International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 11 No. 3 (2022), pp. 175-185.

http://ijgt.ui.ac.ir
www.ui.ac.ir

SYMMETRIC DESIGNS AND PROJECTIVE SPECIAL UNITARY GROUPS $\mathrm{PSU}_5(q)$

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ABSTRACT. In this article, we prove that if a nontrivial symmetric (v, k, λ) design admit a flag-transitive and point-primitive automorphism group G, then the socle X of G cannot be a projective special unitary group of dimension five. As a corollary, we list all exist nineteen non-isomorphism such designs in which $\lambda \in \{1, 2, 3, 4, 6, 12, 16, 18\}$ and $X = \mathrm{PSU}_n(q)$ with $(n, q) \in \{(2, 7), (2, 9), (2, 11), (3, 3), (4, 2)\}$.

1. Introduction

A *symmetric* (v, k, λ) *design* is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ consisting of a set $\mathcal{P}$ of v *points* and a set $\mathcal{B}$ of v *blocks* such that every point is incident with exactly k blocks, and every pair of blocks is incident with exactly λ points. A *nontrivial* symmetric design is one in which $2 < k < v - 1$. A *flag* of $\mathcal{D}$ is an incident pair (α, B) where α and B are a point and a block of $\mathcal{D}$, respectively. An *automorphism* of a symmetric design $\mathcal{D}$ is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of $\mathcal{D}$ is called *flag-transitive* if it is transitive on the set of flags of $\mathcal{D}$. If G is primitive on the point set $\mathcal{P}$, then G is said to be *point-primitive*. We here adopt the standard notation for finite simple groups of Lie type, for example, we use $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{P\Omega}_{2n+1}(q)$ and $\mathrm{P\Omega}^{\pm}_{2n}(q)$ to denote the finite classical simple groups. Symmetric and alternating groups on n letters are denoted by S_n and A_n, respectively. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a nonabelian simple group. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [5, 8, 11, 13].

The main aim of this paper is to study flag-transitive symmetric designs. In [18], Praeger and Zhou study point-imprimitive symmetric (v, k, λ) designs and give a classification of such designs in terms of

Communicated by Jamshid Moori.

MSC(2010): Primary: 05B05; Secondary: 05B25, 20B25.
Keywords: Symmetric design, flag-transitive, point-primitive, automorphism group.
Received: 18 Jun 2020, Accepted: 30 August 2021.
Article Type: Research Paper.
http://dx.doi.org/10.22108/IJGT.2021.123581.1630 .

their parameters. In the case where, a symmetric design admits a flag-transitive and point-primitive automorphism group, for $\lambda \leqslant 100$, the only type of primitive groups might occur is either almost simple, or affine [15, 21]. Although, it is still unknown for larger λ such an automorphism group is of these two types, it is somehow interesting to study such designs whose automorphism group G is an almost simple group with socle X. This paper is part of contribution to classification of symmetric designs admitting flag-transitive and point-primitive finite almost simple automorphism groups of Lie type of small dimension, see [1, 3, 2, 4, 9]. In this paper, we continue this project and study nontrivial symmetric designs admitting flag-transitive and point-primitive automorphism groups whose socle is a projective special unitary group of dimension five.

Theorem 1.1. *Let $\mathcal{D}$ be a nontrivial symmetric (v,k,λ) design. If $G \leqslant \mathrm{Aut}(\mathcal{D})$ is flag-transitive and point-primitive, then the socle of X cannot be $\mathrm{PSU}_5(q)$.*

As a corollary, we obtain all nontrivial symmetric designs admitting an automorphism group whose socle is a projective special unitary group of dimension at most five.

Corollary 1.2. *Let $\mathcal{D}$ be a nontrivial symmetric (v,k,λ) design and let α be a point of $\mathcal{D}$. If G is a flag-transitive and point-primitive almost simple automorphism group of $\mathcal{D}$ with socle $X = \mathrm{PSU}_n(q)$ a projective special unitary group of dimension at most five, then $\lambda \in \{1, 2, 3, 4, 6, 12, 16, 18\}$ and v, k, λ, X, G_α and G are as in one of the lines in* Table 1.

The detailed information about the designs obtained in Corollary 1.2, that is to say, those appear in Table 1, can be found in [2, 4, 6, 10, 15, 17].

