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OPTIMAL MAXIMAL GRAPHS

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Abstract. An optimal labeling of a graph with n vertices and m edges is an injective assignment of the first n nonnegative integers to the vertices, that induces, for each edge, a weight given by the sum of the labels of its end-vertices with the property that the set of all induced weights consists of the first m positive integers. We explore the connection of this labeling with other well-known functions such as super edge-magic and α-labelings. A graph with n vertices is maximal when the number of edges is $2n-3$; all the results included in this work are about maximal graphs. We determine the number of optimally labeled graphs using the adjacency matrix. Several techniques to construct maximal graphs that admit an optimal labeling are introduced as well as a family of outerplanar graphs that can be labeled in this form.

1. Introduction

An *additive vertex labeling* of a graph G of order n and size m is an injective function $f : V(G) \to S$, where S is a set of nonnegative integers, such that each edge uv of G has associated a *weight* defined as $f(u) + f(v)$. The origin of this kind of labeling can be found in the work about magic valuations of Kotzig and Rosa [13] and the paper of Graham and Sloane [9] about harmonious graphs. Kotzig and Rosa said that a graph G of order n and size m has a *magic valuation* with constant k if there exists an injective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, n+m\}$ such that $f(u) + f(v) + f(uv) = k$ for all $uv \in E(G)$. Enomoto et al. [5] said that a magic valuation is a *super edge-magic labeling* if the vertex labels are $1, 2, \ldots, n$. Figueroa-Centeno et al [6] proved that G admits a super edge-magic labeling if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ such that the set of induced weights consists of m consecutive integers.

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Let G be a graph of order n and size m, the function f, as well as G, is said to be *optimal* if $S = \{0, 1, \ldots, n-1\}$ and the set of induced weights is $W = \{1, 2, \ldots, m\}$. Optimal graphs were introduced by Acharya and Hegde [1] under the name strongly indexable; they said that $f : V(G) \to \{0, 1, \ldots, n-1\}$ is *strongly* (k, d)*-indexable* if the set of induced weights is $W = \{k, k+d, \ldots, k+(m-1)d\}$. Thus, an optimal labeling is a strongly $(1, 1)$-indexable labeling. The reason why we use here the name optimal to refer to these labelings is to simplify the notation and emphasize the optimality of the labeling in the sense that uses the smallest possible labels and induces the smallest possible weights, in particular, when it is contrasted with the super edge-magic total labeling.

Suppose that G is a graph of order n, size m, and f is an optimal labeling of G. If the labels $n-2$ and $n-1$ are assigned by f to adjacent vertices, then G has an edge of weight $2n-3$. No other edge of G has a larger weight; then, $2n-3$ is the largest possible value for the size of an optimal graph. When $m = 2n-3$, we say that G is *maximal*. Observe the strong similarity between optimal and super edge-magic labelings of maximal graphs; an optimal labeling can be transformed into a super edge-magic labeling by adding the constant 1 to each vertex label. Thus, all the results obtained in this work about optimal graphs can be written in terms of super edge-magic labelings. We say more about this idea in Section 3.

The new results in this work are presented in four sections. In Section 2 we define the concept of triangular arrangement for an optimal labeling, this idea is based on the adjacency matrix of an optimal graph. We use triangular arrangements to determine the number of optimally labeled graphs of order n. In Section 3 we discuss some general results about optimal labelings in connection with other labelings; we also proved there that in general, the friendship graph is not optimal, in addition we prove the existence of an optimal labeling for a family of outerplanar graphs. In Section 4 we present three new methods to combine optimal maximal graphs to obtain new optimal maximal graphs. In Section 5 we introduce a new technique to build optimal maximal graphs starting with an α-labeled tree. An α-labeling is a difference vertex labeling that in the case of a tree can be easily transformed into an additive vertex labeling. We also calculate the number of optimally labeled graphs that can be generated with this technique.

2. The Number of Optimally Labeled Maximal Graphs

Our goal is to determine the exact number of optimally labeled maximal graphs of order n. In [6], Figueroa-Centeno et al., determined the number of labeled graphs of order n and size m, where the labeling used is super edge-magic. The counting technique used by them is based on the fact that each graph of order n is a subgraph of the complete graph K_n. They presented their result saying that they have found the number of super edge-magic graphs of order n and size m, from our point of view that sentence is not correct; as we said before, they were counting labeled graphs. Independently of that, their formula for the case $m = 2n-3$, is equivalent to the closed formula that we present in the next theorem.

Theorem 2.1. *The number $\theta(n)$ of optimally labeled maximal graphs of order $n \geq 2$ is:*

- *When n is even:* $\theta(n) = \frac{n}{2}! \left(\frac{n-2}{2}!\right)^3$;

- *When n is odd:* $\theta(n) = \frac{n-3}{2}!\left(\frac{n-1}{2}!\right)^3$.

