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INFINITE LOCALLY FINITE SIMPLE GROUPS WITH MANY COMPLEMENTED SUBGROUPS

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Abstract. We prove that the following families of (infinite) groups have complemented subgroup lattice: alternating groups, finitary symmetric groups, Suzuki groups over an infinite locally finite field of characteristic 2, Ree groups over an infinite locally finite field of characteristic 3. We also show that if the Sylow primary subgroups of a locally finite simple group G have complemented subgroup lattice, then this is also the case for G.

1. Introduction

A subgroup H of a group G is said to have a *complement* if there is a subgroup K of G such that $H \cap K = \{1\}$ and $\langle H, K\rangle = G$. We say that the subgroup lattice of a group is *complemented* if each of its subgroups admits a complement. Then we say that a group is a K-*group* if it has a complemented subgroup lattice. There has always been a lot of attention on groups with many complemented subgroups (see for instance [15] or the recent survey [6]) and in 2004, using the classification of finite simple groups, Costantini and Zacher [4] proved that all finite simple groups are K-groups. This result solved a long-standing conjecture and in fact the first steps in this direction were made by Previato [12] in 1982. There, she proved that groups belonging to the following families of (finite) groups have a complemented subgroup lattice: symmetric (alternating) groups $\mathrm{Sym}(n)$ $\big(\mathrm{Alt}(n)\big)$ of degree n, Suzuki groups $\mathrm{Sz}(q)$ over a field of order $q = 2^{2n+1}$ and projective special linear groups $\mathrm{PSL}(n, q)$ of degree n over a finite field of order q. One of the aims of this paper is to show that the same considerations apply to the infinite locally finite analogues of these groups and to Ree groups. We hope this could be a first step towards the proof that all locally finite simple groups are K-groups.

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Finally, we investigate infinite locally finite simple groups with some subgroups which are IK-groups (a very relevant topic in the theory of groups concerns the influence of particular classes of subgroups on the whole group; see for instance [5] and its bibliography). Recall that a group is called an IK-*group* if all its infinite subgroups are complemented; of course, all K-groups and all finite groups are obviously IK-groups, but there are examples of infinite IK-groups which are not K-groups, like for instance the locally dihedral 2-group. It is well known (see [15, example Lemma 3.1.3]) that the class of IK-groups is closed with respect to forming quotients, and it has been proved by Emaldi that abelian IK-groups are either K-groups or Černikov groups (see for instance [6], Theorem 6.5). Our first main result in this context (Theorem 6.2) extends the result of Emaldi to the class of locally nilpotent groups. Finally, we make use of the information we gathered on locally finite simple groups and IK-groups by proving the following theorem.

Theorem 1.1. *Let G be a locally finite simple group. If the Sylow p-subgroups of G are* IK-*groups for all primes p, then G is a* K-*group.*

2. Preliminaries and examples

We refer to [15] as a general reference for the main properties of K-groups, but we just recall a couple of useful facts: (i) if N is any normal subgroup of the K-group G, then also G/N is a K-group (actually, if H is a subgroup of G containing N and admitting a complement, then also H/N admits a complement in G/N); (ii) the direct product of two K-groups is a K-group.

For our purposes we need the following adaptations of [4, Corollary 2.2 and Proposition 2.3] for infinite groups; the proof of these results is omitted since it is essentially identical to those concerning with finite groups.

Lemma 2.1. *Let G be a group and let M be a maximal subgroup of G which is a* K-*group. If $M_G = 1$, then G is a* K-*group.*

Lemma 2.2. *Let G be a simple group. Suppose that $N \leq X \leq M$ are proper subgroups of G such that*

(1) *if $X \leq H < G$, then $H \leq M$,*
(2) *$N \leq Z(M)$,*
(3) *N has prime order, and*
(4) *X/N is a* K-*group.*

Then G is a K-*group.*

It is remarked by Previato in [12, Note 1 at page 1005] that her proof that $\mathrm{PSL}(n, F)$ is a K-group where $n \geq 2$ and F is a finite field can be modified to include the case in which F is a locally finite field. We state this fact as a theorem.