1.1. **Outline of the proof.** We first note that Theorem 1.1 for the case where X is a projective special unitary group of dimension 2, 3 and 4 follows immediately form the main results in [2, 4, 9] with making some comments and remarks in the introduction of Section 3. Therefore, in Section 3, we only need to prove Theorem 1.1 for $X = \mathrm{PSU}_5(q)$ with $q = p^a$, where p is a prime number. Since the group G is point-primitive, it follows that the point-stabiliser $H := G_\alpha$ is maximal in G, where α is a point of $\mathcal{D}$, and so we study the coset actions of G on the set of right cosets of the maximal subgroups H of G. We then continue our argument using case by case analysis. For each maximal subgroup H recorded in Lemma 2.7, we can now find the parameter v by (3.1) deduced from Lemma 2.1. We next apply Lemma 2.6 and detailed information of local (coset) actions of H including subdegrees of G to find polynomial $f(q)$ such that the parameter k divides $c\lambda f(q)$, where $q = p^a$ and c is a divisor of $|\mathrm{Out}(X)|$. Then $mk = c\lambda f(q)$, for some positive integer $m < cf(q)$. Again, by Lemma 2.6(a) and the fact that $mk = c\lambda f(q)$, we find parameters k and λ in terms of m, c and q. If $\lambda v < k^2$ holds for almost all q, then we use Euclidian algorithm and obtain polynomials $h(q)$ and $r(q)$ such that $|H \cap X| = h(q) \cdot (v-1) + r(q)$, and then conclude that the parameter k divides a polynomial $F(a,m,q)$ in terms of m, a, $h(q)$ and $r(q)$, where $q = p^a$. In most cases, the inequality $m \cdot (v-1) < F(a,m,q)$ holds, where the degree of $F(a,m,q)$ in terms of q is less than the degree of v. This inequality restricts the possibilities for q. We now investigate possible parameters obtained in this way to see if any possible design arises. In this paper, we use the software GAP [12] for computational arguments. To

Table 1. Parameters in Corollary 1.2

Line	v	k	λ	X	G_α	G	Designs	References*
1	7	3	1	$\mathrm{PSU}_2(7)$	S_4	$\mathrm{PSU}_2(7)$	PG(2,2)	[2, 15]
2	7	4	2	$\mathrm{PSU}_2(7)$	S_4	$\mathrm{PSU}_2(7)$	Complement of line 1	[2, 15]
3	11	5	2	$\mathrm{PSU}_2(11)$	A_5	$\mathrm{PSU}_2(11)$	Paley	[2, 15]
4	11	6	3	$\mathrm{PSU}_2(11)$	A_5	$\mathrm{PSU}_2(11)$	Complement of line 3	[2, 15]
5	15	8	4	$\mathrm{PSU}_2(9)$	S_4	$\mathrm{PSU}_2(9)$	PG(3,2)	[2, 10]
6	36	21	12	$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_2(7)$	$\mathrm{PSU}_3(3)$	Menon	[6]
7	36	21	12	$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_2(7){:}2$	$\mathrm{PSU}_3(3){:}2$	Menon	[6]
8	36	15	6	$\mathrm{PSU}_4(2)$	S_6	$\mathrm{PSU}_4(2)$	Menon	[4]
9	36	15	6	$\mathrm{PSU}_4(2)$	$\mathrm{S}_6{:}2$	$\mathrm{PSU}_4(2){:}2$	Menon	[4]
10	40	27	18	$\mathrm{PSU}_4(2)$	$3_+^{1+2}{:}2\mathrm{A}_4$	$\mathrm{PSU}_4(2)$	Complement of $\mathrm{PG}_3(3)$	[4]
11	40	27	18	$\mathrm{PSU}_4(2)$	$3_+^{1+2}{:}2\mathrm{A}_4{:}2$	$\mathrm{PSU}_4(2){:}2$	Complement of $\mathrm{PG}_3(3)$	[4]
12	40	27	18	$\mathrm{PSU}_4(2)$	$3^3{:}\mathrm{S}_4$	$\mathrm{PSU}_4(2)$	Complement of Higman design	[4]
13	40	27	18	$\mathrm{PSU}_4(2)$	$3^3{:}\mathrm{S}_4{:}2$	$\mathrm{PSU}_4(2){:}2$	Complement of Higman design	[4]
14	45	12	3	$\mathrm{PSU}_4(2)$	$2{\cdot}(\mathrm{A}_4{\times}\mathrm{A}_4){\cdot}2$	$\mathrm{PSU}_4(2)$	-	[4, 17]
15	45	12	3	$\mathrm{PSU}_4(2)$	$2{\cdot}(\mathrm{A}_4{\times}\mathrm{A}_4){\cdot}2{:}2$	$\mathrm{PSU}_4(2){:}2$	-	[4, 17]
16	63	32	16	$\mathrm{PSU}_3(3)$	$4{\cdot}\mathrm{S}_4$	$\mathrm{PSU}_3(3)$	-	[6]
17	63	32	16	$\mathrm{PSU}_3(3)$	$2_+^{1+4}{\cdot}\mathrm{S}_3$	$\mathrm{PSU}_3(3){:}2$	-	[6]
18	63	32	16	$\mathrm{PSU}_3(3)$	$4^2{:}\mathrm{S}_3$	$\mathrm{PSU}_3(3)$	-	[6]
19	63	32	16	$\mathrm{PSU}_3(3)$	$4^2{:}\mathrm{D}_{12}$	$\mathrm{PSU}_3(3){:}2$	-	[6]

* The last column addresses to references in which a design with the parameters in the line has been constructed.

be more precise, when we have obtained some specific q, then we can obtain v and $|H|$. We next examine the divisors k of $|H|$ and check if $\lambda = k(k-1)/(v-1)$ is a positive integer. Once, we obtain a parameter (v, k, λ), to construct a design, we can use the command `BlockDesigns(v,B,G)` from the software package `design` in `GAP` [12], however, we note that the construction of the designs obtained in this manner can be found in [2, 4, 6, 9, 10, 15, 17] and therein references.