Proof. Suppose that $0, 1, \ldots, n-1$ are the vertices of a maximal graph G. Let $A = (a_{i,j})$ be the adjacency matrix of G, that is, for each $0 \leq i, j \leq n-1$, the cell

$$a_{i,j} = \begin{cases} 1 & \text{if } ij \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Consider the function $\omega : \{a_{i,j} : a_{i,j} \in A\} \to \{0, 1, \ldots, 2n-2\}$, defined as $\omega(a_{i,j}) = i + j$. Clearly, f is a surjective function. For each $0 \leq k \leq 2n-2$, let R_k be the antidiagonal of A formed by all the cells $a_{i,j}$ such that $\omega(a_{i,j}) = k$. Thus, for each $0 \leq k \leq n-1$, R_k and R_{2n-2-k} are formed by exactly k cells. Note that when $k = n-1$, $R_k = R_{2n-2-k}$ and that for each $0 \leq k \leq n-1$, these antidiagonals have exactly $\frac{k}{2}$ cells when k is even and $\frac{k+1}{2}$ cells when k is odd.

Since A is symmetric and the cells on the main diagonal are equal to zero, all the adjacencies of G can be seen on the cells above the main diagonal. From this point, we just focus on the cells located above the main diagonal.

In order to have an optimal labeling of G, for each integer $t \in \{1, 2, \ldots, 2n-3\}$, there must exist an edge of G of weight t. This means that exactly one cell of R_k equals one and every antidiagonal must contain a cell equal to 1.

Therefore, when n is even, for each $r \in \{1, 2, \ldots, \frac{n-2}{2}\}$, the triangular arrangement formed by the cells above the main diagonal has exactly 4 antidiagonals with r cells and 1 antidiagonal with $\frac{n}{2}$ cells. Since only one cell per antidiagonal should be selected in order to have an optimal labeling of a graph G of order n and size $2n-3$, we may calculate the number of optimally labeled graphs to be

$$\theta(n) = 1^4 \cdot 2^4 \cdot \cdots \cdot \left(\frac{n-2}{2}\right)^4 \cdot \frac{n}{2} = \frac{n}{2}!\left(\frac{n-2}{2}!\right)^3$$

Similarly, we may calculate $\theta(n)$ when n is odd; now, for each $1 \leq r \leq \dfrac{n-3}{2}$, there are 4 antidiagonals with r cells and 3 antidiagonals with $\dfrac{n-1}{2}$ cells. Thus,

$$\theta(n) = 1^4 \cdot 2^4 \cdot \cdots \cdot \left(\frac{n-3}{2}\right)^4 \cdot \left(\frac{n-1}{2}\right)^3 = \frac{n-3}{2}!\left(\frac{n-1}{2}!\right)^3$$

□

In Figure 1 we present an optimally labeled graph of order $n = 8$ and size 13 together with the triangular arrangement obtained from the associated adjacency matrix; the numbers on each cell corresponds to the values (or weights) given by the function ω, the adjacencies are given by the highlighted cells.

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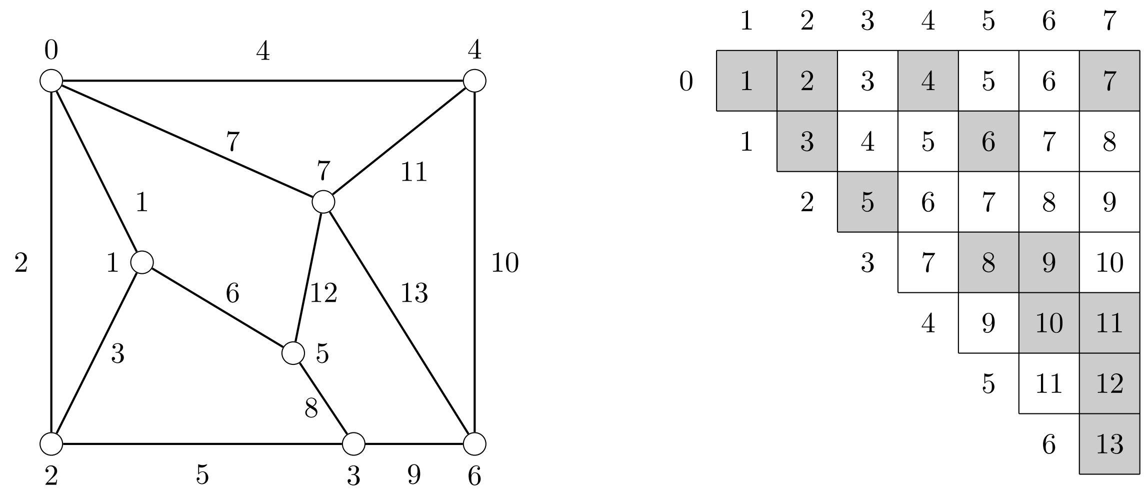