Theorem 2.3. *Let F be a locally finite field and let n be a positive integer. Then* $\mathrm{PSL}(n, F)$ *is a* K-*group.*

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It should be noticed that $\mathrm{SL}(n,F)$ is not in general a K-group. In fact, if $n=2$ and $\mathrm{char}(F)\neq 2$, then either $|F|=3$ and one can easily check that $\mathrm{SL}(2,3)$ is not a K-group, or $|F|>3$ and we know that $\mathrm{SL}(2,F)$ is perfect, so it does not split over its centre; if $n=2$ and $\mathrm{char}(F)=2$, then $\mathrm{SL}(2,F)=\mathrm{PSL}(2,F)$ is a K-group. If $n>2$, then $\mathrm{SL}(n,F)$ is perfect and so it is a K-group if and only if it has a trivial centre.

Furthermore, we observe that there exist simple groups that are not K-groups. Let X be a countable group with no involutions and, using of [11, Theorem 35.1], embed X in a countable simple group G_X such that:

(i) every proper subgroup of G_X is either cyclic of prime order or conjugate to a subgroup of X;
(ii) $G_X=\langle x,y\rangle$ for all $x,y\in G_X$ such that $x\in X$ and $y\notin X$.

Start taking $X=\mathbb{Z}_3$ and $G_0=G_X$. Suppose you have already defined G_λ for all ordinals $\lambda<\alpha<\aleph_1$ and put

$$G_\alpha = G_{G_\lambda}$$

if $\alpha=\lambda+1$, otherwise put

$$G_\alpha=\bigcup_{\lambda<\alpha} G_\lambda.$$

Finally, let $G=G_{\aleph_1}$ be the union of the G_λ's with $\lambda<\aleph_1$.

It is not difficult to see that any proper subgroup of G_λ, for some non-zero limit ordinal λ, is contained in a G_κ for some countable successor ordinal $\kappa<\lambda$ (see [11, proof of Theorem 35.2]). Actually, this fact, in combination with properties (i) and (ii), shows that if λ is any ordinal number $\leq\aleph_1$, then G_λ is a K-group if and only if λ is successor. In particular, $G_{\aleph_0}$ is an example of a (countable) periodic simple group that is not a K-group but whose proper subgroups are K-groups. Of course, any group whose proper subgroups are K-group must be periodic and in the final section of the paper we will see that the above-described situation *is not possible for locally finite simple groups.*

On the other hand, $G_{\aleph_1}$ is an uncountable simple group that is not a K-group, but it is the union of a chain of (countable) subgroups that are K-groups (we remark that all proper subgroups of $G_{\aleph_1}$ are countable). Notice also that if we change the subgroup X we start with, we can modify some features of the above examples. In fact, we could use a non-periodic subgroup to start with and we can arrange in such a way that $G_{\aleph_1}$ is a p-group for some prime p (see [11, actual statement of Theorem 35.1]).

Finally, we notice that $\mathrm{PSL}(2,\mathbb{Z})$ is not a K-group. Indeed, it is known that it is isomorphic to the free product $\mathbb{Z}_2*\mathbb{Z}_3$ (see [1]) and hence (by Kurosh subgroup theorem) it does not split over its commutator subgroup since it does not contain any cyclic subgroup of order 6. It follows that even $\mathrm{SL}(2,\mathbb{Z})$ is a K-group.

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3. Finitary symmetric groups

Let $\mathtt{X}$ be any set. This section deals with the finitary symmetric group $\mathrm{FSym}(\mathtt{X})$ and the alternating group $\mathrm{Alt}(\mathtt{X})$. We refer of [14, Chapter 7] as a general reference for definitions and statements concerning multiple transitivity; we only point out that in the following $\mathtt{X}^{[k]}$ denotes the set of all ordered k-tuples $[\mathtt{x}_1, \mathtt{x}_2, \dots, \mathtt{x}_k]$ consisting of distinct elements $\mathtt{x}_i$ of $\mathtt{X}$.

If $\mathtt{X}$ is finite, [15, Theorem 3.1.2] yields that $\mathrm{FSym}(\mathtt{X}) = \mathrm{Sym}(\mathtt{X})$ and $\mathrm{Alt}(\mathtt{X})$ are K-groups. We will not employ these results in the proofs of the main theorems of this section and in fact our theorems provide a more constructive approach for finding complements for a subgroup either of $\mathrm{Sym}(\mathtt{X})$ or of $\mathrm{Alt}(\mathtt{X})$. Thus, we present two distinct and independent proofs for $\mathrm{FSym}(\mathtt{X})$ and $\mathrm{Alt}(\mathtt{X})$, but it is clear that Theorem 3.1 may be deduced from Theorem 3.3 using Lemma 2.1.