2. Preliminaries

In this section, we state some useful facts in both design theory and group theory. Recall that a group G is called almost simple if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a nonabelian simple group. We start this section with the following elementary and useful fact:

Lemma 2.1. [1, Lemma 2.2] Let G be an almost simple group with socle X, and let H be maximal in G not containing X. Then $G = HX$ and $|H|$ divides $|\mathrm{Out}(X)|{\cdot}|X \cap H|$.

Lemma 2.2. *Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design admitting a flag-transitive and point-primitive almost simple automorphism group G with socle X of Lie type in characteristic p. Suppose*

Table 2. Some subdegrees of almost simple groups with socle $\mathrm{PSU}_5(q)$.

Line	$H \cap X$	c
1	$\hat{}\mathrm{GU}_4(q)$	$(q+1)(q^4-1)$
2	$\hat{}(\mathrm{SU}_3(q) \times \mathrm{SU}_2(q)) : (q+1)$	$(q^2-1)(q^3+1)$

also that the point-stabiliser G_α, not containing X, is not a parabolic subgroup of G. Then $\gcd(p, v-1) = 1$.

Proof. Note that G_α is maximal in G, then by Tits' Lemma [20, 1.6], p divides $|G : G_\alpha| = v$, and so $\gcd(p, v-1) = 1$. □

Lemma 2.3. [14, 3.9] *If $X = \mathrm{PSU}_n(q)$ acts on the set of cosets of a maximal parabolic subgroup, then there is a unique subdegree which is a power of p.*

Lemma 2.4. *Let G be an almost simple group with socle $X = \mathrm{PSU}_5(q)$, and let H be a maximal subgroup of G with $H \cap X$ being as in the second column of* Table 2*. Then the action of G on the cosets of H has subdegrees dividing the numbers c listed in the last column of* Table 2.

Proof. Suppose first that $H \cap X$ is isomorphic to $\hat{}\mathrm{GU}_4(q)$. In this case, H stabilises a pair of non-degenerate subspaces which are mutually orthogonal and span the underlying space V. Thus $H = N_G(W)$ where W is a 1-dimensional non-degenerate subspace. Taking $\alpha = \langle u_1 \rangle$ and $\beta = \langle u_1, u_2 \rangle$, by [19, p. 336], we see that $|G_\alpha : G_{\alpha\beta}|$ divides $(q+1)(q^4-1)$. Suppose now that $H \cap X$ is isomorphic to $\hat{}(\mathrm{SU}_3(q) \times \mathrm{SU}_2(q)) : (q+1)$. Again here H stabilises a pair of non-degenerate subspaces, and so $H = N_G(W)$ where W is a 2-dimensional non-degenerate subspace. Set $\alpha = \langle u_1, u_2 \rangle$ and $\beta = \langle u_1, u_3 \rangle$. Then, by [19, p. 336], $|G_\alpha : G_{\alpha\beta}|$ divides $(q^2-1)(q^3+1)$. □

Lemma 2.5. *Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design. Let G be a flag-transitive automorphism group of $\mathcal{D}$ with simple socle X of Lie type in characteristic p. If the point-stabiliser $H = G_\alpha$ contains a normal quasi-simple subgroup N of Lie type in characteristic p and p does not divide $|Z(N)|$, then k is divisible by $|N{:}M|$, for some maximal subgroup M of H.*

Proof. If B is a block incident with a point α of $\mathcal{D}$, then $k = |H{:}H_B|$, and so $|N{:}N_B|$ divides k. Now, let M be a maximal subgroup of N such that $N_B \leqslant M$. Then $|N{:}M|$ must divide k, so k is divisible by $|N{:}M|$. □

Lemma 2.6. [2, Lemma 2.1] Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point in $\mathcal{P}$ and $H := G_\alpha$, then

(a) $k(k-1) = \lambda(v-1)$;
(b) $4\lambda(v-1)+1$ is square;
(c) $k \mid |H|$ and $\lambda v < k^2$;
(d) $k \mid \gcd(\lambda(v-1), |H|)$;

(e) $k \mid \lambda d$, for all nontrivial subdegrees d of G.

If a group G acts primitively on a set $\mathcal{P}$ with $|\mathcal{P}| \geqslant 2$ and $\alpha \in \mathcal{P}$, then the point-stabiliser G_α is maximal in G [11, Corollary 1.5A]. Therefore, in our study, we need a list of all maximal subgroups of almost simple group G with socle $X := \mathrm{PSU}_5(q)$. Note that if H is a maximal subgroup of G, then $H \cap X$ is not necessarily maximal in X in which case H is called a *novelty*. By [7, Tables 8.20 and 8.21], the complete list of maximal subgroups of an almost simple group G with socle $\mathrm{PSU}_5(q)$ are known, and in this case, there arise only three novelties.