FIGURE 1. Optimally labeled graph and its adjacency matrix

The sequence formed by the consecutive values of $\theta(n)$ corresponds to the sequence A327904 in OEIS.

n	Optimally labeled graphs of order n	n	Optimally labeled graphs of order n
2	1	3	1
4	2	5	8
6	48	7	432
8	5,184	9	82,944
10	1,658,880	11	41,472,000
12	1,244,160,000	13	44,789,760,000
14	1,881,169,920,000	15	92,177,326,080,000
16	5,161,930,260,480,000	17	330,363,536,670,720,000
18	23,786,174,640,291,800,000	19	1,926,680,145,863,640,000,000
20	173,401,213,127,728,000,000,000	21	17,340,121,312,772,800,000,000,000

TABLE 1. Number of optimally labeled maximal graphs of order n

3. General Results

Let G be a graph of order n and size m. An additive vertex labeling f of G is said to be *super edge-magic* if $S = \{1, 2, \dots, n\}$ and the set of induced weights is formed by m consecutive integers. When such a labeling exists, the graph G is also called *super edge-magic*. If f is an optimal labeling of G, then it can be transformed into a super edge-magic labeling by adding the constant 1 to each label. Graham and Sloane [9] called a connected graph of order n and size $m \geq n$ *harmonious* if it is possible to label the vertices v with distinct elements $f(v)$ of $\mathbb{Z}_m$ (the set of integers modulo m) in such a way that, when each edge uv is labeled with $f(u) + f(v)$, the resulting edge labels (or weights) are distinct. Among other

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results, they proved that the cycle C_n is harmonious if and only if $n \geq 3$ is odd. The cycle C_n is a graph of order and size equal to n, therefore this labeling is in fact super edge-magic. This result was also proven, years later, by Enomoto et al., in [5]. The smallest weight induced by the harmonious (or super edge-magic) labeling of C_n given in [9] is $\frac{n+1}{2}$ which is larger than 1 when $n \geq 5$. This implies that not every super edge-magic labeling can be transformed into an optimal labeling.

In 1992, Jungreis and Reid [12] proved that the Cartesian product of two paths is an α-graph (see Section 5), they also showed how the α-labeling of $P_m \times P_n$ can be transformed into a harmonious labeling. In the specific case of the ladder $L_n = P_2 \times P_n$, with n odd, this α-labeling can be easily modified to obtain a super edge-magic labeling. On the other side, the ladder is not optimal; in order to see this fact we must recall that, because L_n is triangle-free, in a hypothetical optimal labeling of L_n, the vertex labeled 0 must be adjacent to the vertices labeled $1, 2$ and 3 to obtain the weights 1, 2, and 3; which implies that L_n does not have an edge of weight 4. Thus, the ladder is another example of a super edge-magic graph that is not optimal. So, we have that optimal labelings are more restrictive than other additive vertex labelings. Several interesting results about optimal labelings can be found in [3], [8], [10], [11], [15].

For $n \geq 1$, the *friendship graph* is $F_n = nK_2 + K_1$. Then, F_n has order $2n+1$ and size $3n$, with one vertex of degree $2n$ and $2n$ vertices of degree 2. In [2], Acharya and Germina conjectured that any cactus (or Husini tree) with $4t$ blocks is optimal; they showed an example to support this conjecture, exhibiting an optimal labeling of F_4. They also proved that F_5 is not optimal. In the next result, we prove that F_n is not optimal when $n \geq 5$.

Proposition 3.1. *The friendship graph F_n is not optimal when $n \geq 5$.*

Proof. By contradiction. Suppose that there exists an optimal labeling f of F_n. Let $V(F_n) = \{v_0, v_1, \ldots, v_{2n}\}$, where $\deg(v_0) = 2n$ and for $i \geq 1$, $\deg(v_i) = 2$. Note that $f(v_0) \leq n$, otherwise F_n would have an edge of weight larger than $3n$ because the vertex labeled $2n$ is adjacent to v_0. Moreover, since the weights 1 and 2 can only be produced with the combinations $0+1$ and $0+2$, we have that $f(v_0) \in \{0, 1, 2\}$. We analyze the three options.

Case 1: When $f(v_0) = 0$. Thus, the weights of the edges incident to v_0 are $1, 2, \ldots, 2n$. This implies that the edges connecting the vertices of degree 2 have weights $2n+1, 2n+2, \ldots, 3n$. Without loss of generality, we assume momentarily that $f(v_i) = i$. Note that v_1 and v_{2n} must be connected; if that is not he case, the edge formed by v_1 and its neighbor of degree 2 has weight less than $2n+1$, which is not possible. Consequently, the edge formed by v_2 and its neighbor of degree 2 has weight at most $2n+1$. But this contradicts the fact that for each $w \in \{1, 2, \ldots, 3n\}$, F_n has only one edge of weight w.

