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THE RECOGNITION OF FINITE SIMPLE GROUPS WITH NO ELEMENTS OF ORDER 10 BY THEIR ELEMENT ORDERS

HUAIYU HE AND WUJIE SHI*

Abstract. The spectrum of a finite group is the set of its element orders. H is said to be a finite cover of G if G is a homomorphic image of H and H is finite. The main aim of this article is to characterize the finite simple groups with no elements of order 10 by its spectrum among covers. At the same time, above simple groups are completely classified. At last, some results on the recognition by spectrum of above groups are also achieved.

1. **Introduction**

Throughout this paper, all groups are assumed to be finite and all simple groups are non-abelian. Let G be a finite group, $\pi(G)$ be the set of all prime divisors of its order, and $\pi_e(G)$ be the spectrum of G, i.e., the set of its element orders. The set $\pi_e(G)$ of a finite group G defines the *Gruenberg-Kegel graph* GK(G) (or prime graph) whose vertices are prime divisors of the order of G, and two primes p, q are adjacent if G has an element of order pq. Of course, $\pi_e(G)$ defines $GK(G)$ uniquely, so $GK(G) = GK(L)$ if $\pi_e(G) = \pi_e(L)$.

A group G is said to be *recognizable* by $\pi_e(G)$ if every finite group H with $\pi_e(G) = \pi_e(H)$ is isomorphic to G. In other words, G is recognizable if $h(G) = 1$ where $h(G)$ is the number of pairwise non-isomorphic groups H with $\pi_e(G) = \pi_e(H)$. A group G is *non-recognizable* by $\pi_e(G)$ if $h(G) = \infty$, and is *almost recognizable* by $\pi_e(G)$ if $h(G) < \infty$. If every finite group H with $\pi_e(G) = \pi_e(H)$ has

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*Corresponding author.

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a unique non-abelian composition factor S and $S \cong G$, the group G is said to be *quasirecognizable*. H is said to be a *finite cover* of G if G is a homomorphic image of H and H is finite. G is said to be *recognizable among covers* if $\pi_e(H) \neq \pi_e(G)$ for any proper finite cover H of G. Obviously, if a non-abelian finite simple group L is simultaneously quasirecognizable and recognizable among cover, then every finite group with same spectrum to L is isomorphic to a group G with $L \leq G \leq Aut(L)$, in particular, L is almost recognizable.

In [13], the finite simple groups without elements of order 6 and 10 is obtained. In [4], the finite simple groups with no elements of order 6 is gave out. Comparing above results, the finite simple groups with no elements of order 10 is discussed, and its classification is also obtained.

Since Wujie Shi showed in 1986 that the alternating group of degree 5 is recognizable by spectrum [14], the recognition problem has been investigated for a numerous number of simple groups. Then Wujie Shi supposed that:

Conjecture 1.1. *All the simple groups are recognizable by spectrum and order.*

Investigations on this subject have 30 years' history and resulted in more than a hundred papers of numerous authors. Then Shi's conjecture is confirmed in [16]. What happens if we omit the order of the simple group in above conjecture? Then Shi's conjecture is not true. For example, $h(Alt_6) = \infty$. Is there an substitutable condition? In [1], the finite simple groups with no elements of order 6 are characterized by its spectrum. In [5], a series of simple classical groups are recognizable among covers. Thus, according to above line of thinking, the finite simple groups with no elements of order 10 are discussed.

If q is a natural number, r is an odd prime, and $(r, q) = 1$, then by $e(r, q)$ we denote a minimal natural number n with $q^n \equiv 1 \pmod r$. The prime r with $e(r, q) = n$ is called a primitive prime divisor of $q^n - 1$, denoted by r_n. In [22] , Zsigmondy ensured the existence of this prime, but a primitive prime divisor is not uniquely determined.

Lemma 1.2. *Let q and m be natural numbers greater than 1. There exists a prime divisor r of $q^m - 1$ such that r does not divide $q^i - 1$ for all $i < m$, except for the following cases:*

(1) *$m = 6$ and $q = 2$;*
(2) *$m = 2$ and $q = 2^t - 1$ for some natural number t.*

A group is C_{pp} if the centralizer of elements of order p are $p-$groups. Obviously, there are no elements of order 10 in C_{pp} groups if $p \in \{2, 5\}$. In [19, Theorem 4-5]*, Williams gave out all the C_{55} groups and C_{22} groups.*

Lemma 1.3. [19, Theorem 4] *If G is a simple group of Lie type in odd characteristic, or alternating or sporadic, then G is a C_{55} group if and only if G is one of the following: Alt_7, M_{11}, M_{12}, $PSL_2(q)$ $(q = 5^m, 2 \cdot 5^m - 1$ or $2 \cdot 5_m + 1)$ or $C_2(q)$ $(q^2 = 2 \cdot 5^m - 1)$.*

Remark 1.4. *There is a misprint in Lemma1.3: M_{12} should be replaced by M_{22}. Since M_{12} has a subgroup $2 \times S_5$ (see [3]), and M_{22} contains a nilpotent Hall $5-$subgroup which is isolated* [19, Table 2a], *then M_{22} should be a C_{55} group.*

Based on the classification of finite simple groups, the finite simple groups without elements of order 10 are gave out.