Lemma 2.7. *Let G be an almost simple group with socle $X = \mathrm{PSU}_5(q)$, and let H be a maximal subgroup of G not containing X. Then $H \cap X$ is isomorphic to one of the subgroups listed in* Table 3.

Proof. The maximal subgroups H of G can be read off from [7, Tables 8.20 and 8.21]. □

Table 3. The subgroups $H \cap X$ of $X = \mathrm{PSU}_5(q)$ in Lemma 2.7.

Line	$H \cap X$	Comments
1	$\hat{}[q]^{1+6} : \mathrm{SU}_3(q) : (q^2-1)$	
2	$\hat{}[q]^{4+4} : \mathrm{GL}_2(q^2)$	
3	$\hat{}\mathrm{GU}_4(q)$	
4	$\hat{}(\mathrm{SU}_3(q) \times \mathrm{SU}_2(q)) : (q+1)$	
5	$\hat{}(q+1)^4 : \mathrm{S}_5$	
6	$\hat{}[\frac{q^5+1}{q+1}] : 5$	$q \geqslant 3$
7	$\hat{}\mathrm{SU}_5(q_0) \cdot \gcd(\frac{q+1}{q_0+1}, 5)$	$q = q_0^r$, r odd prime
8	$\mathrm{SO}_5(q)$	q odd
9	$\hat{}[5^3] : \mathrm{Sp}_2(5)$	$q = p \equiv 4 \pmod 5$ or $q = p^2$ and $p \equiv 2, 3 \pmod 5$
10	$\mathrm{PSL}_2(11)$	$q = p \equiv 2, 6, 7, 8, 10 \pmod{11}$
11	$\mathrm{PSU}_4(2)$	$q = p \equiv 5 \pmod 6$

3. **Proof of the main result**

In this section, we prove Theorem 1.1 in the following lemmas. We first recall from Subsection 1.1 that the assertion for the case where $X = \mathrm{PSU}_n(q)$ with $n = 2, 3, 4$ can be deduced from [2, 4, 9]. By revisiting [9], we obtain the missing designs in lines 6-7 and 16-19 of Table 1.

In what follows, we suppose that $\mathcal{D}$ is a nontrivial symmetric (v, k, λ) design and G is an almost simple automorphism group with simple socle $X = \mathrm{PSU}_5(q)$, where $q = p^a$ with p prime, that is to say, $X \lhd G \leqslant \mathrm{Aut}(X)$. Suppose also that $V = \mathbb{F}_q^5$ is the underlying vector space of X over the finite field $\mathbb{F}_q$ of size q. If G is a point-primitive automorphism group of $\mathcal{D}$, then the point-stabiliser $H = G_\alpha$ is maximal in G. Let $H_0 = H \cap X$. Then by Lemma 2.7, the subgroup H_0 is isomorphic to one of the subgroups recorded in Table 3, and so Lemma 2.1 implies that

$$v = \frac{|X|}{|H_0|} = \frac{q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)}{\gcd(5,q+1)\cdot|H_0|}. \tag{3.1}$$

Note that $|\mathrm{Out}(X)| = 2a\cdot\gcd(5,q+1)$. Therefore, by Lemmas 2.1(b) and 2.6(c),

$$k \mid 2a\cdot\gcd(5,q+1)\cdot|H_0|. \tag{3.2}$$

We now run through all possible subgroups H_0 recorded in Table 3, and obtain the only possible cases mentioned in Theorem 1.1.

Lemma 3.1. *The subgroup H_0 cannot be isomorphic to $\hat{}[q]^{1+6}:\mathrm{SU}_3(q):(q^2-1)$.*

Proof. By (3.1), we have that $v = q^7+q^5+q^2+1$. It follows from Lemmas 2.6(e) and 2.3 that k divides λq^2. Let now m be a positive integer such that $mk = \lambda q^2$. Since $\lambda < k$, we have that $m < q^2$. By Lemma 2.6(a), $k(k-1) = \lambda(v-1)$, and so $\lambda q^2(k-1) = m\lambda(q^7+q^5+q^2)$. Thus,

$$k = m\cdot(q^5+q^3+1)+1 \text{ and } \lambda = m^2(q^3+q)+\frac{m^2+m}{q^2}. \tag{3.3}$$

Since λ is integer, (3.3) implies that $q^2 \mid m^2+m$. Recall that $\gcd(m,m+1)=1$ and $m<q^2$. Therefore, q^2 must divide $m+1$, and so $m = q^2-1$. It follows from (3.3) that $k = (q^2-1)(q^5+q^3+1)+1 = q^2(q^5-q+1)$. By (3.2), k divides $2aq^{10}(q^3+1)(q^2-1)^2$. Therefore q^5-q+1 must divide $2a(q^3+1)$. Thus $q^5-q+1 \leqslant 2a(q^3+1)$, which is impossible. □

Lemma 3.2. *The subgroup H_0 cannot be isomorphic to $\hat{}[q]^{4+4}:\mathrm{GL}_2(q^2)$.*