Case 2: When $f(v_0) = 1$. Recall that the only option to have an edge of weight 2 is by assigning the labels 0 and 2 on two adjacent vertices. Thus, the weights of the edges incident to v_0 are $1, 2, \ldots, 2n+1$. In addition, we have the labels $3, 4, \ldots, 2n$ to produce edges with weights in $W = \{2n+2, 2n+3, \ldots, 3n\}$. Note that $3 + x \in W$ if and only if $x \in \{2n-1, 2n\}$.

If the vertices labeled 3 and $2n-1$ are connected, then we get an edge of weight $2n+2$. So, the vertices labeled 4 and $2n$ must be connected, otherwise the weight of the edge connecting the vertex labeled 4 with the other vertex of degree 2 has weight smaller than $2n+3$. Similarly, we conclude that the vertices labeled 5 and $2n-2$ are connected with an edge of weight $2n+3$. Observe that the largest label to assign on the vertex adjacent to the vertex labeled 6 is $2n+3$; that means that the largest weight on this edge is $2n+3$, but this weight (as well as all the weights from 1 up to $2n+4$) has been obtained before. Thus, the vertex labeled 6 cannot be connected with any of the still available vertices, which is a contradiction.

If the vertices 3 and $2n$ are connected, producing an edge of weight $2n+3$, we are forced to connect the vertices labeled 4 and $2n-2$, to generate an edge of weight $2n+2$. But this implies that the vertices labeled 5 and $2n-1$ must be connected to generate an edge of weight $2n+4$. Consequently, the largest label available for the vertex adjacent to the vertex labeled 6 is $2n-3$, creating an edge of weight $2n+3$, but this weight was already obtained; so, we have another contradiction.

Hence, the friendship graph F_n is not optimal when $n \geq 5$. □

Suppose that G is an optimal maximal graph of order n. Then G has a subgraph isomorphic to C_3, which vertices are labeled $(0, 1, 2)$, or isomorphic to $K_{1,3}$, which vertices are labeled $(0; 1, 2, 3)$. But this characteristic can be extended because there are only two possibilities to obtain the three largest weights: a C_3 which vertices are labeled $(n-3, n-2, n-1)$, or a $K_{1,3}$ which vertices are labeled $(n-1; n-2, n-3, n-4)$.

In [2], Acharya and Germina studied the existence of an optimal labeling for some maximal outerplanar graphs; assuming that G is a maximal outerplanar graph of order n, they prove the following:

(1) For $0 \leq n \leq 7$, the graph G is optimal if and only if G is other than the outerplanar graph of order 6 which 3 chords form a triangle.
(2) If G has a vertex of degree $n-1$, then G is optimal if and only if $n \leq 7$.
(3) For $n > 7$, if G is optimal, then G is super edge-magic.
(4) For $n > 7$, if G has maximum degree $\lfloor \frac{n+4}{2} \rfloor$ and exactly two vertices of degree 2, then G is optimal.

The k-cell snake polyiamonds form a family of maximal outerplanar graphs. *Polyiamonds* are geometrical shapes constructed from unit equilateral triangles, called *cells*, joined edge-to-edge on a triangular grid. Note that when these structures are analyzed from a graph theoretical perspective, the associated graphs cannot have vertices of degree larger than 6. In a *snake polyiamond*, each cell shares at most two edges with other cells. In [4], Barrientos proved that all snake polyiamonds are harmonious graphs. The labeling used to prove this result assigns the labels $0, 1, \ldots, n-1$ and induces the weights $1, 2, \ldots, 2n-3$. Therefore, that labeling of these snakes is in fact an optimal labeling. The pattern used by this labeling can be easily observed in the graphs on Figure 2.

Proposition 3.2. *If G is a maximal outerplanar graph of order $n \geq 3$ such that $\Delta(G) \leq 6$, then G is optimal.*

By imposing the condition $\Delta(G) \leq 6$ we guarantee that G is in fact a snake polyiamond. The result in this last proposition contradicts the following statement proved in [2].

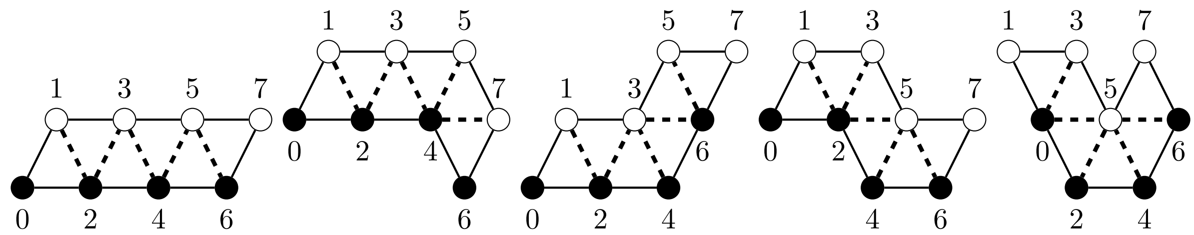

FIGURE 2. Optimal labeling of all 6-cell snake polyiamonds

"Let $G = (V, E)$ be a maximal outerplanar graph with $p > 7$. Let $H = (u_1, u_2, \ldots, u_p)$ be a Hamiltonian cycle in G. Then, there exists an equitable partition $\{V_1, V_2\}$ of $V(G)$ such that no chord of H has both its ends in V_1 or V_2 if and only if $\Delta(G) = \lfloor \frac{p}{2} \rfloor + 2$ and there exist exactly two vertices of degree 2."