Theorem 3.1. *Let $\mathtt{X}$ be any set. Then $G = \mathrm{FSym}(\mathtt{X})$ is a* K-*group.*

Proof. Let H be a non-trivial proper subgroup of G. Suppose first that H is not transitive on $\mathtt{X}$ and let $\mathtt{Y}$ be a set of representatives for the orbits of H in $\mathtt{X}$; in particular, $|\mathtt{Y}| > 1$. Since H is non-trivial, there is $\overline{\mathtt{y}} \in \mathtt{Y}$ such that $|\mathtt{Z}| > 1$, where $\mathtt{Z} = H \cdot \overline{\mathtt{y}}$. Let $C = C_H(\mathtt{X} \setminus \mathtt{Z})$ and let $\mathtt{W}$ be a set of representatives for the orbits of C in $\mathtt{Z}$ such that $\overline{\mathtt{y}} \in \mathtt{W}$. Now, let K be the subgroup of G generated by

$$\big\{(\mathtt{y}', \mathtt{y}''),\, (\mathtt{w}', \mathtt{w}'') \,:\, \mathtt{y}', \mathtt{y}'' \in \mathtt{Y},\, \mathtt{w}', \mathtt{w}'' \in \mathtt{W}\big\}.$$

It is easy to see that $H \cap K = 1$. Let $L = \langle H, K\rangle$ and choose $\overline{\mathtt{y}}' \in \mathtt{Y} \setminus \{\overline{\mathtt{y}}\}$; in particular, the transposition $(\overline{\mathtt{y}}, \overline{\mathtt{y}}')$ lies in L. Take $\mathtt{z} \in \mathtt{Z}$ and write $\{\mathtt{w}\} = \mathtt{W} \cap \big(C \cdot \mathtt{z}\big)$. Since L contains $(\mathtt{w}, \overline{\mathtt{y}})$, it also contains $(\mathtt{w}, \overline{\mathtt{y}}')$. Since $C \leq L$, then $(\mathtt{z}, \overline{\mathtt{y}}')$ lies in L for all $\mathtt{z} \in \mathtt{Z}$. Thus, if $\mathtt{z}' \neq \mathtt{z}'' \in Z$, then $(\mathtt{z}', \overline{\mathtt{y}}')$ and $(\mathtt{z}'', \overline{\mathtt{y}}')$ are in L, and so also $(\mathtt{z}', \mathtt{z}'')$ lies in L. It follows that L contains $C_G(\mathtt{X} \setminus \mathtt{Z})$. Thus, since $(\overline{\mathtt{y}}, \overline{\mathtt{y}}')$ lies in L, we also have $(\mathtt{z}, \mathtt{u}) \in L$ for all $\mathtt{z} \in \mathtt{Z}$ and $\mathtt{u} \in H \cdot \overline{\mathtt{y}}'$. From here it is almost immediate to see that L contains all transpositions of G and so $L = G$.

Now suppose that H is transitive but not 2-transitive. Let $\mathtt{x} \in \mathtt{X}$ and let $\mathtt{Y}$ be a set of representatives for the orbits of H in $\mathtt{X}^{[2]}$ consisting of pairs whose first component is $\mathtt{x}$ (this is certainly possible due to the transitivity of H). Let $\mathtt{Z}$ be the subset of $\mathtt{X}$ consisting of all elements $\mathtt{z}$ such that $[\mathtt{x}, \mathtt{z}]$ belongs to $\mathtt{Y}$ and put $K = C_G(\mathtt{X} \setminus \mathtt{Z}) \simeq \mathrm{FSym}(\mathtt{Z})$. It is clear that $H \cap K = 1$. Let $L = \langle H, K\rangle$. Since K contains a transposition and L is 2-transitive, it follows that $L = G$.

Finally, suppose that H is 2-transitive and let $(\mathtt{a}, \mathtt{b})$ be any transposition in $G \setminus H$. Then

$$\langle H, (\mathtt{a}, \mathtt{b})\rangle = G$$

and $H \cap \langle (\mathtt{a}, \mathtt{b})\rangle = 1$. □

The following result is well known, and its proof can for instance be found in [7].

