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BOUNDS FOR THE PEBBLING NUMBER OF PRODUCT GRAPHS

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Abstract. Let G be a connected graph. Given a configuration of a fixed number of pebbles on the vertex set of G, a pebbling move on G is the process of removing two pebbles from a vertex and adding one pebble on an adjacent vertex. The pebbling number of G, denoted by $\pi(G)$, is defined to be the least number of pebbles to guarantee that there is a sequence of pebbling movement that places at least one pebble on each vertex v, for any configuration of pebbles on G. In this paper, we improve the upper bound of $\pi(G\Box H)$ from $2\pi(G)\pi(H)$ to $\left(2-\frac{1}{\min\{\pi(G),\pi(H)\}}\right)\pi(G)\pi(H)$ where $\pi(G)$, $\pi(H)$ and $\pi(G\Box H)$ are the pebbling number of graphs G, H and the Cartesian product graph $G\Box H$, respectively. Moreover, we also discuss such bound for strong product graphs, cross product graphs and coronas.

1. Introduction

Throughout this paper, all graphs are considered to be connected, finite and simple. The number of vertices of a graph G is denoted by $|G|$. For basic definitions and terminologies not mentioned here, we refer the reader to the book of West [13].

Let G and H be graphs. The *Cartesian product* of G and H, denoted by $G\Box H$, is the graph with the vertex set $V(G)\times V(H)$ and the edge set $\{(u,v_1)(u,v_2) : u\in V(G) \text{ and } v_1v_2\in E(H)\}\cup\{(u_1,v)(u_2,v) : u_1u_2\in E(G) \text{ and } v\in V(H)\}$ (see [7]). The *cross (or direct) product* of G and H, denoted by $G\times H$, is the graph with the vertex set $V(G)\times V(H)$ and the edge set $\{(u_1,v_1)(u_2,v_2) : u_1u_2\in E(G) \text{ and } v_1v_2\in E(H)\}$ (see [7]). The *strong product* of G and H, denoted by $G\boxtimes H$, is the graph with the vertex set $V(G)\times V(H)$ and the edge set $E(G\Box H)\cup E(G\times H)$ (see [7]). The *coronas* of G and H, denoted by $G\bowtie H$, is the graph with the vertex set $V(G)\cup(V(G)\times V(H))$ and the edge set $E(G)\cup\{u(u,v) :$

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$u \in V(G)$ and $v \in V(H)\} \cup \{(u, v_1)(u, v_2) : u \in V(G)$ and $v_1v_2 \in E(H)\}$ (see [8]). For more detail treatments of graph products, we refer the reader to [7].

Let G be a graph. A *(pebbling) configuration* on G is defined to be a function $D : V(G) \to \mathbb{N} \cup \{0\}$ and we can say that D is a configuration of $\sum_{v\in V(G)} D(v)$ pebbles on G. Given a configuration on G, a pebbling move consists of removing two pebbles from a vertex and placing a single pebble on an adjacent vertex. For a configuration D on G and a vertex $r \in V(G)$, D is said to be *r-solvable* when there exists a sequence of pebbling moves that places at least one pebble on r. Furthermore, a configuration is *solvable* whenever it is r-solvable for every vertex r. It is *unsolvable* otherwise. Given a configuration D on a graph G, we call $\sum_{v\in V(G)} D(v)$ the *size* of D and denoted by $|D|$. The *pebbling number* of a connected graph G, denoted by $\pi(G)$, is the smallest integer m such that D is solvable for every configuration D on G with $|D| \geq m$. We remind a fact due to Chung [2] that $|G| \leq \pi(G)$. Analogously, the 2-*pebbling number* of G, denoted by $\pi_{(2)}(G)$, is defined to be the least number of pebbles to guarantee that for any configuration of pebbles on G and arbitrary vertex v, there is a sequence of pebbling moves that places at least 2 pebbles on v [2]. It is obvious that $\pi(G) \leq \pi_{(2)}(G) \leq 2\pi(G)$. In the product graphs between graphs G and H, $|D_x|$ denotes $\sum_{v\in V(H)} D(x, v)$ for each $x \in V(G)$. For a survey of graph pebbling, we refer the reader to [5],[6] and [9].