Note that all graphs depicted in Figure 2 are maximal and outerplanar, their vertices are separated in two classes, each chord has its end vertices on different classes, in addition, each graph has exactly two vertices of degree 2, but the maximum degree not always equals $\lfloor \frac{p}{2} \rfloor + 2$, that in this case is 6.

4. **New Constructions**

In this section we present four alternatives to construct optimal maximal graphs. Constructions 1 and 2 are strongly related but they are not exactly the same. The outcome of the first construction is a graph that present some sort of symmetry or regularity. Even considering that the Construction 2 supersedes Construction 1 we have decided to include it here to introduce the technique used in Construction 2.

Proposition 4.1. *For $i = 1, 2$, let G_i be an optimal maximal graph of order n_i. There exists an optimal maximal graph of order $n_1 + n_2 - 1$ that contains G_1 and G_2 as induced subgraphs.*

Proof. Suppose that G_1 and G_2 have been optimally labeled. Add to every label on G_2 the constant $n_1 - 1$; thus, the labels on G_2 are $n_1 - 1, n_1, \ldots, n_1 + n_2 - 2$ and the induced weights are $2n_1 - 1, 2n_1, \ldots, 2(n_1 + n_2 - 1) - 3$. Since G_1 has been optimally labeled, the labels on G_1 are $0, 1, \ldots, n_1 - 1$ and the induced weights are $1, 2, \ldots, 2n_1 - 3$. Both graphs have a vertex labeled $n_1 - 1$, then the amalgamation of these two vertices results in a graph of order $n_1 + n_2 - 1$ with a labeling that assigns the labels $0, 1, \ldots, n_1 + n_2 - 2$; in addition, the weights induced by this labeling form the set $\{1, 2, \ldots, 2(n_1 + n_2 - 1) - 3\} - \{2n_1 - 2\}$. Clearly, G_1 and G_2 are induced subgraphs of the newly formed graph. The vertex labeled 0 is in G_1 and the vertex labeled $2n_1 - 2$ is in G_2. Therefore, we may connect them, without duplicating any edge, to obtain an optimally labeled maximal graph. □

Note that 0 and $2n_1 - 2$ are not the only pair of labeled vertices that can be connected to produce an optimally labeled graph. If $\nu = \min\{n_1, n_2\}$, there are $\nu - 1$ pairs of vertices that can be connected to form the final graph. We show an example of this fact in Figure 3, where any of the dashed lines can be used to produce an edge of weight 14.

Construction 1: Suppose that G is an optimal maximal graph of order n. Let f be an optimal labeling of G. We take two labeled copies of G, say G_1 and G_2. Let M be the graph of order $2n$ obtained

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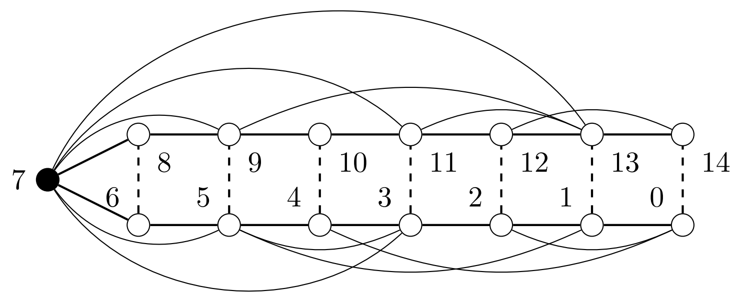

FIGURE 3. Optimally labeled graphs with maximum circumference

by connecting the vertices u, v, w of G_1, which corresponding labels are $n-3, n-2$ and $n-1$, to their replicas in G_2. The graph M has size $2(2n-3)+3 = 2(2n)-3$, that is, M is maximal. An optimal labeling of M is obtained by relabeling G_2 with the labeling $\overline{f}$ shifted n units, i.e., the final label of each $v \in G_2$ is $2n-1-f(v)$. Thus, there is no repetition of labels between G_1 and G_2 and the weights induced on G_2 are $2n+1, 2n+2, \dots, 4n-3$. Note that the replicas of u, v and w have labels $n+1, n+2, n$, respectively. Therefore, when each of these vertices is connected with its replica, we generate three edges which corresponding weights are $2n-2, 2n$, and $2n-1$.