Lemma 3.2. *Let $\mathtt{X}$ be any set and let G be any subgroup of $\mathrm{Alt}(\mathtt{X})$ generated by 3-cycles. Then there is a family $\{\mathtt{X}_i\}_{i \in I}$ of disjoint subsets of $\mathtt{X}$ such that $G = \mathrm{Dr}_{i \in I} \mathrm{Alt}(\mathtt{X}_i)$.*

Theorem 3.3. *Let $\mathtt{X}$ be any set. Then $G = \mathrm{Alt}(\mathtt{X})$ is a* K-*group.*

Proof. The result is obvious when $|\mathtt{X}| \leq 3$, so we may assume $|\mathtt{X}| > 3$. Let H be a non-trivial proper subgroup of G.

Suppose first that H *is* transitive and let $\mathtt{x}, \mathtt{y}, \mathtt{z}$ be distinct elements of $\mathtt{X}$ such that the 3-cycle $(\mathtt{x}, \mathtt{y}, \mathtt{z})$ does not belong to H. Put $K_1 = \langle(\mathtt{x}, \mathtt{y}, \mathtt{z})\rangle$, so, clearly, $H \cap K_1 = \{1\}$. If $\langle H, K_1\rangle$ is a proper subgroup of G, then K_1^H is also a proper subgroup of G. Now, we may apply Lemma 3.2 and obtain a family $\{\mathtt{X}_i\}_{i\in I}$ of disjoint subsets of $\mathtt{X}$ such that:

(1) $\mathtt{X}$ is the union of the members of the family $\{\mathtt{X}_i\}_{i\in I}$,
(2) $|\mathtt{X}_i| \geq 3$ for all $i \in I$, $|I| \geq 2$, and
(3) $K_1^H = \mathrm{Dr}_{i\in I}\,\mathrm{Alt}(\mathtt{X}_i)$.

Let j be the only element of I such that $(\mathtt{x}, \mathtt{y}, \mathtt{z}) \in \mathrm{Alt}(\mathtt{X}_j)$. For each $i \in I \setminus \{j\}$ choose $\mathtt{x}_i \in \mathtt{X}_i$ and let $\mathtt{Y}$ be the set of all these chosen elements. Define

$$K = \langle(\mathtt{a}, \mathtt{b}, \mathtt{c}) \,:\, \mathtt{a}, \mathtt{b}, \mathtt{c} \in \mathtt{Y} \cup \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}\rangle.$$

It easily follows from Lemma 3.2 that $\langle H, K\rangle = G$ and $H \cap K = \{1\}$. Thus K is the required complement.

Suppose now that H is *not* transitive and let $\mathcal{X}$ be the set of all orbits of H in $\mathtt{X}$. Fix $\mathtt{Y} \in \mathcal{X}$ such that $|\mathtt{Y}| \geq 2$ and let $\mathtt{Z}$ be a set of representatives for the family $\mathcal{X} \setminus \{\mathtt{Y}\}$. If $\mathtt{Y} = \{\mathtt{y}, \mathtt{y}'\}$, define

$$K = \langle(\mathtt{y}, \mathtt{y}', \mathtt{z}) \,:\, \mathtt{z} \in \mathtt{Z}\rangle$$

and notice that $H \cap K = \{1\}$, while $\langle H, K\rangle \geq K^H = G$ by Lemma 3.2 and (3). Thus, we may assume that all orbits of H on $\mathtt{X}$ have either 1 or ≥ 3 elements. Let $\mathtt{W}$ be a set of representatives for the orbits of $C = C_H(\mathtt{X} \setminus \mathtt{Y})$ in $\mathtt{Y}$.

If $|\mathtt{W}| = 1$, choose two distinct elements $\mathtt{y}, \mathtt{y}'$ in $\mathtt{Y}$ and put

$$K = \langle(\mathtt{y}, \mathtt{y}', \mathtt{z}) \,:\, \mathtt{z} \in \mathtt{Z}\rangle.$$

It is clear that $H \cap K = \{1\}$. Moreover, for each $\mathtt{z} \in \mathtt{Z}$, it follows from Lemma 3.2 that $\langle H, K\rangle$ contains $\mathrm{Alt}(\mathtt{Y} \cup \{\mathtt{z}\})$. Therefore $\langle H, K\rangle$ contains $\mathrm{Alt}(\mathtt{Y} \cup \mathtt{U})$ for each $\mathtt{U} \in \mathcal{X}$ and hence $\langle H, K\rangle = G$.