A center of interest of graph pebbling is Graham's conjecture which is proposed by Chung [2]. This was proven for special cases of products between graphs from specific families such as paths [2], cycles [4], complete graphs [2], complete bipartite graphs [3], etc. In more general, it was also discussed between a graph and a particular graph, for example, complete graphs (see [2] and [11]), complete bipartite graphs (see [10]), complete multipartite graphs (see [12]), etc. However, it remains open in general.

Conjecture 1.1. [2, Graham] *For graphs G and H, we have*

$$\pi(G \square H) \leq \pi(G)\pi(H).$$

In general, a recent result concerning with this conjecture is proposed in the paper of Asplund, Hurlbert and Kenter [1].

Theorem 1.2. [1] *For graphs G and H, we have*

$$\pi(G \square H) \leq 2\pi(G)\pi(H).$$

The purpose of this paper is to improve this upper bound for Cartesian product graphs and also for other product graphs.

2. Main results

Before we begin this section with an observation of a configuration introduced by Pleanmani [10] and [11] on a Cartesian product graph, we note that $G \square H$ is connected if and only if G and H are both connected. Furthermore, this product is commutative up to isomorphism [7]. For a positive integer n greater than 1 and a graph G, we define the nth power of G with respect to the Cartesian product by $G^1 = G$ and $G^n = G \square G^{n-1}$.

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Lemma 2.1. [10, 11] *Let $G = (V, E)$ and H be connected graphs, S be a subset of V and D be a configuration on $G\Box H$. Then we have*

$$n \sum_{x \in V} \left\lfloor \frac{|D_x|}{n} \right\rfloor \geq |D| - (n-1)|G|.$$

We note that this lemma also holds for $G \boxtimes H$, $G \times H$ and $G \bowtie H$ by appropriately choosing of G and H.

Next, we introduce an observation of Asplund, Hurlbert and Kenter [1] which is useful for the proof of our main result.

Lemma 2.2. [1] *For graphs G and H, let D be a configuration on $G\Box H$. If*

$$\sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{\pi(H)} \right\rfloor \geq \pi(G),$$

then D is solvable.

We are now ready for proving the main result.

Proposition 2.3. *For graphs G and H, we have*

$$\pi(G\Box H) \leq \pi(G)\pi(H) + \min\{(\pi(G)-1)|H|, (\pi(H)-1)|G|\}.$$

Proof. We first show that $\pi(G\Box H) \leq \pi(G)\pi(H) + (\pi(H)-1)|G|$. Let D be a configuration of $\pi(G)\pi(H) + (\pi(H)-1)|G|$ pebbles on $H\Box G$. By Lemma 2.1, we have

$$\begin{aligned} \pi(H) \sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{\pi(H)} \right\rfloor &\geq \pi(G)\pi(H) + (\pi(H)-1)|G| - (\pi(H)-1)|G| \\ &= \pi(G)\pi(H). \end{aligned}$$

This implies that $\sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{\pi(H)} \right\rfloor \geq \pi(G)$. By Lemma 2.2, D is solvable. Hence $\pi(G\Box H) \leq \pi(G)\pi(H) + (\pi(H)-1)|G|$. Similarly, we can show that $\pi(H\Box G) \leq \pi(H)\pi(G) + (\pi(G)-1)|H|$. Since $\pi(G\Box H) = \pi(H\Box G)$, the result follows. □

Theorem 2.4. *For graphs G and H, we have*

$$\pi(G\Box H) \leq \left(2 - \frac{1}{\min\{\pi(G), \pi(H)\}}\right) \pi(G)\pi(H).$$

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Proof. By Proposition 2.3,

$$\begin{aligned}
\pi(G\square H) &\leq \pi(G)\pi(H) + \min\{(\pi(G)-1)|H|, (\pi(H)-1)|G|\} \\
&\leq \pi(G)\pi(H) + \min\{(\pi(G)-1)\pi(H), (\pi(H)-1)\pi(G)\} \\
&= \min\{\pi(G)\pi(H) + (\pi(G)-1)\pi(H), \pi(G)\pi(H) + (\pi(H)-1)\pi(G)\} \\
&= \min\{2\pi(G)\pi(H) - \pi(H), 2\pi(G)\pi(H) - \pi(G)\} \\
&= \min\left\{\left(2 - \frac{1}{\pi(G)}\right)\pi(G)\pi(H), \left(2 - \frac{1}{\pi(H)}\right)\pi(G)\pi(H)\right\} \\
&= \left(2 - \max\left\{\frac{1}{\pi(H)}, \frac{1}{\pi(G)}\right\}\right)\pi(G)\pi(H) \\
&= \left(2 - \frac{1}{\min\{\pi(H), \pi(G)\}}\right)\pi(G)\pi(H).
\end{aligned}$$

□

We now provide a sharp bound for the nth power of a graph.