In Figure 4 we show an example of an optimal maximal graph obtained using this construction.

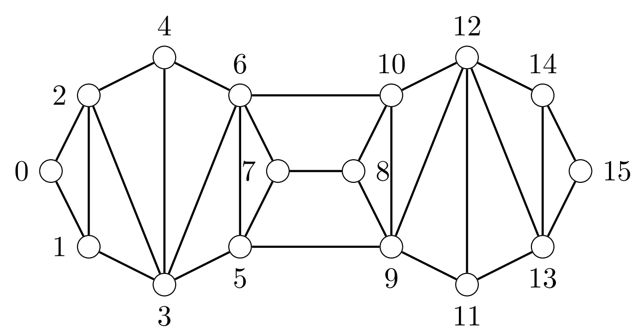

FIGURE 4. Optimal maximal graph built using Construction 1

Construction 2: For $i = 1, 2$, let G_i be an optimal maximal graph of order n_i and f_i be an optimal labeling of G_i. Suppose that u_i, v_i and w_i are the vertices of G_i such that

$$f_1(u_1) = n_1 - 3, f_1(v_1) = n_1 - 2, f_1(w_1) = n_1 - 1, f_2(u_2) = 1, f_2(v_2) = 2, \text{and } f_2(w_2) = 0.$$

Let R be the graph of order $n_1 + n_2$ obtained by connecting u_1 to u_2, v_1 to v_2, and w_1 to w_2. Then, R has size $2n_1 - 3 + 2n_2 - 3 + 3 = 2(n_1 + n_2) - 3$, i.e., R is maximal. Furthermore, R is optimal. Indeed, an optimal labeling f of R is obtained by keeping the labeling of G_1 and shifting n_1 units the labeling of G_2. In other terms, for any vertex x in R,

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in V(G_1), \\ f_2(x) + n_1 & \text{if } x \in V(G_2). \end{cases}$$

Note that f is an injective function that assigns the labels $0, 1, \ldots, n_1 - 1$ to the vertices of G_1 and $n_1, n_1 + 1, \ldots, n_1 + n_2 - 1$ to the vertices of G_2. The weights on the edges of G_1 are $1, 2, \ldots, 2n_1 - 3$ and $2n_1 + 1, 2n_1 + 2, \ldots, 2(n_1 + n_2) - 3$ on the edges of G_2. In addition, the weights of the edges connecting G_1 and G_2 are:

$$f(u_1) + f(u_2) = n_1 - 3 + n_1 + 1 = 2n_1 - 2,$$
$$f(v_1) + f(v_2) = n_1 - 2 + n_1 + 2 = 2n_1,$$
$$f(w_1) + f(w_2) = n_1 - 1 + n_1 = 2n_1 - 1.$$

Consequently, the weights on the edges of R form the set $\{1, 2, \ldots, 2(n_1 + n_2) - 3\}$. Therefore, R is optimal.

In the following proposition we show that there are several ways to connect with three edges the graphs G_1 and G_2.

Proposition 4.2. *For $i = 1, 2$, let G_i be an optimal maximal graph of order n_i and $n = min\{n_1, n_2\}$. There are $(n-1)n^2$ ways to connect G_1 to G_2 with three edges to obtain an optimal maximal graph of order $n_1 + n_2$.*

Proof. Suppose that G_1 and G_2 have been optimally labeled and that the labeling of G_2 has been shifted n units. Without loss of generality we assume that $n_1 \leq n_2$. Since G_1 and G_2 are maximal, $2n - 3$ is the largest weight on an edge of G_1, while $2n + 1$ is the smallest weight on an edge of G_2. Therefore, the triangular arrangement associated with the adjacency matrix of $G_1 \cup G_2$ has all the adjacencies of G_1 in the first $2n_1 - 3$ antidiagonals and all the adjacencies of G_2 in the last $2n_2 - 3$ antidiagonals. In other terms, for each $i \in \{0, 1, 2\}$, the $(2n - i)$th antidiagonal does not contain an adjacency. The selection of one cell on each of these antidiagonals corresponds to the introduction of three edges connecting the two graphs. Recall that the kth antidiagonal has $\frac{k}{2}$ cells when k is even. Based on the facts that $n_1 \leq n_2$, and $k = 2n_1 - 2 \leq (2(n_1 + n_2) - 3) - 1$ is even, we conclude that R_k has $\frac{2n-2}{2} = n - 1$ cells, while R_{k+1} and R_{k+2} have n cells each. Thus, the number of ways to select three adjacencies on these antidiagonals is $(n-1)n^2$. □

In the following result we present another method to obtain an optimal maximal graph of order $n + 1$ from an optimal maximal graph of order n.