Assume that $|\mathtt{W}| \geq 2$ and put

$$K = \langle(\mathtt{a}, \mathtt{b}, \mathtt{c}) \,:\, \mathtt{a}, \mathtt{b}, \mathtt{c} \in \mathtt{W} \cup \mathtt{Z}\rangle.$$

As in the above paragraph, we can prove that $H \cap K = \{1\}$ and $\langle H, K\rangle = G$. □

We observe that the situation is dramatically different if we consider full symmetric groups. In fact, let $\mathtt{X}$ be an infinite set and let $G = \mathrm{Sym}(\mathtt{X})$ be the full symmetric group on $\mathtt{X}$. Then G is not a K-group. In fact, suppose by contradiction that G is a K-group and let K be a complement of $S = \mathrm{FSym}(\mathtt{X})$ in G. Clearly, $S = \langle(\mathtt{x}, \mathtt{y})\rangle \ltimes A$, where $A = \mathrm{Alt}(\mathtt{X})$ and $\mathtt{x}, \mathtt{y}$ are distinct elements of $\mathtt{X}$, so $G = \langle(\mathtt{x}, \mathtt{y})\rangle \ltimes (KA)$ since A is a characteristic subgroup of S and this is a normal subgroup of G. It follows that KA is a subgroup of G of index 2, so it is normal and this contradicts [2].

4. Suzuki groups

Let F be an infinite locally finite field of characteristic 2 admitting an automorphism θ such that $\theta^2 = 2$ (in the following, $F^\times$ denotes the multiplicative group of F). As in the finite case, the group $\mathrm{Sz}(F)$ can be seen as the subgroup of $\mathrm{SL}(4,F)$ generated by the permutation matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

the diagonal matrices

$$\overline{f} = \begin{pmatrix} f^{1+\theta^{-1}} & 0 & 0 & 0 \\ 0 & f^{\theta^{-1}} & 0 & 0 \\ 0 & 0 & f^{-\theta^{-1}} & 0 \\ 0 & 0 & 0 & f^{-1-\theta^{-1}} \end{pmatrix} \quad (f \in F^\times)$$

and the matrices

$$|a,b| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1+\theta} + b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & b & a & 1 \end{pmatrix} \quad (a, b \in F).$$

Since F is countable, it is the union of a chain of finite subfields

$$2^3 \leq |F_1|, \quad F_1 \leq F_2 \leq \cdots \leq F_n \leq F_{n+1} \leq \cdots$$

and so $G = \mathrm{Sz}(F)$ is the union of the chain of finite subgroups

$$G_1 = \mathrm{Sz}(F_1) \leq G_2 = \mathrm{Sz}(F_2) \leq \cdots \leq G_n = \mathrm{Sz}(F_n) \leq G_{n+1} = \mathrm{Sz}(F_{n+1}) \leq \cdots$$

Let $S = \langle |a,b| \,:\, a,b \in F\rangle$ and $D = \langle \overline{f} \,:\, f \in F^\times\rangle$. It is easy to see that $S_i = S \cap G_i$ is a Sylow 2-subgroup of G_i and $SD \cap G_i$ is the normalizer of S_i. Moreover, it follows from the well known structure of finite Suzuki groups (see [16]) that $S' = \Omega_1(S) \leq Z(S)$, S' and S/S' are (countably) infinite elementary abelian 2-groups, and every proper subgroup of G containing S' must be contained in SD. Since S' is a K-group, it follows from Lemma 2.2 that G itself is a K-group. We have proved the following result.

Theorem 4.1. *$\mathrm{Sz}(F)$ is a* K*-group for every (infinite) locally finite field F of characteristic* 2 *admitting an automorphism θ such that $\theta^2 = 2$.*

Recall that if G is a group, the symbol $\pi(G)$ denotes the set of all prime numbers p such that G admits elements of order p.

