A_2 \cup A_3$, where $A_1 = \mathcal{P}_2(V_1)$, $A_2 = \mathcal{P}_2(V_2)$ and $A_3 = \{\{i,j\} | \{i,j\} \in V_1 \times V_2\}$. For $i \in V_1$, $i \times V_2 = \{\{i,j\} | j \in V_2\}$ is called the $i^{th}$ *layer* of the vertices of $A_3$ and we denote it by $Z_i$. Further, for $j \in V_2$, $V_1 \times j = \{\{i,j\} | i \in V_1\}$ is called the $j^{th}$ *column* of vertices of $A_3$. We define the graphs $G_i, 1 \leq i \leq 6$ as follows:

$$\begin{aligned}
V(G_1) &= A_1 &;\quad & E(G_1) = \{XY | X, Y \in A_1 \textit{ and } X \cap Y = \emptyset\} \\
V(G_2) &= A_2 &;\quad & E(G_2) = \{XY | X, Y \in A_2 \textit{ and } X \cap Y = \emptyset\} \\
V(G_3) &= A_3 &;\quad & E(G_3) = \{XY | X, Y \in A_3 \textit{ and } X \cap Y = \emptyset\} \\
V(G_4) &= A_1 \cup A_2 &;\quad & E(G_4) = \{XY | X \in A_1, Y \in A_2 \textit{ and } X \cap Y = \emptyset\} \\
V(G_5) &= A_1 \cup A_3 &;\quad & E(G_5) = \{XY | X \in A_1, Y \in A_3 \textit{ and } X \cap Y = \emptyset\} \\
V(G_6) &= A_2 \cup A_3 &;\quad & E(G_6) = \{XY | X \in A_2, Y \in A_3 \textit{ and } X \cap Y = \emptyset\}
\end{aligned}$$

We observe that, $G_1 \cong KG_{n_1,2}$, $G_2 \cong KG_{n_2,2}$, $G_3 \cong K_{n_1} \times K_{n_2}$, $G_4 \cong K_{|A_1|,|A_2|}$, $G_5 \cong \langle E(A_1, A_3) \rangle$, $G_6 \cong \langle E(A_2, A_3) \rangle$ and $KG_{n,2} = \oplus_{i=1}^{6} G_i$.

**Proposition 2.3.** *If $G$ is a bipartite graph such that every vertex in one of the parts has a degree which is multiple of 3, then $G$ is claw-decomposable.*

*Proof.* Without loss of generality, we assume that $deg_G x_i \equiv 0 (\text{mod } 3)$ for every $x_i \in X$. Then $deg_G x_i = 3q_i$, for some positive integer $q_i$. We fix each $x_i$ as a center vertex $q_i$ times, to get a claw-decomposition of $G$. □

## 3. Claw-decomposition of $KG_{n,2}$

In this section, we prove that $KG_{n,2}$ is claw-decomposable, for all $n \geq 6$.

**Lemma 3.1.** *$KG_{6,2}$ is claw-decomposable.*

*Proof.* Let $n_1, n_2 = 3$. By using the above construction, we have $V_1 = \{1, 2, 3\}$, $V_2 = \{4, 5, 6\}$, and $V(KG_{6,2}) = A_1 \cup A_2 \cup A_3$, where $A_1 = \mathcal{P}_2(V_1) = \{\{1,2\}, \{1,3\}, \{2,3\}\}$, $A_2 = \mathcal{P}_2(V_2) = \{\{4,5\}, \{4,6\}, \{5,6\}\}$ and $A_3 = \{\{i,j\} | \{i,j\} \in V_1 \times V_2\}$. Therefore, $|A_1| = |A_2| = 3$ and $|A_3| = 9$. In $A_3$, there are 3 layers and each layer has 3 vertices. We observe that, $G_1 \cong G_2 \cong KG_{3,2}$, $G_3 \cong K_3 \times K_3$, $G_4 \cong K_{3,3}$, $G_5 \cong \langle E(A_1, A_3) \rangle$, $G_6 \cong \langle E(A_2, A_3) \rangle$ and $KG_{6,2} = \oplus_{i=1}^{6} G_i$. The graphs $G_1$ and $G_2$ are null graphs. Now, we show that each subgraph $G_3, G_4, G_5$ and $G_6$ of $KG_{6,2}$ is claw-decomposable. As $G_4$ is a simple 3-regular bipartite graph, it is claw-decomposable by Proposition 2.3. In $G_5$, the vertex set $A_3$ has 3 layers and each layer has 3 vertices. Each vertex of $A_1$ is adjacent to all the three vertices of exactly one layer of $A_3$, see Figure 3.1. So, the degree of each vertex of $A_1$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_5$. In $G_6$, the vertex set $A_3$ has