Corollary 2.5. *For any graph G and positive integer n, we have*

$$\begin{aligned}
\pi(G^n) &\leq \frac{(\pi(G)-1)(\pi(G)+|G|)^n + |G|}{\pi(G)+|G|-1} \\
&= (\pi(G)+|G|)^n - |G|\left(\frac{(\pi(G)+|G|)^n - 1}{\pi(G)+|G|-1}\right).
\end{aligned}$$

Proof. It is clear that the inequality holds for $n = 1$. We show that $\pi(G^n) \leq \pi(G)(\pi(G)+|G|)^{n-1} - |G|\sum_{i=0}^{n-2}(\pi(G)+|G|)^i$ by mathematical induction on n with $n \geq 2$. By Proposition 2.3, we have

$$\begin{aligned}
\pi(G^2) &\leq \pi(G)(\pi(G)+|G|) - |G| \\
&= \pi(G)(\pi(G)+|G|)^{2-1} - |G|\sum_{i=0}^{2-2}(\pi(G)+|G|)^i.
\end{aligned}$$

This means that the inequality holds for $n = 2$. Assume that $\pi(G^k) \leq \pi(G)(\pi(G)+|G|)^{k-1} - |G|\sum_{i=0}^{k-2}(\pi(G)+|G|)^i$ for $k \geq 2$. By Proposition 2.3 and our induction hypothesis, we have

$$\begin{aligned}
\pi(G^{k+1}) = \pi(G\square G^k) &\leq \pi(G^k)(\pi(G)+|G|) - |G| \\
&\leq \left(\pi(G)(\pi(G)+|G|)^{k-1} - |G|\sum_{i=0}^{k-2}(\pi(G)+|G|)^i\right)(\pi(G)+|G|) - |G| \\
&= \pi(G)(\pi(G)+|G|)^{(k+1)-1} - |G|\sum_{i=1}^{(k+1)-2}(\pi(G)+|G|)^i - |G|(\pi(G)+|G|)^0 \\
&= \pi(G)(\pi(G)+|G|)^{(k+1)-1} - |G|\sum_{i=0}^{(k+1)-2}(\pi(G)+|G|)^i.
\end{aligned}$$

Hence, for $n \geq 2$, we get

$$\begin{aligned}\pi(G^n) &\leq \pi(G)(\pi(G)+|G|)^{n-1} - |G|\sum_{i=0}^{n-2}(\pi(G)+|G|)^i \\ &= \pi(G)(\pi(G)+|G|)^{n-1} - |G|\left(\frac{(\pi(G)+|G|)^{n-1}-1}{\pi(G)+|G|-1}\right) \\ &= \frac{(\pi(G)-1)(\pi(G)+|G|)^n+|G|}{\pi(G)+|G|-1}.\end{aligned}$$

□

Note that this bound is best possible, that is, we obtain the equation when $n = 1$.

3. Pebbling on other binary graph constructions

In this supplementary section, we apply the same technique to improve the upper bounds for the pebbling number of other product graphs, i.e., strong product graphs, cross product graphs and coronas which are well-discussed in the work of Asplund, Hurlbert and Kenter [1]. For convenience, we split them into 3 subsections as follows.

3.1. Cross products.

Naturally, we start this subsection with an observation of Asplund, Hurlbert and Kenter [1]. Before that, we remind a basic fact about the connectedness of a cross product graphs, that is, the cross product of nontrivial graphs G and H is connected if and only if G and H are connected and at least one of them is nonbipartite [7].

Lemma 3.1. [1] *For nontrivial graphs G and H with H is nonbipartite, let D be a configuration on $G \times H$. For each $x \in V(G)$, let v_x be an adjacent vertex of x in G. Arbitrarily order the vertices of G, and reset $|D_x|$ to be the number of pebbles on $(\{x, v_x\} \times V(H)) \setminus \bigcup_{y>x}(\{y, v_y\} \times V(H))$, so that each pebble is counted once, and $\sum_{x \in V(G)} |D_x| = |D|$. If*

$$\sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{2(\pi(H'))^2} \right\rfloor \geq \pi(G'),$$

then D is solvable.