Proof. According to (3.1), we have that $v = q^8+q^5+q^3+1$. By Lemmas 2.6(e) and 2.3, k divides λq^3. If m is a positive integer such that $mk = \lambda q^3$, then since $\lambda < k$, we have that $m < q^3$, and again by Lemma 2.6(a), we must have $\lambda q(k-1) = m\lambda(q^5+q^2+1)$. Thus, $k = m\cdot(q^5+q^2+1)+1$ and

$$\lambda = m^2q^2+\frac{m^2(q^2+1)+m}{q^3}.$$

This implies that q^3 divides $m^2(q^2+1)+m$. Recall that $m<q^3$. Therefore, q^3 must divide $m\cdot(q^2+1)+1$. Let n be a positive integer such that $m\cdot(q^2+1)+1 = nq^3$. Note that $m<q^3$. Thus $nq^3 = m\cdot(q^2+1)+1 < q^3(q^2+1)+1$, and so $n \leqslant q^2+1$. Also, we have that

$$m = \frac{nq^3-1}{q^2+1} = nq-\frac{nq+1}{q^2+1}.$$

Since m is integer, q^2+1 must divide $nq+1$. Let s be a positive integer that $nq+1 = s\cdot(q^2+1)$. Note that $n \leqslant q^2+1$. Therefore $s\cdot(q^2+1) = nq+1 \leqslant q(q^2+1)+1$, and so $s \leqslant q$. As $nq+1 = s\cdot(q^2+1)$, it follows that q divides $s-1$, and this contradicts the fact that $s \leqslant q$. □

Lemma 3.3. *The subgroup H_0 cannot be isomorphic to $\hat{}\mathrm{GU}_4(q)$.*

Proof. We note by (3.1) that $v = q^4(q^4 - q^3 + q^2 - q + 1)$. By Lemmas 2.4 and 2.6(e), k divides $\lambda(v-1) = \lambda(q^5+q+1)(q^2+1)(q-1)$. Therefore, k divides $\lambda(q^2+1)(q-1)$. Let m be a positive integer that $mk = \lambda f(q)$, where $f(q) = (q^2+1)(q-1)$. Then by Lemma 2.6(a), we have that

$$k = m \cdot (q^5 + q + 1) + 1 \text{ and } \lambda = m^2(q^2+q) + \frac{m^2(2q+1)+m}{(q^2+1)(q-1)}. \tag{3.4}$$

where $m < (q^2+1)(q-1)$. By Lemma 2.5, we conclude that k is divisible by the index of a maximal subgroup of $\mathrm{PSU}_4(q)$. It follows from [7, Table 8.10] that k are divisible by q^3+1 or q^3. If q^3 would divide k, then by (3.4), q^3 should divide $m \cdot (q+1) + 1$. Let n_1 be a positive integer such that $m \cdot (q+1) + 1 = n_1 q^3$. Then

$$m = \frac{n_1 q^3 - 1}{q+1} = n_1 \cdot (q^2 - q + 1) - \frac{n_1 + 1}{q+1}.$$

Since m is a positive integer, $q+1$ would divide n_1+1, and so $n_1 > q-1$. Recall that $m < (q^2+1)(q-1)$. Then $n_1q^3 = m\cdot(q+1)+1 < (q^2+1)(q-1)(q+1)+1$, and so $n_1 < q$, which is a contradiction. If q^3+1 divides k, then by (3.4), q^3+1 must divide $m\cdot(q^2-q-1)+1$. If $q = 2$, then 9 must divide $m+1$, where $m < 5$, which is impossible. Let now n_2 be a positive integer such that $m \cdot (q+1) + 1 = n_2 \cdot (q^3+1)$. As $m < (q^2+1)(q-1)$, we have that $n_2 < q$. Moreover,

$$m = \frac{n_2 \cdot (q^3+1) - 1}{q^2 - q - 1} = n_2 \cdot (q+1) + \frac{2n_2 \cdot (q+1) - 1}{q^2 - q - 1}. \tag{3.5}$$

Since m is an integer number, $q^2 - q - 1$ must divide $2n_2 \cdot (q+1) - 1$. Let u be a positive integer number such that $2n_2 \cdot (q+1) - 1 = u \cdot (q^2 - q - 1)$. Recall that $n_2 < q$. Then $u \cdot (q^2-q-1) = 2n_2 \cdot (q+1) - 1 < 2q^2 + 2q - 1$, and so $u \leqslant 3$. If $u = 1$, then $2n_2 \cdot (q+1) - 1 = q^2 - q - 1$, and so $q+1$ must divide $q^2 - q$, which is impossible. If $u = 2$, then $2n_2 \cdot (q+1) - 1 = 2(q^2-q-1)$, and so $q+1$ must divide $2q^2 - 2q - 1 = 2(q+1)(q-2) + 3$, which is impossible. If $u = 3$, then $2n_2 \cdot (q+1) - 1 = 3(q^2-q-1)$, and so $q+1$ must divide $3q^2 - 3q - 2 = 3(q+1)(q-2) + 4$. Thus $q+1$ divides 4, and so $q = 3$. In which case $n_2 = 2$, and by (3.5), $m = 55/4$, which is impossible. □

Lemma 3.4. *The subgroup H_0 cannot be isomorphic to* $\hat{}(\mathrm{SU}_3(q) \times \mathrm{SU}_2(q)) : (q+1)$.