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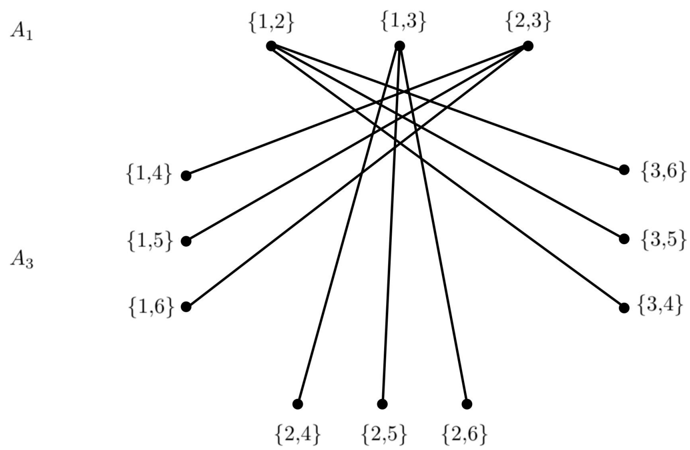


FIGURE 3.1. The subgraph $G_5$ of $KG_{6,2}$

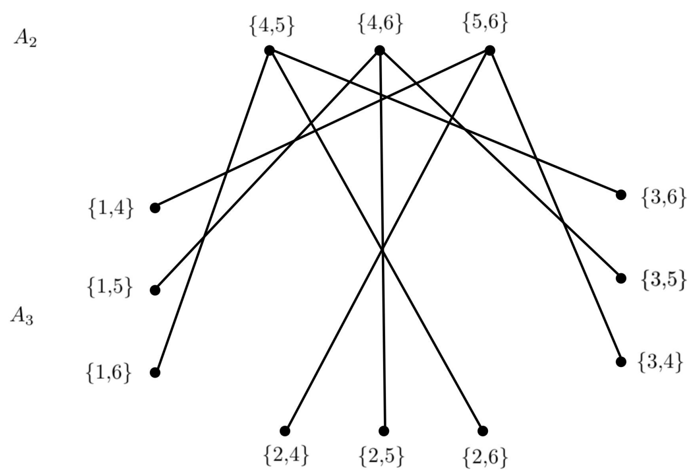


FIGURE 3.2. The subgraph $G_6$ of $KG_{6,2}$

3 layers and each layer has 3 vertices. Each vertex of $A_2$ is adjacent to all the three vertices of exactly one column of $A_3$, see Figure 3.2. So, the degree of each vertex of $A_2$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_6$. Finally, the graph $G_3 \cong K_3 \times K_3$ can be decomposed into 6 copies of claws as follows: $(\{2,4\};\{1,5\},\{1,6\},\{3,5\})$, $(\{2,5\};\{1,6\},\{1,4\},\{3,6\})$, $(\{2,6\};\{1,4\},\{1,5\},\{3,4\})$, $(\{3,4\};\{2,5\},\{1,5\},\{1,6\})$, $(\{3,5\};\{2,6\},\{1,4\},\{1,6\})$, $(\{3,6\};\{2,4\},\{1,4\},\{1,5\})$. This completes the proof of the lemma. □