Asplund, Hurlbert and Kenter [1] determined a bound for the pebbling number of a cross product of graphs in terms of parameters of their connected spanning bipartite subgraphs.

Theorem 3.2. [1] *For a nontrivial graph G and a nonbipartite graph H, let G' and H' be connected spanning bipartite subgraphs of G and H, respectively. Then we have*

$$\pi(G \times H) \leq 2(\pi(G') + |G|)(\pi(H'))^2.$$

We subtract this bound by the order of the graph G and obtain a new bound which can be seen in the following theorem.

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Theorem 3.3. *For a nontrivial graph G and a nonbipartite graph H, let G' and H' be connected spanning bipartite subgraphs of G and H, respectively. Then we have*

$$\pi(G \times H) \leq 2(\pi(G') + |G|)(\pi(H'))^2 - |G|.$$

Proof. Let D be a configuration of $2(\pi(G') + |G|)(\pi(H'))^2 - |G|$ pebbles on $G \times H$. By Lemma 2.1, we have

$$\begin{aligned} 2(\pi(H'))^2 \sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{2(\pi(H'))^2} \right\rfloor &\geq 2(\pi(G') + |G|)(\pi(H'))^2 - |G| - (2(\pi(H'))^2 - 1)|G| \\ &= 2(\pi(H'))^2 \pi(G'). \end{aligned}$$

This implies that $\sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{2(\pi(H'))^2} \right\rfloor \geq \pi(G')$. By Lemma 3.1, D is solvable. Hence $\pi(G \times H) \leq 2(\pi(G') + |G|)(\pi(H'))^2 - |G|$, as required. □

3.2. **Strong products.** In this subsection, we start with some basic notions about strong product graphs. The strong product of graphs G and H is connected if and only if they are both connected. Furthermore, this product is associative and commutative up to isomorphism. Analogously, for a positive integer n greater than 1 and a graph G, we define the nth power of G with respect to the strong product by $G^1 = G$ and $G^n = G \boxtimes G^{n-1}$. Asplund, Hurlbert and Kenter [1] used the following fact to prove Theorem 3.5, and we also combine it with Lemma 2.1 to prove Theorem 3.6.

Lemma 3.4. [1] *For graphs G and H, let D be a configuration on $G \boxtimes H$. If*

$$\sum_{x \in V(G)} \left\lfloor \frac{|D_x|}{\pi(H) + 1} \right\rfloor \geq \left\lceil \frac{\pi(G)}{2} \right\rceil,$$

then D is solvable.

We apply the commutativity of the strong product of graphs to Proposition 4.3 of Asplund, Hurlbert and Kenter [1] to yield the following result.

Theorem 3.5. [1] *For graphs G and H, we have*

$$\pi(G \boxtimes H) \leq \frac{1}{2}(\pi(G) + 1)(\pi(H) + 1) + \min\{|G|(\pi(H) + 1), |H|(\pi(G) + 1)\}.$$

We improve this result by using of Lemma 2.1 and Lemma 3.4 together with the fact that $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor$ for every positive integer n as follows.

Theorem 3.6. *For graphs G and H, we have*

$$\begin{aligned} \pi(G \boxtimes H) &\leq \min\left\{ \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G), \left(\frac{1}{2}\pi(H) + |H|\right)\pi(G) + \frac{1}{2}\pi(H) \right\} + \frac{1}{2} \\ &= \frac{1}{2}(\pi(G) + 1)(\pi(H) + 1) + \min\left\{ \left(|G| - \frac{1}{2}\right)\pi(H), \left(|H| - \frac{1}{2}\right)\pi(G) \right\}. \end{aligned}$$

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Proof. We show that $\pi(G \boxtimes H) \leq \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G)$. Let D be a configuration of $\left\lceil \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G)\right\rceil$ pebbles on $G \boxtimes H$. By Lemma 2.1, we have

$$\begin{aligned}(\pi(H)+1)\sum_{x\in V(G)}\left\lfloor \frac{|D_x|}{\pi(H)+1}\right\rfloor &\geq \left\lceil \left(\frac{1}{2}\pi(G)+|G|\right)\pi(H)+\frac{1}{2}\pi(G)\right\rceil - ((\pi(H)+1)-1)|G| \\ &\geq \left(\frac{1}{2}\pi(G)+|G|\right)\pi(H)+\frac{1}{2}\pi(G) - ((\pi(H)+1)-1)|G| \\ &= \frac{1}{2}\pi(G)(\pi(H)+1).\end{aligned}$$