Proposition 4.3. *Suppose G is an optimally labeled maximal graph of order $n \geq 3$. An optimal maximal graph is obtained by subdividing the edge of weight $2n - 3$ and connecting the new vertex to the vertex labeled $n - 3$.*

Proof. Let f be an optimal labeling of G. Because G is maximal, there exists $e = uv \in E(G)$ such that $f(u) = n - 2$ and $f(v) = n - 1$, that is, an edge of weight $2n - 3$. If e is subdivided by introducing a new vertex labeled n, the weights of the edges of the new graph form the set $\{1, 2, \dots, 2n-1\} \setminus \{2n-3\}$. By connecting the new vertex to the vertex labeled $n-3$ by f, we create an edge of weight $2n-3$. Thus, the new graph has order $n+1$ and size $2n-1$, because only one vertex and two edges have been introduced into the structure of G. In other terms, the new graph is maximal. In addition, the labeling of this graph is optimal because assigns the labels $0, 1, \dots, n$ and induces the weights $1, 2, \dots, 2n-1$. □

In Figure 5 we show two optimal graphs obtained applying this method multiple times to the optimal labeling of C_3. Note that for $n \geq 6$, the graphs obtained in this way have degree sequence $3^6\, 4^{n-6}$.

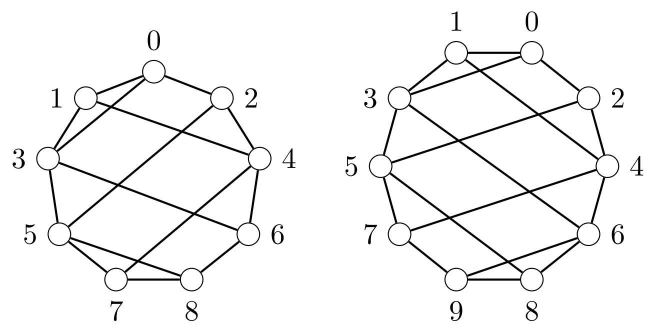

FIGURE 5. Optimally labeled graphs with maximum circumference

5. Optimal and α-labelings

In this section we explore a relationship between optimal labelings of maximal graphs and the strongest type of difference vertex labeling. A *difference vertex labeling* of a graph G of order n and size m is a function $f : V(G) \to S$, where S is a set of nonnegative integers, such that each edge uv of G has associated a *weight* defined as $|f(u) - f(v)|$. This type of labeling was introduced by Rosa [14]; among the labelings defined by Rosa, the kind called graceful is the most widely studied. The labeling f is said to be *graceful* if it is injective, $S = \{0, 1, \dots, m\}$, and the set of induced weights is $\{1, 2, \dots, m\}$. A graceful labeling is called an α-labeling if there exists an integer λ, called its *boundary value*, such that for each edge uv of G, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. When such a labeling exists for a graph G, we say that G is an *α-graph*. Note that every α-graph must be bipartite. This variety of graceful labeling has connections with several types other types of labelings. The readers interested in this subject are referred to Gallian [7], where they can find a comprehensive collection of results on this research area. In the next theorem we introduce another method to generate optimal maximal graphs, the method uses α-labeled trees that become the spanning trees of the resulting optimal graphs.

Theorem 5.1. *Every α-tree is a spanning tree of an optimal maximal graph.*

Proof. Let f be an α-labeling with boundary value λ of a tree T of size n. Then the labels on each stable set of T form the sets $\{0, 1, \dots, \lambda\}$ and $\{\lambda+1, \lambda+2, \dots, n\}$. Let $g : V(T) \to \{0, 1, \dots, n\}$ a new labeling of T defined by:

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$$g(v) = \begin{cases} f(v) & \text{if } f(v) \leq \lambda, \\ n + \lambda + 1 - f(v) & \text{if } f(v) > \lambda. \end{cases}$$

Note that g is an injective function and that the labels on each stable set are still the same. Suppose that uv is the edge of T such that $f(v) - f(u) = w$, where $w \in \{1, 2, \ldots, n\}$. If we consider g to be an additive vertex labeling, then the weight of uv is:

$$\begin{aligned} g(u) + g(v) &= f(u) + n + \lambda + 1 - f(v) \\ &= n + \lambda + 1 - (f(v) - f(u)) \\ &= n + \lambda + 1 - w. \end{aligned}$$

Since $w \in \{1, 2, \ldots, n\}$, we have that $n + \lambda + 1 - w \in \{\lambda + 1, \lambda + 2, \ldots, n + \lambda\}$. Recall that the size of an optimal maximal graph of order $n + 1$ is $2n - 1$. If the vertex labeled 0 is connected with the vertices labeled $1, 2, \ldots, \lambda$, we create λ edges which weights are $1, 2, \ldots, \lambda$. If the vertex labeled n is connected with the vertices labeled $\lambda + 1, \lambda + 2, \ldots, n - 1$, we form $n - \lambda - 1$ edges which weights are $n + \lambda + 1, n + \lambda + 2, \ldots, 2n - 1$. These edges do not exist in T because they are connecting vertices in the same stable set. Therefore, the graph obtained in this manner has the same vertex set that T, consequently, T is one of its spanning trees. □