**Lemma 3.2.** *$KG_{7,2}$ is claw-decomposable.*

*Proof.* Let $n_1 = 3, n_2 = 4$. Then, we have $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6,7\}$, and $V(KG_{7,2}) = A_1 \cup A_2 \cup A_3$, where $A_1 = \mathcal{P}_2(V_1) = \{\{1,2\},\{1,3\},\{2,3\}\}$, $A_2 = \mathcal{P}_2(V_2) = \{\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}\}$ and $A_3 = \{\{i,j\}|\{i,j\} \in V_1 \times V_2\}$. Therefore, $|A_1| = 3, |A_2| = 6$ and $|A_3| = 12$. In $A_3$, there are 3 layers and each layer has 4 vertices. We observe that $G_1 \cong KG_{3,2}$, $G_2 \cong KG_{4,2}$, $G_3 \cong K_3 \times K_4$, $G_4 \cong K_{3,6}$, $G_5 \cong \langle E(A_1, A_3)\rangle$, $G_6 \cong \langle E(A_2, A_3)\rangle$ and $KG_{7,2} = \oplus_{i=1}^{6} G_i$. The graph $G_1$ is a null graph. In $G_2 \cup G_4$, consider the claws $S_1 : (\{4,5\};\{6,7\},\{1,2\},\{1,3\})$, $S_2 : (\{4,6\};\{5,7\},\{1,2\},\{1,3\})$

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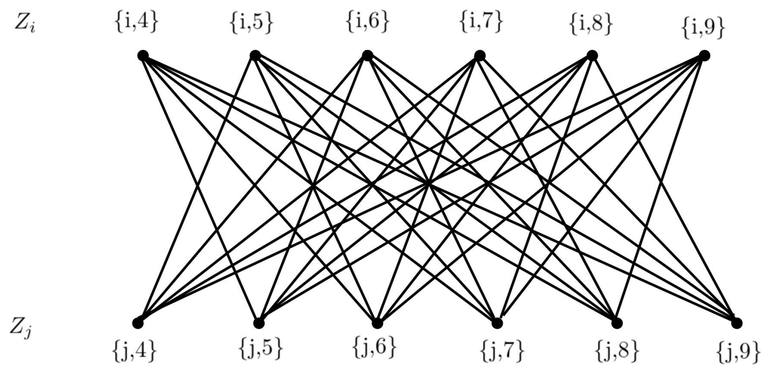


FIGURE 3.3. The subgraph $\langle E(Z_i, Z_j)\rangle$ of $KG_{9,2}$

and $S_3 : (\{4,7\}; \{5,6\}, \{1,2\}, \{1,3\})$. In $(G_2 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$, the degree of the vertex $\{1,2\}$ is 3, $\{1,3\}$ is 3 and $\{2,3\}$ is 6. Now, by fixing each vertex of $A_1$ as a center vertex we get a claw-decomposition in $(G_2 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$. In $G_6$, the degree of each vertex of $A_3$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_6$. Finally, the graph $G_3 \cup G_5$ can be decomposed into 16 claws as follows: $(\{1,2\}; \{3,4\}, \{3,5\}, \{3,6\})$, $(\{1,3\}; \{2,5\}, \{2,6\}, \{2,7\})$, $(\{2,3\}; \{1,4\}, \{1,6\}, \{1,7\})$, $(\{2,7\}; \{3,4\}, \{3,5\}, \{3,6\})$, $(\{3,7\}; \{1,2\}, \{1,4\}, \{1,5\})$, $(\{1,7\}; \{2,4\}, \{2,5\}, \{3,6\})$, $(\{3,4\}; \{2,5\}, \{1,6\}, \{1,7\})$, $(\{3,5\}; \{1,4\}, \{1,6\}, \{1,7\})$, $(\{1,4\}; \{2,5\}, \{2,6\}, \{2,7\})$, $(\{1,5\}; \{2,4\}, \{2,6\}, \{2,7\})$, $(\{1,6\}; \{2,4\}, \{2,5\}, \{2,7\})$, $(\{3,7\}; \{2,5\}, \{2,6\}, \{1,6\})$, $(\{2,6\}; \{3,4\}, \{3,5\}, \{1,7\})$, $(\{2,4\}; \{3,5\}, \{3,7\}, \{1,3\})$, $(\{1,5\}; \{2,3\}, \{3,4\}, \{3,6\})$, $(\{3,6\}; \{1,4\}, \{2,4\}, \{2,5\})$. This completes the proof of the lemma. □