This implies that $\sum_{x\in V(G)}\left\lfloor \frac{|D_x|}{\pi(H)+1}\right\rfloor \geq \frac{\pi(G)}{2}$. It follows that $\sum_{x\in V(G)}\left\lfloor \frac{|D_x|}{\pi(H)+1}\right\rfloor \geq \left\lceil \frac{\pi(G)}{2}\right\rceil$. By Lemma 3.4, D is solvable. Therefore, $\pi(G \boxtimes H) \leq \left\lceil \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G)\right\rceil$ $= \left\lfloor \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G) + \frac{1}{2}\right\rfloor \leq \left(\frac{1}{2}\pi(G) + |G|\right)\pi(H) + \frac{1}{2}\pi(G) + \frac{1}{2}$. Similarly, we can show that $\pi(H \boxtimes G) \leq \left(\frac{1}{2}\pi(H) + |H|\right)\pi(G) + \frac{1}{2}\pi(H) + \frac{1}{2}$. Hence the result follows since $\pi(G \boxtimes H) = \pi(H \boxtimes G)$. □

Next, we determine an upper bound for the nth power of a graph. Moreover, we can see that the bound is tight, especially, we get the equation for $n = 1$.

Corollary 3.7. *For a graph G and a positive integer n, we have*

$$\pi(G^n) \leq \frac{\left((\pi(G))^2 + (2|G|-1)\pi(G) + 1\right)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} - 1}{\pi(G) + 2|G| - 2}.$$

Proof. It is not hard to see that the inequality holds for $n = 1$. Next, we show that $\pi(G^n) \leq \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} + \frac{1}{2}(\pi(G)+1)\sum_{i=0}^{n-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i$ by mathematical induction on $n \geq 2$. By Theorem 3.6, we have

$$\begin{aligned}\pi(G^2) &\leq \left(\frac{1}{2}\pi(G)+|G|\right)\pi(G)+\frac{1}{2}\pi(G)+\frac{1}{2} \\ &= \pi(G)\left(\frac{1}{2}\pi(G)+|G|\right)^{2-1}+\frac{1}{2}(\pi(G)+1)\sum_{i=0}^{2-2}\left(\frac{1}{2}\pi(G)+|G|\right)^i.\end{aligned}$$

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This means that the inequality holds for $n = 2$. Assume that $\pi(G^k) \leq \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{k-1} + \frac{1}{2}(\pi(G) + 1)\sum_{i=0}^{k-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i$ for $k \geq 2$. By Theorem 3.6 and our induction hypothesis, we have

$$\begin{aligned}
\pi(G^{k+1}) =& \pi(G \boxtimes G^k) \leq \left(\frac{1}{2}\pi(G) + |G|\right)\pi(G^k) + \frac{1}{2}\pi(G) + \frac{1}{2} \\
&\leq \left(\frac{1}{2}\pi(G) + |G|\right)\left(\pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{k-1} + \frac{1}{2}(\pi(G) + 1)\sum_{i=0}^{k-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i\right) \\
&\quad + \frac{1}{2}\pi(G) + \frac{1}{2} \\
&= \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{(k+1)-1} + \frac{1}{2}(\pi(G) + 1)\sum_{i=1}^{(k+1)-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i \\
&\quad + \frac{1}{2}(\pi(G) + 1) \\
&= \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{(k+1)-1} + \frac{1}{2}(\pi(G) + 1)\sum_{i=0}^{(k+1)-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i.
\end{aligned}$$