Proof. In this case, $v = q^6(q^4-q^3+q^2-q+1)(q^2+1)$ by (3.1). It follows from Lemmas 2.4 and 2.6(e) that k must divide $\lambda(q^2-1)(q^3+1)$. On the other hand, k divides $\lambda(v-1) = \lambda(q^2-q+1)(q^{10}+q^8-q^7+q^4+q^3-q-1)$. Therefore, k divides $\lambda(q^2-q+1)\cdot\gcd(q^{10}+q^8-q^7+q^4+q^3-q-1, (q-1)(q+1)^2)$. Note that $\gcd(q^{10}+q^8-q^7+q^4+q^3-q-1, (q-1)(q+1)^2)$ divides 9. Let m be a positive integer that $mk = 9\lambda f(q)$, where $f(q) = q^2 - q + 1$. Then by Lemma 2.6(a), we have that

$$k = 1 + \frac{m \cdot (q^{10} + q^8 - q^7 + q^4 + q^3 - q - 1)}{9}, \tag{3.6}$$

where $m < 9(q^2-q+1)$. Note by (3.2) that k divides $2ag(q)$, where $g(q) = q^4(q^3+1)(q^2-1)^2(q+1)$. Then, by (3.6), we must have

$$m \cdot (q^{10} + q^8 - q^7 + q^4 + q^3 - q - 1) + 9 \text{ divides } 18ag(q). \tag{3.7}$$

http://dx.doi.org/10.22108/IJGT.2021.123581.1630

Table 4. Possible value for k and v when $q \in \{2, 3\}$.

q	2	3	4
v	1408	8404641	3562930176
k divides	19440	61440	300000

Let now $r(q) = q^9 - 6q^8 + 4q^7 + 3q^6 + q^5 - 3q^4 - 4q^3 - 2q^2 + 2q + 3$ and $h(q) = q^2 + q - 3$. Then $18amh(q)[m\cdot(q^{10}+q^8-q^7+q^4+q^3-q-1)+9]-18amg(q) = 18am[r(q)+9h(q)]$, and so (3.7) implies that $m\cdot(q^{10}+q^8-q^7+q^4+q^3-q-1)+9$ divides $18am[r(q)+9h(q)]$. Thus $m\cdot(q^{10}+q^8-q^7+q^4+q^3-q-1)+9 \leqslant 18am|r(q)+9h(q)|$, and so $q^{10}+q^8-q^7+q^4+q^3-q-1 < 18a|q^9-6q^8+4q^7+3q^6+q^5-3q^4-4q^3+16q^2+20q-45|$. This inequality holds only for $q \in \{2, 3, 4, 8, 9, 16, 25, 27, 32, 64\}$. For these values of q, considering the fact that $m < 9(q^2-q+1)$, there is no possible parameters k satisfying (3.7), which is a contradiction. □

Lemma 3.5. *The subgroup H_0 cannot be isomorphic to neither $\hat{}(q+1)^4 : \mathrm{S}_5$, nor $\hat{}[\frac{q^5+1}{q+1}] : 5$.*

Proof. Let first H_0 be isomorphic to $\hat{}(q+1)^4 : \mathrm{S}_5$. By (3.1), we have $v = q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)/[120\cdot(q+1)^4]$, and since $|\mathrm{Out}(X)| = 2a\cdot\gcd(5, q+1)$, it follows from (3.2) that k divides $240a(q+1)^4$. By [16, 22] and Lemma 2.6(c), we may assume that λ is at least 4, and so

$$\frac{q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1)}{30\cdot(q+1)^4} \leqslant \lambda v < k^2 \leqslant 57600a^2(q+1)^8.$$

This implies that $q^{10}(q^5+1)(q^4-1)(q^3+1)(q^2-1) < 1728000a^2(q+1)^{12}$. This inequality is true only when $q \in \{2, 3, 4\}$. Since k is a divisor of $240a(q+1)^4$, for each such $q = p^a$, the possible values of k and v are listed in Table 4. This is a contradiction as for each k and v as in Table 4, the fraction $k(k-1)/(v-1)$ is not integer.