**Lemma 3.3.** *If $n \in \{8, 9, 10, 11\}$, then $KG_{n,2}$ is claw-decomposable.*

*Proof.* We split the proof into the following three cases.

**Case 1:** $n = 9$. Let $n_1 = 3$ and $n_2 = 6$. Then, the graph $G_1$ is a null graph. The graph $G_2$ is claw-decomposable, by Lemma 3.1. In $G_3$, there are 3 layers and each layer has 6 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle, 1 \le i < j \le 3$ of $G_3$ form a crown graph $C_{6,5}$, see Figure 3.3. By Theorem 2.1, the graph $G_3$ is claw-decomposable. The graph $G_4 \cong K_{3,15}$ is claw-decomposable, by Theorem 2.2. In $G_5$, the degree of each vertex of $A_1$ is exactly 6. By Proposition 2.3, we get a claw-decomposition of $G_5$. In $G_6$, the degree of each vertex of $A_2$ is exactly 12. By Proposition 2.3, we get a claw-decomposition of $G_6$. This completes the proof of Case 1.

**Case 2:** $n = 11$. Let $n_1 = 7$ and $n_2 = 4$. The graph $G_1$ is claw-decomposable, by Lemma 3.2. In $G_2 \cup G_4$, consider the claws $S_1 : (\{8,9\}; \{10,11\}, \{1,2\}, \{1,3\})$, $S_2 : (\{8,10\}; \{9,11\}, \{1,2\}, \{1,3\})$ and $S_3 : (\{8,11\}; \{9,10\}, \{1,2\}, \{1,3\})$. In $(G_2 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$, the degree of each vertex of $A_1$ is exactly 6 except two vertices $\{1,2\}$ and $\{1,3\}$. The degree of these two vertices is exactly 3. Now, by fixing each vertex of $A_1$ as a center vertex we get a claw-decomposition in $(G_2 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$. In $G_3$, the vertex set $A_3$ has 7 layers and each layer has 4 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle, 1 \le i < j \le 7$ form a crown graph $C_{4,3}$. By Theorem 2.1, the graph $G_3$ is claw-decomposable. In $G_5$, the degree of each vertex of $A_3$ is exactly 15. By Proposition 2.3, we get