Theorem 1.5. *If G is a finite non-abelian simple group with no elements of order 10, then G is isomorphic to one of the following groups:*

(a) M_{11}, M_{22}, M_{23}.

(b) Alt_n, $5 \leq n \leq 8$.

(c) ${}^2B_2(q)$,${}^2G_2(q)$, ${}^3D_4(q)$ with $5|q^2+1$, $G_2(q)$ with $5|q^2+1$.

(d) $A_1(q)$ where $20 \nmid (q+1)$ and $20 \nmid (q-1)$, $A_2(q)$ where $5 \mid q^2+1$ or $q = 2^n \equiv -1 \pmod 5$, $A_3(q)$ where $5 \mid q^2+1$ and $4 \mid q^2(q-1)$, $A_4(2^n)$ with odd n, ${}^2A_2(q)$ where $5 \mid q^2+1$ or $q = 2^n \equiv 1 \pmod 5$, ${}^2A_3(q)$ where $5 \mid q^2+1$ and $4 \mid q^2(q+1)$, ${}^2A_4(2^n)$ with odd n, $C_2(q)$ with $5 \mid q^2+1$.

Proof. (a) Suppose G is a sporadic group. It is obviously that $5 \in \pi(G)$ in this case, then 2 must be non-adjacent to 5 in $GK(G)$, therefore, G is isomorphic to M_{11}, M_{22}, M_{23}. Otherwise, $10 \in \pi_e(G)$ [18, Table 2].

(b) If $r \in \pi(G)$ is an odd prime and G is an alternating group of degree n, then 2 is non-adjacent to r if and only if $4 + r > n$, therefore, $n < 9$.

(c) Let G be a finite simple exceptional group of Lie type over a field of characteristic p. As we know $5 \mid p^4 - 1$ if $p \neq 5$, so $5 \in \{p, r_1, r_2, r_4\}$.

If $5 \notin \pi(G)$, then $p \neq 5$, and G could be isomorphic to ${}^2G_2(q)$, ${}^3D_4(q)$ and $G_2(q)$ by the order of G and $2-$independence numbers [18]. Since $5 \mid 3^{2(2n+1)} + 1$, then G can be ${}^2G_2(q)$ [18, Table 5]. If $5|q^2+1$, then G can be ${}^3D_4(q)$ and $G_2(q)$ [18, Table 5-7].

Let $5 \in \pi(G)$. Obviously $p = 2$, otherwise 2 is adjacent to 5 in $GK(G)$ [18, Table 7]. Therefore G could possibly be isomorphic to ${}^2B_2(q)$ and ${}^2F_4(q)$. But in $GK({}^2F_4(q))$ 2 is adjacent to 5 and in $GK({}^2B_2(q))$ 2 is non-adjacent to 5 [18, Table 5]. Thus, $G \cong {}^2B_2(q)$.

(d) Let G be a finite simple classical group over a field of characteristic p, then $5 \in \{p, r_1, r_2, r_4\}$.

If $5 \notin \pi(G)$, then G is isomorphic to $A_1(q)$, $A_2(q)$ and ${}^2A_2(q)$ by the order of G, in which $5 = r_4$.

Let $5 \in \pi(G)$. If $p = 5$, then G is isomorphic to $A_1(q)$ by $10 \notin \pi_e(G)$ [18, Table 6].

If $p = 2$, then G is isomorphic to the following groups:$A_1(q)$ with $5 \in \{r_1, r_2\}$, $A_2(q)$ with $5 = r_2$, $A_3(q)$ and $A_4(q)$ with $5 = r_4$, ${}^2A_2(q)$ with $5 = r_1$, ${}^2A_3(q)$ and ${}^2A_4(q)$ with $5 = r_4$, $C_2(q)$ with $5 = r_4$([18, Table 4-6]) If $5 \mid 2^{2n} + 1$, then n is odd.

When $p \neq 2$ and $p \neq 5$, by the adjacency criterion of $GK(G)$ in [18], we study all the possibilities of G. Notice that $5 \in \{r_1, r_2, r_4\}$. Obviously, by $10 \notin \pi_e(G)$, G is not isomorphic to $D_n(q)$ or ${}^2D_n(q)$. If G is isomorphic to $B_n(q)$ or $C_n(q)$, then $n = 2$ and $5 = r_4$. If $G \cong {}^2A_n(q)$, then $n = 3$, $5 = r_4$ and $4 \mid q+1$. If $G \cong A_n(q)$, then $n = 3$ or $n = 1$. When $n = 3$, we have $5 = r_4$ and $4 \mid q-1$ since 2 is not adjacent to 5 in $GK(G)$. Similarly, When $n = 1$ we have $4 \mid q-1$ with $5 = r_2$ or $4 \mid q-3$ with $5 = r_1$, therefore $\frac{q\pm1}{10}$ is an odd.