Let now H_0 be isomorphic to $\hat{}[\frac{q^5+1}{q+1}] : 5$. In this case (3.1) implies that $v = q^{10}(q^4-1)(q^3+1)(q^2-1)(q+1)/5$, and since $|\mathrm{Out}(X)| = 2a\cdot\gcd(q+1, 5)$, it follows from (3.2) that k divides $2a(q^4-q^3+q^2-q+1)$. Again by Lemma 2.6(c), we have that $q^{10}(q^4-1)(q^3+1)(q^2-1)(q+1) \leqslant 5\lambda v < 5k^2 \leqslant 20a^2\cdot(q^4-q^3+q^2-q+1)^2$, and so $q^{10}(q^4-1)(q^3+1)(q^2-1)(q+1) < 20a^2\cdot(q^4-q^3+q^2-q+1)^2$, which is impossible. □

Lemma 3.6. *The subgroup H_0 cannot be isomorphic to $\hat{}\mathrm{SU}_5(q_0)\cdot\gcd(\frac{q+1}{q_0+1}, 5)$, where $q = q_0^r$ and r is a odd prime number.*

Proof. By (3.1), we have that

$$v = \frac{1}{d}\cdot\frac{q_0^{10r}(q_0^{5r}+1)(q_0^{4r}-1)(q_0^{3r}+1)(q_0^{2r}-1)}{q_0^{10}(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)},$$

where $d = \gcd(\frac{q+1}{q_0+1}, 5)$. Note by (3.2) that k divides $10aq_0^{10}(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)$. We may assume by [16, 22] that $\lambda \geqslant 4$. Moreover, $d \in \{1, 5\}$, and $a^2 \leqslant q_0^r$ as $q = q_0^r$ with r an odd prime number. Since $\lambda v < k^2$, by Lemma 2.6(b), we must have $q_0^{10r}(q_0^{5r}+1)(q_0^{4r}-1)(q_0^{3r}+1)(q_0^{2r}-1) <$

TABLE 5. The pairs $(X, H \cap X)$ in Lemma 3.8

X	$H \cap X$	v	k divides
$\mathrm{PSU}_5(2)$	$\mathrm{PSL}_2(11)$	20736	1320
$\mathrm{PSU}_5(4)$	$\hat{}5^{1+2}_{+} : \mathrm{Sp}_2(5)$	3562930176	60000
$\mathrm{PSU}_5(9)$	$\hat{}5^{1+2}_{+} : \mathrm{Sp}_2(5)$	1051720694280527616	60000

$100q_0^{30+r}(q_0^5+1)^3(q_0^4-1)^3(q_0^3+1)^3(q_0^2-1)^3$. Note that $q_0^{24r-1} \leqslant q_0^{10r}(q_0^{5r}+1)(q_0^{4r}-1)(q_0^{3r}+1)(q_0^{2r}-1)$ and $q_0^{30+r}(q_0^5+1)^3(q_0^4-1)^3(q_0^3+1)^3(q_0^2-1)^3 \leqslant q_0^{72+r}$. Then $q_0^{23r-1} < 100q_0^{72}$, and so $r = 3$. Thus (3.1) implies that

$$v = \frac{q_0^{20}(q_0^{15}+1)(q_0^{12}-1)(q_0^9+1)(q_0^6-1)}{(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)\cdot \gcd(q_0^2-q_0+1,5)}. \tag{3.8}$$

By (3.2), k divides $2adq_0^{10}(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)$, where $d = \gcd(q_0^2-q_0+1,5)$. Then by Lemma 2.6(c), we have that $\lambda q_0^{20}(q_0^{15}+1)(q_0^{12}-1)(q_0^9+1)(q_0^6-1) < 4a^2d^3q_0^{30}(q_0^5+1)^3(q_0^4-1)^3(q_0^3+1)^3(q_0^2-1)^3$. Therefore, $\lambda < 4a^2d^3$. Since k divides $2adq_0^{10}(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)$ and $v-1$ is coprime to q_0, k must divide $2\lambda ad(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)$. We use again Lemma 2.6(c), and so $\lambda v < k^2 \leqslant 4\lambda^2a^2d^2(q_0^5+1)^2(q_0^4-1)^2(q_0^3+1)^2(q_0^2-1)^2$. Thus (3.8) implies that

$$q_0^{47} < \frac{q_0^{20}(q_0^{15}+1)(q_0^{12}-1)(q_0^9+1)(q_0^6-1)}{(q_0^5+1)(q_0^4-1)(q_0^3+1)(q_0^2-1)} < 4\lambda a^2d^3. \tag{3.9}$$

Since $\lambda < 4a^2d^3$, it follows form (3.9) that $q_0^{47} < 16a^4d^6$, where $d = \gcd(q_0^2-q_0+1,5)$, which is impossible. □

Lemma 3.7. *The subgroup H_0 cannot be isomorphic to $\hat{}\mathrm{SO}_5(q)$ with q odd.*

Proof. In this case, by (3.1), we have that $v = q^6(q^5+1)(q^3+1)$. It follows from (3.2) that k divides $2ag(q)$, where $g(q) = q^4(q^4-1)(q^2-1)$. Moreover, Lemma 2.6(a) implies that k divides $\lambda(v-1)$. Let $f(q) = 3(q-1)^2$. Then $\gcd(v-1, 2q^4(q^4-1)(q^2-1))$ divides $f(q)$, and so k is a divisor of $\lambda af(q)$. Suppose that m is a positive integer such that $mk = \lambda af(q)$. Since now $k(k-1) = \lambda(v-1)$, it follows that $k = 1 + m\cdot(v-1)/af(q)$, and since $k \mid 2ag(q)$, we must have $m\cdot(v-1)+af(q) \mid 2a^2f(q)g(q)$. Therefore, $q^6(q^5+1)(q^3+1) < 2a^2f(q)g(q)$ for q odd, and this does not give rise to any possible parameters. □

Lemma 3.8. *The subgroup H_0 cannot be isomorphic to the subgroups as in the lines* 14-16 *of* Table 3.