Corollary 4.2. *Let G be an infinite, locally finite simple group which is linear. If $3 \notin \pi(G)$, then G is a* K*-group.*

Proof. It follows from [9, Theorem 1.L.2] that G is countable and hence it is the union of a chain of finite simple subgroups by [9, Lemma 4.5 and Proposition 4.6]. However, it is well known that Suzuki groups are the only non-abelian finite simple groups whose order is not divided by 3. Thus G is the union of a chain of finite simple subgroups each of which is isomorphic to some Suzuki group. Now, Theorem A of [8] shows that $G \simeq \mathrm{Sz}(F)$ for some infinite locally finite field F and hence Theorem 4.1 yields that G is a K-group. □

5. Ree groups

Let F be an infinite locally finite field of characteristic 3 admitting an automorphism θ such that $3\theta^2 = 1$. As in the finite case (see [10]), the group $G = \mathrm{Ree}(F)$ is a subgroup of $\mathrm{SL}(7, F)$ generated by the following elements:

$$\alpha(t) = \begin{pmatrix} 1 & -t^\theta & -t^{\theta+1} & 0 & t^{3\theta+1} & 0 & t^{4\theta+2} \\ 0 & 1 & -t & -t^{\theta+1} & -t^{2\theta+1} & t^{2\theta+2} & t^{3\theta+2} \\ 0 & 0 & 1 & -t^\theta & -t^{2\theta} & t^{2\theta+1} & -t^{3\theta+1} \\ 0 & 0 & 0 & 1 & -t^{-\theta} & t^{\theta+1} & 0 \\ 0 & 0 & 0 & 0 & 1 & t & -t^{\theta+1} \\ 0 & 0 & 0 & 0 & 0 & 1 & t^\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (t \in F),$$

$$\beta(t) = \begin{pmatrix} 1 & 0 & t^\theta & 0 & t & 0 & -t^{\theta+1} \\ 0 & 1 & 0 & -t^\theta & 0 & -t^{2\theta} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 1 & 0 & -t^\theta & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -t^\theta \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (t \in F), \quad \gamma(t) = \begin{pmatrix} 1 & 0 & 0 & -t^\theta & 0 & -t & -t^{2\theta} \\ 0 & 1 & 0 & 0 & -t^{-\theta} & 0 & t \\ 0 & 0 & 1 & 0 & 0 & t^\theta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -t^\theta \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (t \in F),$$

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h(t) = \begin{pmatrix} t^\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{1-\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{2\theta-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{1-2\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^{\theta-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{-\theta} \end{pmatrix} \quad (t \in F^\times),$$

Since F is countable, it is the union of a chain of finite subfields

$$3^3 \leq |F_1|, \quad F_1 \leq F_2 \leq \cdots \leq F_n \leq F_{n+1} \leq \cdots$$

and so G is the union of the chain of finite subgroups

$$G_1 = \mathrm{Ree}(F_1) \leq G_2 = \mathrm{Ree}(F_2) \leq \cdots \leq G_n = \mathrm{Ree}(F_n) \leq G_{n+1} = \mathrm{Ree}(F_{n+1}) \leq \ldots$$

Let $x = \mathrm{diag}(-1, 1, -1, 1, -1, 1, -1) \in G$ and let $X = C_G(x)$. It easily follows from results in [10] that $X = \langle x \rangle \times U$, where $U = \langle \beta(t), \beta(t)^\tau \, : \, t \in F \rangle$ is such that $U_i = U \cap G_i \simeq \mathrm{PSL}(2, F_i)$ for all i. A well known theorem of Hartley and Shute [8] implies that $U \simeq \mathrm{PSL}(2, F)$. Since the direct product of two K-groups is a K-group, it follows from Theorem 2.3 that X is a K-group. Moreover, $\langle x\rangle \times U_i$ is maximal in G_i for all i (see [10, Theorem 1]) and hence X is a maximal subgroup of G. Finally, Lemma 2.1 yields that G itself is a K-group.

Theorem 5.1. *Let F be an infinite locally finite field of characteristic 3 admitting an automorphism θ such that $3\theta^2 = 1$. Then ${}^2\,\mathrm{G}_2(F) \equiv \mathrm{Ree}(F)$ is a K-group.*

6. Locally finite simple groups with many proper subgroups that are K-groups

Lemma 6.1. *Let G be an infinite, locally finite simple group whose Sylow p-subgroups are either Černikov or abelian. Then G isomorphic to $\mathrm{PSL}(2,F)$ for some infinite locally finite field F.*

Proof. Let H be a countably infinite simple subgroup of G and let Q be a Sylow 2-subgroup of H. If Q is abelian, then H is isomorphic either with $\mathrm{PSL}(2,F)$ or with ${}^2\,\mathrm{G}_2(F) \equiv \mathrm{Ree}(F)$ for a suitable infinite locally finite field F (see [9, Theorem 4.19, Theorem 4.23 and the discussion at page 141]). However, the latter case is impossible. In fact, in that case, the Sylow 3-subgroups of H are non-abelian (see for instance [17]) and H certainly contains an infinite Sylow 3-subgroup L. Thus L is Černikov, but, on the other hand, H is linear over a field of characteristic 3 and so it must have finite exponent, which means that L is finite, a contradiction. Therefore H is isomorphic with $\mathrm{PSL}(2,F_H)$ for a suitable infinite locally finite field F_H. The arbitrariness of H yields that G is isomorphic with $\mathrm{PSL}(2,F)$ for some infinite locally finite field F (see [9, Theorems 4.4 and 4.18]).