At the same time, it is known that $\pi_e(A_1(q))$ is exactly the set of all divisors of the following numbers :$\frac{q^2+1}{(2,p-1)}$, $\frac{q^2-1}{(2,p-1)}$ and p. Thus, if $G \cong A_1(q)$ then $20 \nmid (q+1)$ and $20 \nmid (q-1)$. If $G \cong A_3(q)$, then $5 \mid q^2+1$ and $p = 2$, or $5 \mid q^2+1$ and $4 \mid q-1$, and which is to say $5 \mid q^2+1$ and $4 \mid q^2(q-1)$. If $G \cong^2 A_3(q)$, then $5 \mid q^2+1$ and $p = 2$, or $5 \mid q^2+1$ and $4 \mid q+1$, and which is to say $5 \mid q^2+1$ and $4 \mid q^2(q+1)$. If $G \cong C_2(q)$, then $5 \mid q^2+1$ and $p = 2$, or $5 \mid q^2+1$ and $p \notin \{2,5\}$, which is to say $5 \mid q^2+1$. The theorem is proved. □

Next, it's time to discuss the characterization of finite non-abelian simple groups with no elements of order 10.

Theorem 1.6. *If G is a finite non-abelian simple group with no elements of order 10 other than ${}^3D_4(2)$, ${}^2A_4(2)$, ${}^2A_2(q)$ where q is a special Mersenne prime, then G is recognizable by spectrum among covers.*

Proof. By Theorem 1.5, [5, Corollary 1.1] and [7, Theorem 1], we only need to discuss the recognition of ${}^2A_2(q)$.

Let $G \simeq {}^2A_2(q)$. Then either $5 \mid q^2+1$, or $5 \mid q-1$ and $p = 2$. If $p = 2$, then G is recognizable by its spectrum (see [12]), therefore, G is recognizable by spectrum among covers. Let p be odd and let S be a finite group with the same spectrum as G. By [20], if q is not a special Mersenne prime (that is, q is a Mersenne prime such that q^2-q+1 is also prime), then G is recognizable among covers. But if q is a special Mersenne prime, then G is not recognizable among covers for $q > 3$ [20, Lemma 11]. Observe that there are special Mersenne primes q such that 5 divides q^2+1, for example, $q = 7$. On the other hand, ${}^2A_2(3)$ is non-recognizable (see [8]). Thus ${}^2A_2(q)$ with no no elements of order 10 is recognizable by spectrum among covers except ${}^2A_2(q)$, where q is a special Mersenne prime.

Thus, G is recognizable by spectrum among covers. □

By [9] and [17], we know the following groups are recognizable by spectrum: sporadic groups except J_2 , Alt_5, Alt_7, Alt_8, ${}^2B_2(q)$, ${}^2G_2(q)$, $G_2(q)$, $A_1(q)$ $(q \neq 9)$, and $C_2(q)(q = 3^{2m+1} > 3)$. At the same time, $A_4(2^n)$ [15] and ${}^2A_4(2^{n+1})$ [6] are also recognizable. At the other hand, ${}^3D_4(2)$ [10], $A_1(9)$ [2], $A_2(3)$ and $C_2(q)(q = 3$ or $q \neq 3^{2m+1} > 3)$ are non-recognizable ([11]). ${}^3D_4(q)(q > 2)$ ([17]), $A_3(q)$ and ${}^2A_3(q)$ [7] are almost recognizable . By the characterization of $A_2(q)$ [21] and ${}^2A_2(q)$ [20], we can achieve the following conclusion.

Let G be a finite non-abelian simple group with no elements of order 10.

(a) Let G be not a finite simple classical group. Then G is recognizable by its spectrum except Alt_6 and ${}^3D_4(q)$. At the other hand, Alt_6 and ${}^3D_4(2)$ are non-recognizable, ${}^3D_4(q)(q \neq 2)$ is almost recognizable.

(b) Let G be a finite simple classical group. Then G is recognizable by its spectrum except the following: $A_1(9)$, $A_2(3)$, $A_2(q)$ $(q \equiv 1,5,7,9(mod\ 12))$, ${}^2A_2(q)$ $(q \equiv 3,5,7,11(mod\ 12))$, $A_3(q)$ $(q \neq$

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2^m), ${}^2A_3(2)$, ${}^2A_3(q)$ ($q \neq 2^m$), ${}^2A_4(2)$, $C_2(3)$, $C_2(q)$ ($q \neq 3^{2m+1}$). On the other hand, $A_1(9)$, $A_2(3)$, ${}^2A_2(q)$ where q is a special Mersenne prime, ${}^2A_3(2)$, ${}^2A_4(2)$, $C_2(3)$ and $C_2(q)$ ($q \neq 3^{2m+1}$) are non-recognizable; $A_2(q)$ ($q > 3$), ${}^2A_2(q)$ where q is not a special Mersenne prime, $A_3(q)$, and ${}^2A_3(q)$ ($q > 2$) are almost recognizable.

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Huaiyu He

Department of Economics and Management, Shanghai University of Political Science and Law, Shanghai 201701, China

he-huaiyu@163.com

Wujie Shi

Department of Mathematics, Chongqing University of Arts and Sciences, Chongqing 402160, China [3] School of Mathematics, Suzhou University, Suzhou 215006, China

shiwujie@outlook.com