Proof. Let H_0 be isomorphic to one of the subgroups in the lines 9-11 of Table 3. Since $|X| \leqslant |\mathrm{Out}(X)|^2\cdot|H\cap X|^3$, we only need to consider the pairs $(X, H\cap X)$ in Table 5. For each such $H\cap X$, by (3.1), we obtain v as in the third column of Table 5. Recall that k is a divisor of $2a\cdot\gcd(5,q+1)\cdot|H\cap X|$ which is recorded in the fourth column of Table 5. This is a contradiction as for each k and v as in Table 5, the fraction $k(k-1)/(v-1)$ is not integer. □

Proof of Theorem 1.1. The proof follows immediately from Lemmas 3.1–3.8. □

Proof of Corollary 1.2. The proof follows immediately from Theorem 1.1 and the main results in [2, 4, 9]. □

Acknowledgments

The authors would like to thank anonymous referees for providing us helpful and constructive comments and suggestions.

References

[1] S. H. Alavi and M. Bayat, Flag-transitive point-primitive symmetric designs and three dimensional projective special linear groups, *Bull. Iranian Math. Soc.*, **42** (2016) 201–221.

[2] S. H. Alavi, M. Bayat and A. Daneshkhah, Symmetric designs admitting flag-transitive and point-primitive automorphism groups associated to two dimensional projective special groups, *Des. Codes Cryptogr.*, **79** (2016) 337–351.

[3] S. H. Alavi, M. Bayat and A. Daneshkhah, *Finite exceptional groups of Lie type and symmetric designs*, (Submitted), 2020.

[4] S. H. Alavi, M. Bayat, A. Daneshkhah and S. Zang Zarin , Symmetric designs and four dimensional projective special unitary groups, *Discrete Math.*, **342** (2019) 1159–1169.

[5] T. Beth, D. Jungnickel and H. Lenz, *Design theory*, **I**, Second edition, Encyclopedia of Mathematics and its Applications, **69**, Cambridge University Press, Cambridge, 1999.

[6] S. Braić, A. Golemac, J. Mandić and T. Vučičić, Primitive symmetric designs with up to 2500 points, *J. Combin. Des.*, **19** (2011) 463–474.

[7] John N. Bray, Derek F. Holt and Colva M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, With a foreword by Martin Liebeck. London Mathematical Society Lecture Note Series, **407**, Cambridge University Press, Cambridge, 2013.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985.

[9] A. Daneshkhah and S. Zang Zarin, Flag-transitive point-primitive symmetric designs and three dimensional projective special unitary groups, *Bull. Korean Math. Soc.*, **54** (2017) 2029–2041.

[10] U. Dempwolff, Primitive rank 3 groups on symmetric designs, *Des. Codes Cryptogr.*, **22** (2001) 191–207.

[11] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, **163**, Springer-Verlag, New York, 1996.

[12] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.0*, 2020.

[13] D. R. Hughes and F. C. Piper, *Design theory*, Second edition. Cambridge University Press, Cambridge, 1988.

[14] Martin W. Liebeck, J. Saxl and Gary M. Seitz, On the overgroups of irreducible subgroups of the finite classical groups, *Proc. London Math. Soc. (3)*, **55** (1987) 507–537.

[15] E. O'Reilly-Regueiro, On primitivity and reduction for flag-transitive symmetric designs, *J. Combin. Theory Ser. A*, **109** (2005) 135–148.

[16] E. O'Reilly-Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle, *J. Algebraic Combin.*, **26** (2007) 529–552.

[17] Cheryl E. Praeger, The flag-transitive symmetric designs with 45 points, blocks of size 12 and 3 blocks on every point pair, *Des. Codes Cryptogr.*, **44** (2007) 115–132.

[18] Cheryl E. Praeger and S. Zhou, Imprimitive flag-transitive symmetric designs, *J. Combin. Theory Ser. A*, **113** (2006) 1381–1395.

[19] J. Saxl, On finite linear spaces with almost simple flag-transitive automorphism groups, *J. Combin. Theory Ser. A*, **100** (2002) 322–348.

[20] Gary M. Seitz, Flag-transitive subgroups of Chevalley groups, *Ann. of Math. (2)*, **97** (1973) 27–56.

[21] D. Tian and S. Zhou, Flag-transitive point-primitive symmetric (v, k, λ) designs with λ at most 100, *J. Combin. Des.*, **21** (2013) 127–141.

[22] S. Zhou, H. Dong and W. Fang, Finite classical groups and flag-transitive triplanes, *Discrete Math.*, **309** (2009) 5183–5195.

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