By mathematical induction on $n \geq 2$, we get

$$\begin{aligned}
&\pi(G^n) \leq \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} + \frac{1}{2}(\pi(G) + 1)\sum_{i=0}^{n-2}\left(\frac{1}{2}\pi(G) + |G|\right)^i \\
&= \pi(G)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} + \frac{1}{2}(\pi(G) + 1)\left(\frac{\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} - 1}{\frac{1}{2}\pi(G) + |G| - 1}\right) \\
&= \frac{\left(\frac{1}{2}(\pi(G))^2 + |G|\pi(G) - \pi(G)\right)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} + \frac{1}{2}(\pi(G) + 1)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} - 1}{\frac{1}{2}\pi(G) + |G| - 1} \\
&= \frac{\left((\pi(G))^2 + (2|G| - 1)\pi(G) + 1\right)\left(\frac{1}{2}\pi(G) + |G|\right)^{n-1} - 1}{\pi(G) + 2|G| - 2}.
\end{aligned}$$

Hence the result follows. □

3.3. **Coronas.** In this subsection, we note that $G \bowtie H$ is connected if and only if G is connected. At this point, we modify the idea of Asplund, Hurlbert and Kenter [1] to yield a useful observation for coronas.

Lemma 3.8. [1] *For a connected graph G and a graph H, let D be a configuration on $G \bowtie H$ and r be a vertex of $G \bowtie H$. Then D is r-solvable if (i) or (ii) holds where*

(i) $r \in V(G)$ and

$$\sum_{x \in V(G) \times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor \geq \pi(G);$$

(ii) $r \in V(G) \times V(H)$ and

$$\sum_{x \in V(G) \times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor \geq \pi_{(2)}(G).$$

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They leave an upper bound for the pebbling number of coronas as follows.

Proposition 3.9. [1] *For a connected graph G and a graph H, we have*

$$\pi(G \bowtie H) \leq 4\pi(G) + |G||H|.$$

Now, we propose an alternative bound which is related to the 2-pebbling number.

Proposition 3.10. *For a connected graph G and a graph H, we have*

$$\pi(G \bowtie H) \leq 3\pi_{(2)}(G) + |G||H|.$$

Proof. Let r be a vertex of $G \bowtie H$ and D be a configuration of $3\pi_{(2)}(G) + |G||H|$ pebbles on $G \bowtie H$. We denote $\sum_{x\in V(G)} D(x) \geq \pi(G)$ by $|D_G|$.
Case 1: $r \in V(G)$.
If $|D_G| \geq \pi(G)$, then D is obviously r-solvable. So we can assume that $|D_G| \leq \pi(G) - 1$. By Lemma 2.1, we have

$$\begin{aligned} 2\sum_{x\in V(G)\times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor &\geq (|D| - |D_G|) - |V(G)\times V(H)| \\ &\geq (|D| - \pi(G) + 1) - |G||H| \\ &= 3\pi_{(2)}(G) + |G||H| - \pi(G) + 1 - |G||H| \\ &= 3\pi_{(2)}(G) - \pi(G) + 1 \\ &\geq 3\pi(G) - \pi(G) + 1 \\ &= 2\pi(G) + 1. \end{aligned}$$

This implies that $\sum_{x\in V(G)\times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor \geq \pi(G)$. Thus D is r-solvable by Lemma 3.8.
Case 2: $r \in V(G) \times V(H)$.
If $|D_G| \geq \pi_{(2)}(G)$, then D is obviously r-solvable. So we can assume that $|D_G| \leq \pi_{(2)}(G) - 1$. By Lemma 2.1, we have

$$\begin{aligned} 2\sum_{x\in V(G)\times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor &\geq (|D| - |D_G|) - |V(G)\times V(H)| \\ &\geq (|D| - \pi_{(2)}(G) + 1) - |G||H| \\ &= 3\pi_{(2)}(G) + |G||H| - \pi_{(2)}(G) + 1 - |G||H| \\ &= 3\pi_{(2)}(G) - \pi_{(2)}(G) + 1 \\ &= 2\pi_{(2)}(G) + 1. \end{aligned}$$

This implies that $\sum_{x\in V(G)\times V(H)} \left\lfloor \frac{D(x)}{2} \right\rfloor \geq \pi_{(2)}(G)$. Thus D is r-solvable by Lemma 3.8.
From both cases, we therefore conclude that D is solvable and the result follows. □

Since $4\pi(G)$ and $3\pi_{(2)}(G)$, in general, are not comparable, we combine two results above together and summarize them into the following theorem.

Theorem 3.11. *For a connected graph G and a graph H, we have*

$$\pi(G \bowtie H) \leq \min\{3\pi_{(2)}(G), 4\pi(G)\} + |G||H|.$$

Proof. By Proposition 3.9 and 3.10, the result follows. □

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