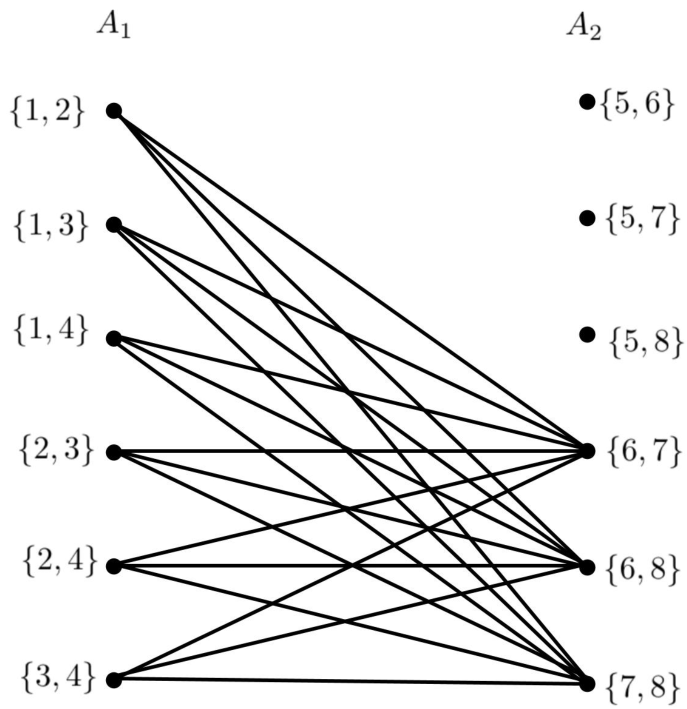


FIGURE 3.4. The subgraph $(G_1 \cup G_2 \cup G_4) \setminus E(\bigcup_{i=1}^{8} S_i)$ of $KG_{8,2}$

a claw-decomposition of $G_5$. In $G_6$, the degree of each vertex of $A_3$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_6$. This completes the proof of Case 2.

**Case 3:** $n \in \{8, 10\}$. Let $n_1$=4 and $n_2$=4 or 6. The graph $G_3$ has 4 layers and each layer has $n_2$ (either 4 or 6) vertices. Each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \le i < j \le 4$ form a crown graph $C_{4,3}$ (if $n_2$=4) or $C_{6,5}$ (if $n_2$=6). By Theorem 2.1, the graph $G_3$ is claw-decomposable. Now, we show that, $G_1 \cup G_2 \cup G_4$ is claw-decomposable. If $n_1$=$n_2$=4, then consider the claws $S_1 : (\{1,2\}; \{3,4\}, \{5,6\}, \{5,7\})$, $S_2 : (\{1,3\}; \{2,4\}, \{5,6\}, \{5,7\})$, $S_3 : (\{1,4\}; \{2,3\}, \{5,6\}, \{5,7\})$, $S_4 : (\{2,3\}; \{5,6\}, \{5,7\}, \{5,8\})$, $S_5 : (\{5,6\}; \{2,4\}, \{3,4\}, \{7,8\})$, $S_6 : (\{5,7\}; \{2,4\}, \{3,4\}, \{6,8\})$, $S_7 : (\{5,8\}; \{1,2\}, \{6,7\}, \{1,3\})$, $S_8 : (\{5,8\}; \{1,4\}, \{2,4\}, \{3,4\})$ in $G_1 \cup G_2 \cup G_4$. In $(G_1 \cup G_2 \cup G_4) \setminus E(\bigcup_{i=1}^{8} S_i)$, the degree of each vertex of $A_1$ is exactly 3, see Figure 3.4. By Proposition 2.3, we get a claw-decomposition in $(G_1 \cup G_2 \cup G_4) \setminus E(\bigcup_{i=1}^{8} S_i)$. If $n_1$=4, $n_2$=6. The graph $G_2$ is claw-decomposable, by Lemma 3.1. In $G_1 \cup G_4$, consider the claws $S_1 : (\{1,2\}; \{3,4\}, \{5,6\}, \{5,7\})$, $S_2 : (\{1,3\}; \{2,4\}, \{5,6\}, \{5,7\})$ and $S_3 : (\{1,4\}; \{2,3\}, \{5,6\}, \{5,7\})$. In $(G_1 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$, the degree of each vertex of $A_2$ is exactly 6 except two vertices $\{5,6\}$ and $\{5,7\}$. The degree of these two vertices is exactly 3. By Proposition 2.3, we get a claw-decomposition in $(G_1 \cup G_4) \setminus E(S_1 \cup S_2 \cup S_3)$. In $G_5$, there are 4 layers and each layer has $n_2$(4 or 6) vertices. Note that, the degree of each vertex of $A_3$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_5$. Finally, it is enough to prove that, $G_6$ is claw-decomposable. If $n_2 = 4$, then the vertex set $A_3$ has 4 layers and each layer has 4 vertices. The degree of each vertex of $A_3$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_6$.