In general, there is more that one pair of labels of T that produce the same weight that we are looking for, then there is more that one graph that has T as a spanning tree. To verify the accuracy of this statement we go back to Section 2 where we introduced the idea of the triangular arrangement associated to the optimal labeling of a maximal graph. Recall that in the present case the maximal graph has order $n+1$ because any of its spanning trees (in particular T) has size n; hence, there are $2n-1$ antidiagonals. We must also remember that for each $1 \leq i \leq 2n - 1$, the antidiagonals R_i and R_{2n-i} contain the same number of cells and that for each $1 \leq i \leq n - 1$, R_i has $\lceil \frac{i}{2} \rceil$ cells.

The labeling g of T uses the antidiagonals from $R_{\lambda+1}$ up to $R_{\lambda+n}$. Thus, counting the possible distributions of the adjacencies of the maximal graph on the last $n - \lambda - 1$ antidiagonals is the same that counting using the first $n - \lambda - 1$ antidiagonals. Since the number of cells in R_i depends on the parity of i, we need to analyze four cases.

Case 1: When both n and λ are odd. The number of possible distributions of the adjacencies on the first λ antidiagonals is $\frac{\lambda+1}{2}\left(\frac{\lambda-1}{2}!\right)^2$, and on the first $n - \lambda - 1$, it is $\frac{n-\lambda}{2}\left(\frac{n-\lambda-2}{2}!\right)^2$. Therefore, there are $\frac{(n-\lambda)(\lambda+1)}{4}\left(\frac{\lambda-1}{2}!\frac{n-\lambda-2}{2}!\right)^2$ optimally labeled maximal graphs that have T as spanning tree.

Case 2: When n is even and λ is odd. On the first λ antidiagonals, the number is the same that before; on the first $n - \lambda - 1$ the number is $\left(\frac{n-\lambda-1}{2}!\right)^2$. Thus, there are $\frac{\lambda+1}{2}\left(\frac{\lambda-1}{2}!\frac{n-\lambda-1}{2}!\right)^2$ optimally labeled maximal graphs that have T as spanning tree.

Case 3: When n is odd and λ is even. For the first λ antidiagonals, there are $\left(\frac{\lambda}{2}!\right)^2$ possible distributions of the adjacencies; for the first $n - \lambda - 1$, the number is $\left(\frac{n-\lambda-1}{2}!\right)^2$. Then, $\left(\frac{\lambda}{2}!\frac{n-\lambda-1}{2}!\right)^2$ is the number of optimally labeled maximal graphs that have T as spanning tree.

Case 4: When both n and λ are even. On the first λ antidiagonals, the number is as in Case 3; on the first $n-\lambda-1$, the number is $\frac{n-\lambda}{2}\left(\frac{n-\lambda-2}{2}!\right)^2$. Consequently, the number of optimally labeled maximal graphs that have T as spanning tree is $\frac{n-\lambda}{2}\left(\frac{\lambda}{2}!\frac{n-\lambda-2}{2}!\right)^2$.

In Figure 6 we show this fact presenting four optimal maximal graphs that have the same spanning tree. Thicker lines are used for the edges of the spanning tree.

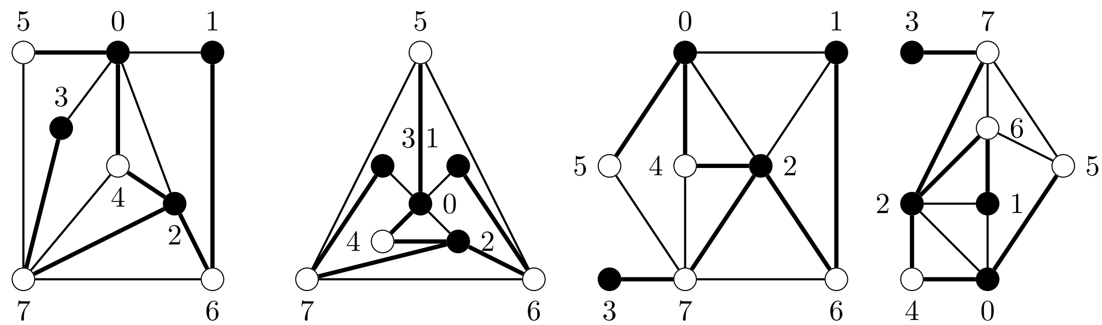

FIGURE 6. Optimally labeled maximal graphs with the same spanning tree

Before closing this section and this work, we must mention that not all the time these maximal graphs are nonisomorphic, but as labeled graphs they are different. Also, it is possible that the same maximal graph can be constructed with more that one α-tree. However, this technique has the benefit that it is a lot easier to find the required labeling of a tree than the labeling of a maximal graph.

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