Suppose now that Q is finite, so G is a group of Lie type over an infinite locally finite field F of characteristic p by [8, Theorem B]. Let P be a Sylow p-subgroup of G. It follows from [18, Corollary 9.7] that P is infinite. Moreover, P is of finite exponent (being G linear) and so it is even abelian. Using the argument employed in the first part of the proof, we can assume $p > 2$.

First, we analyze untwisted Chevalley groups. If G is of type G_2 and $p = 3$, it follows from [8, Lemma 4.5] that P has nilpotency class 3 and this is a contradiction. Assume therefore that G is one of the remaining types A_n $(n > 1)$, B_n $(n > 1)$, C_n $(n > 2)$, D_n $(n > 3)$, E_6, E_7, E_8, F_4, or G_2 $(p > 3)$. Now, it follows from [8, Lemma 4.2] that the nilpotency class of P is the height $h(R)$ of the unique highest root R as described in [3, Theorem 5.3.3]. By [3, Proposition 10.2.5] (and the remark below it) this class is always ≥ 2, which is not possible.

Suppose G is a twisted Chevalley group. [8, Lemma 4.8] shows that G is not of type ${}^3\,\mathrm{D}_4$. Moreover, since $p > 2$, we have that G is neither of type ${}^2\,\mathrm{F}_4$ nor of type ${}^2\,\mathrm{B}_2$. Furthermore, if G is of type ${}^2\,\mathrm{A}_n$ $(n > 1)$, ${}^2\,\mathrm{D}_n$ $(n > 3)$ or ${}^2\,\mathrm{E}_6$, it follows from [8, Lemma 4.7] that the nilpotency class of a maximal p-subgroup of G is the same as in the corresponding untwisted case. Thus, another application of [3, Proposition 10.2.5] (and the remark below it) completes the proof of the statement. □

Theorem 6.2. *Let G be a locally nilpotent group whose infinite subgroups are complemented. Then G is either abelian or Černikov.*

Proof. If N is any infinite normal subgroup of G, then G/N is a K-group and so it follows from [15, Lemma 3.1.1] that G/N is abelian (recall that maximal subgroups of a locally nilpotent group are

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normal); in particular, $G' \leq N$. Let L be the intersection of all infinite normal subgroups of G, so $G' \leq L$.

If $G = N_1 \times N_2$, where N_1 is infinite, then $G' \leq L \leq N_1$ and hence N_2 is abelian. If also N_2 were infinite, then $G' = \{1\}$, so we may assume that N_2 is finite. This shows that G cannot be the direct product of infinitely many non-trivial groups and so that we may assume that G is directly indecomposable. In particular, if G is periodic, it is even a primary group.

Suppose now G' is finite (and non-trivial) and G is infinite. Then G is an FC-group and by our work above $G' \leq C_G(x^G)$ for all $x \in G$. Thus $G' \leq Z(G)$, so G is nilpotent. Moreover, of [6, Theorem 6.5] yields that G is periodic (= p-group for some prime p) and that G/G' is either Černikov or an (infinite) elementary abelian group. In the former case, G is Černikov, so we may assume the latter case holds. Let M_1 be a maximal subgroup of G'. Then M_1 is normal in G and G'/M_1 has order p. Choose a normal subgroup M/M_1 of G/M_1 that is maximal with respect to $M/M_1 \cap G'/M_1 = \{1\}$. Write $\overline{G} = G/M$; of course, $\overline{G}'$ has order p and lies in the centre of $\overline{G}$. Let $\overline{X}$ be any infinite subgroup of $\overline{G}$ such that $|\overline{G} : \overline{X}| = \infty$ and $\overline{X} \geq \overline{G}'$. Then there is a complement $\overline{K}$ of $\overline{X}$ in $\overline{G}$; clearly, $\overline{K}$ is an infinite elementary abelian p-subgroup. Write $\overline{K} = \langle \overline{k} \rangle \times \overline{Y}$ for some non-trivial element $\overline{k}$ of $\overline{K}$ and some (infinite) subgroup $\overline{Y}$. Then $\overline{U} = \overline{G}' \times \overline{Y}$ admits a complement $\overline{H}$ in $\overline{G}$ and $\overline{Z} = \overline{H} \cap (\overline{K} \times \overline{G}')$ is a $\overline{G}$-invariant complement of $\overline{U}$ in $\overline{K} \times \overline{G}'$ (in fact, it is normalized by $\overline{H}$ and centralized by $\overline{U}$). This contradicts the choice of M.