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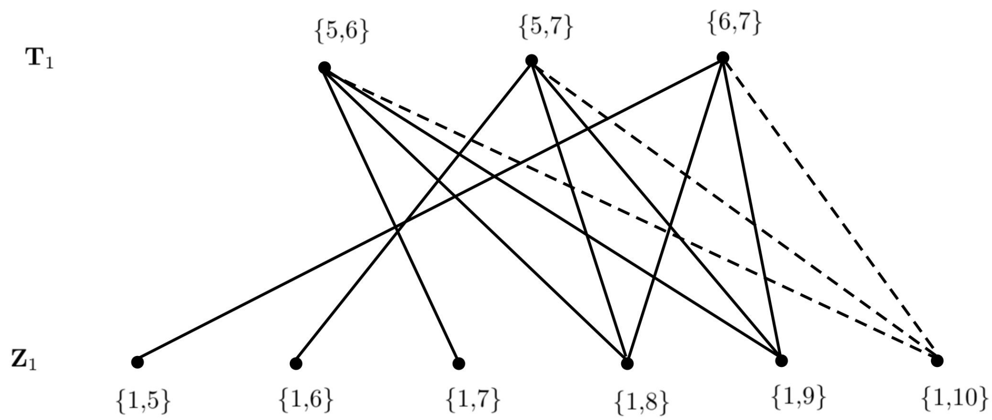


FIGURE 3.5. Claw-decomposition of $\langle E(T_1, Z_1)\rangle$ in $KG_{10,2}$

If $n_2$=6, then the vertex set $A_3$ has 4 layers and each layer has 6 vertices and $|A_2|$=15. The vertex set $A_2$ can be partitioned into 5 sets, say $T_1 = \{\{5,6\},\{5,7\},\{6,7\}\}$, $T_2 = \{\{8,9\},\{8,10\},\{9,10\}\}$, $T_3 = \{\{5,8\},\{5,9\},\{5,10\}\}$, $T_4 = \{\{6,8\},\{6,9\},\{6,10\}\}$ and $T_5 = \{\{7,8\},\{7,9\},\{7,10\}\}$, so that $A_2 = \bigcup_{i=1}^{5} T_i$. Then $G_6 = \oplus_{i=1}^{5} \oplus_{j=1}^{4} \langle E(T_i, Z_j)\rangle$. We prove that each subgraph $\langle E(T_i, Z_j)\rangle$, where $1 \le i \le 5$ and $1 \le j \le 4$ is claw-decomposable. Consider a subgraph $\langle E(T_i, Z_j)\rangle$, we choose a vertex $\{j, y'\} \in Z_j$ as a center vertex of a star, say $S$, when $y' \neq x$ and $y' \neq y$, for all $\{x, y\} \in T_i$. In $\langle E(T_i, Z_j)\rangle \setminus S$, the degree of each vertex of $T_i$ is exactly three. By Proposition 2.3, we get a claw-decomposition. A claw-decomposition of $\langle E(T_1, Z_1)\rangle$ is given in Figure 3.5. Hence $\langle E(T_i, Z_j)\rangle$, $1 \le i \le 5$ and $1 \le j \le 4$ is claw-decomposable. This completes the proof. □

**Theorem 3.4.** *For all $n \ge 6$, the graph $KG_{n,2}$ is claw-decomposable.*

*Proof.* By using our construction, we write $KG_{n,2} = \oplus_{i=1}^{6} G_i$ and we prove that each $G_i$, $1 \le i \le 6$ is claw-decomposable. Let $t \ge 1$ be a positive integer. We divide the proof into the following two cases.