Assume thus that G' is infinite, so that $G' = L$ and hence all proper G-invariant subgroups of G' are finite. The consideration of a maximal normal series of G from $\{1\}$ to G' shows that $G' \leq Z_\omega(G)$, so G is hypercentral. Moreover, $Z(G)$ has no aperiodic element, otherwise G' would be infinite cyclic and $(G')^2$ would be an infinite proper G-invariant subgroup of G'. This argument actually shows that $G' \cap Z_n(G)$ is finite for any non-negative integer n, so G' is periodic and hence G is such since G/G' is an abelian K-group. It follows from [13, Theorem 10.23] that G' is Černikov and hence even divisible. Thus $|G : C_G(G')|$ is finite (see [13], Corollary to Lemma 3.28) and $C = C_G(G')$ is nilpotent of class ≤ 2. Actually, C' is finite. In fact, C/C' is a direct product of locally cyclic primary groups and hence C' is finite since it is a homomorphic image of $C/C' \otimes C/C'$, which is elementary abelian. Now, if C is infinite, then it contains an infinite subgroup of finite exponent which is G-invariant and hence $L = G'$ has finite exponent, a contradiction. Thus C is finite and so such is G. □

Theorem 1.1 is now an obvious combination of the main result of [4], Theorem 2.3, Lemma 6.1 and Theorem 6.2.

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References

[1] R. C. Alperin, $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, *Amer. Math. Monthly*, **100** (1993) 385–386.

[2] R. Baer, Die Kompositionsreihe der Gruppe aller eineindeutigen Abbildungen einer unendlichen Menge auf sich, *Stud. Math.*, **5** (1934) 15–17.

[3] R. W. Carter, *Simple groups of Lie type*, Wiley, New York, 1972.

[4] M. Costantini and G. Zacher, The finite simple groups have complemented subgroup lattices, *Pacific J. Math.*, **213** (2004) 245–251.

[5] M. Ferrara and M. Trombetti, A local study of group classes, *Note Mat.*, **42** (2020) 1–20.

[6] M. Ferrara and M. Trombetti, Infinite groups with many complemented subgroups, *Beitr. Alg. Geom*, to appear; doi:10.1007/s13366-021-00597-w.

[7] J. Hall, Infinite alternating groups as finitary linear transformation groups, *J. Algebra*, **119** (1988) 337–359.

[8] B. Hartley and G. Shute, Monomorphisms and direct limits of finite groups of Lie type, *Quart. J. Math. Oxford (2)*, **35** (1984) 49–71.

[9] O. H. Kegel and B. A. F. Wehrfritz, *Locally finite groups*, North-Holland, London, 1973.

[10] V. M. Levchuk and Ya. N. Nuzhin, Structure of Ree groups, *Algebra and Logic*, **24** (1985) 16–26.

[11] A. Yu. Ol'shanskij, *Geometry of defining relations in groups*, Kluwer Academy Publishers, Dordrecht, 1991.

[12] E. Previato, Some families of simple groups whose lattices are complemented, *Boll. U. M. I.*, **6** (1982) 1003–1014.

[13] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin, 1972.

[14] D. J. S. Robinson, *A course in the theory of groups*, Springer, Berlin, 1995.

[15] R. Schmidt, *Subgroup lattices of groups*, De Gruyter, Berlin, 1994.

[16] M. Suzuki, On a class of doubly transitive groups, *Ann. Math.*, **75** (1962) 105–145.

[17] H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.*, **121** (1966) 62–89.

[18] B. A. F. Wehrfritz, *Infinite linear groups*, Springer, Berlin, 1973.

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