**Case 1:** $n \equiv 0, 1, 3, 4 (\mathrm{mod}\ 6)$. Let

$$n = \begin{cases} 6t \quad , & \text{if } n \equiv 0 (\mathrm{mod}\ 6) \\ 1 + 6t \ , & \text{if } n \equiv 1 (\mathrm{mod}\ 6) \\ 3 + 6t \ , & \text{if } n \equiv 3 (\mathrm{mod}\ 6) \\ 4 + 6t \ , & \text{if } n \equiv 4 (\mathrm{mod}\ 6) \end{cases}$$

and let $n_2$=6. Then $n_1 = n - n_2$. We apply mathematical induction on $t$, to prove that $KG_{n,2}$ is claw-decomposable, when $n \ge 6$. If $t = 1$, then $n \in \{6, 7, 9, 10\}$. By Lemma 3.1, 3.2 and 3.3, we have the result. Assume that, the result is true for all $t < k$. Now, we prove that $KG_{n,2}$ is claw-decomposable, when $t = k$. The graph $G_1$ is claw-decomposable, by the induction hypothesis and $G_2$ is claw-decomposable, by Lemma 3.1. In $G_3$, the vertex set $A_3$ has $n_1$ layers and each layer has 6 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \le i < j \le n_1$ form a crown graph $C_{6,5}$. By Theorem

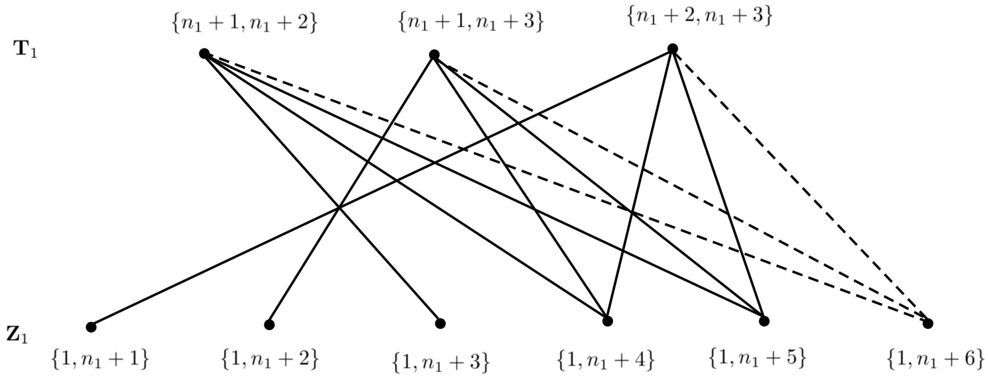


FIGURE 3.6. Claw-decomposition of $\langle E(T_1, Z_1)\rangle$ in $KG_{n,2}$

2.1, the graph $G_3$ is claw-decomposable. The graph $G_4$ is claw-decomposable, by Theorem 2.2. In $G_5$, the vertex set $A_3$ has $n_1$ layers and each layer has 6 vertices. We observe that, each vertex of $A_1$ is adjacent to $(n_1-2)$ layers. So, the degree of each vertex of $A_1$ is $6(n_1-2)$. Now, by fixing each vertex of $A_1$ as a center vertex (ofcourse, $2(n_1-2)$ times) we get a claw-decomposition of $G_5$. In $G_6$, the vertex set $A_3$ has $n_1$ layers and each layer has 6 vertices and $|A_2|$=15. The vertex set $A_2$ can be partitioned into 5 sets, say $T_1=\{\{n_1+1,n_1+2\},\{n_1+1,n_1+3\},\{n_1+2,n_1+3\}\}$, $T_2=\{\{n_1+4,n_1+5\},\{n_1+4,n_1+6\},\{n_1+5,n_1+6\}\}$, $T_3=\{\{n_1+1,n_1+4\},\{n_1+1,n_1+5\},\{n_1+1,n_1+6\}\}$, $T_4=\{\{n_1+2,n_1+4\},\{n_1+2,n_1+5\},\{n_1+2,n_1+6\}\}$ and $T_5=\{\{n_1+3,n_1+4\},\{n_1+3,n_1+5\},\{n_1+3,n_1+6\}\}$, so that $A_2=\bigcup_{i=1}^{5} T_i$. Then $G_6$=$\oplus_{i=1}^{5}\oplus_{j=1}^{n_1}\langle E(T_i,Z_j)\rangle$. We prove that each subgraph $\langle E(T_i,Z_j)\rangle$, where $1\le i\le 5$ and $1\le j\le n_1$ is claw-decomposable. Consider a subgraph $\langle E(T_i,Z_j)\rangle$, we choose a vertex $\{j,y^{'}\}\in Z_j$ as a center vertex of a star, say $S$, when $y^{'}\neq x$ and $y^{'}\neq y$, for all $\{x,y\}\in T_i$. In $\langle E(T_i,Z_j)\rangle\setminus S$, the degree of each vertex of $T_i$ is exactly three. By Proposition 2.3, we get a claw-decomposition. A claw-decomposition of $\langle E(T_1,Z_1)\rangle$ is given in Figure 3.6. Hence $\langle E(T_i,Z_j)\rangle$, $1\le i\le 5$ and $1\le j\le n_1$ is claw-decomposable. Thus, the graph $KG_{n,2}$ is claw-decomposable, for any $n\ge 6$ and $n\equiv 0,1,3,4 (\mathrm{mod}\ 6)$. This completes the proof of Case 1.

**Case 2:** $n\equiv 2,5(\mathrm{mod}\ 6)$. Let

$$n=\begin{cases}2+6t\ , & \text{if } n\equiv 2(\mathrm{mod}\ 6)\\ 5+6t\ , & \text{if } n\equiv 5(\mathrm{mod}\ 6)\end{cases}$$

and let $n_2$=4. Then $n_1=n-n_2$. If $t=1$, then $n\in\{8,11\}$. By Lemma 3.3, the graph $KG_{8,2}$ and $KG_{11,2}$ are claw-decomposable. Now, if $t\ge 2$, then the graph $G_1$ is claw-decomposable, by Case 1. In $G_2\cup G_4$, consider the claws $S_1:(\{n_1+1,n_1+2\};\{n_1+3,n_1+4\},\{1,2\},\{1,3\})$, $S_2:(\{n_1+1,n_1+3\};\{n_1+2,n_1+4\},\{1,2\},\{1,3\})$ and $S_3:(\{n_1+1,n_1+4\};\{n_1+2,n_1+3\},\{1,2\},\{1,3\})$. In $(G_2\cup G_4)\setminus E(S_1\cup S_2\cup S_3)$, the degree of each vertex of $A_1$ is exactly 6 except two vertices $\{1,2\}$ and $\{1,3\}$. The degree of these 2 vertices are exactly 3. Now, by fixing each vertex of $A_1$ as a center vertex we get a claw-decomposition in $(G_2\cup G_4)\setminus E(S_1\cup S_2\cup S_3)$. In $G_3$, the vertex set $A_3$ has $n_1$ layers and each layer has 4 vertices. Each subgraph $\langle E(Z_i,Z_j)\rangle$, $1\le i<j\le n_1$ form a crown graph

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$C_{4,3}$. Hence by Theorem 2.1, the graph $G_3$ is claw-decomposable. In $G_5$, the degree of each vertex of $A_3$ is $\frac{n_1(n_1-1)}{2} - (n_1 - 1) = \frac{(n_1-1)(n_1-2)}{2}$ and note that $n_1 - 1 \equiv 0 (\bmod\ 3)$. By Proposition 2.3, we get a claw-decomposition of $G_5$. In $G_6$, the vertex set $A_3$ has $n_1$ layers and each layer has 4 vertices. Note that, the degree of each vertex of $A_3$ is exactly 3. By Proposition 2.3, we get a claw-decomposition of $G_6$. This completes the proof. □

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