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INTEGRITY OF GRAPH OPERATIONS

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Abstract. A communication network can be considered to be highly vulnerable to disruption if the failure of few members (nodes or links) can result in no members being able to communicate with very many others. These communication networks can be modeled through graphs. There are several graph-theoretic parameters to describe the stability of graphs. But, these parameters are not sufficient to study stability of graphs. This leads to the concept of integrity of a graph. In this paper, we obtain the integrity of some graph operations and some special graphs which can help us to reconstruct the given network in such a way that it is more stable than the earlier one.

1. Introduction

In communication networks, we require greater degrees of stability or less vulnerability. The vulnerability measures resistance of the network to the disruption in operation after the failure of certain stations or communication links. The stability of a communication network is of prime importance to network designers. As the network starts losing links or nodes, ultimately there is a loss in its efficiency. Thus, communication networks must be assembled to be as stable as possible, not only with respect to the initial interruption, but also with respect to the possible reconstruction of the network. The communication network can be represented as an undirected graph. Tree, hypercube and star graph are popular communication networks. If we model a network through graph, then there are many graph theoretical parameters to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The best known

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measure of reliability of a graph is its vertex-connectivity $\kappa(G)$ defined to be the minimum number of vertices whose removal results in a disconnected or trivial graph. The difficulty with these parameters is that they do not consider what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in order to attempt to survive with this difficulty. The connectivity of the two different graphs may be same, but the orders of theirs largest components need not be same. Then they may differ in respect to stability. Now, how can we measure this property? The idea behind the answer is the concept of integrity, which is different from connectivity.

The concept of integrity was introduced as a measure of graph vulnerability. The integrity of a graph G is

$$I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\},$$

where $m(G-S)$ denotes the order of a largest component of $G-S$. This concept was introduced by Barefoot et al. in [4]. The set $S \subset V(G)$ is called the I-set, if $I(G) = |S| + m(G-S)$. For more details refer [2, 3, 4, 5, 7, 8, 9, 10, 11, 12].

Let G be a finite graph with n vertices and m edges is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of G is called a vertex cover for G. The minimum cardinality among the vertex covers for G is the vertex covering number α_0 of G. In this paper, we consider, nontrivial finite undirected graph with no loops and no multiple edges, and we denote P_n, C_n, K_n, $K_{a,b}$, $K_{1,n}$ and $W_{1,n}$ for a path, a cycle, a complete graph, a complete bipartite graph, a star and a wheel, respectively. The symbol $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x, $\lfloor x \rfloor$ denotes the greatest integer that is smaller than or equal to x and the absolute value of x is denoted by $|x|$. The graph theoretic terminologies and notations can be found in [6, 14, 16].

2. Preliminaries

In this section, we list some of the known results about integrity of graphs.

Theorem 2.1. [2] *The integrity of*

(a) *complete graph K_n is $I(K_n) = n$,*
(b) *the null graph is $I(\overline{K_n}) = 1$,*
(c) *the star is $I(K_{1,n}) = 2$,*
(d) *the path is $I(P_n) = \lceil 2\sqrt{n+1} \rceil - 2$,*
(e) *the cycle is $I(C_n) = \lceil 2\sqrt{n} \rceil - 1$,*
(f) *the complete bipartite graph is $I(K_{a,b}) = 1 + \min\{a, b\}$.*

Theorem 2.2. [1] *For $n \geq 3$, let $a = \lfloor \sqrt{n+1} \rfloor$ and $b = \lceil 2\sqrt{n+1} \rceil$. Then,*

$$I(K_2 \times P_n) = \begin{cases} 2I(P_n) - 1 & if\ (n+1) \leq a(b - a - \frac{1}{2}), \\ 2I(P_n) & otherwise. \end{cases}$$

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Theorem 2.3. [1] *For $n \geq 3$, let $a = \lfloor \sqrt{n} \rfloor$ and $b = \lceil 2\sqrt{n} \rceil$. Then,*

$$I(K_2 \times C_n) = \begin{cases} 2I(C_n) - 1 & if\ (n+1) \leq a(b - a - \frac{1}{2}), \\ 2I(C_n) & otherwise. \end{cases}$$

Theorem 2.4. [10] *For $2 \leq p \leq q$,*

$$I(K_p \times K_q) = pq - \max_{i \leq j \leq p} \left\lfloor \frac{q(p-j)}{p} \right\rfloor.$$

3. Main results

In this section, we compute integrity of graph operations such as product, corona, complement, subdivision of graphs and some special graphs.

3.1. Integrity of some graph operations.

Definition 3.1. *The product* [14] *$G \times H$ of graphs G (which has n_1 vertices, m_1 edges) and H (which has n_2 vertices, m_2 edges) has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if and only if [$a = b$ and $xy \in E(H)$] or [$x = y$ and $ab \in E(G)$]. It follows from the definition of product that $G \times H$ has $n_1 n_2$ vertices and $n_1 m_2 + n_2 m_1$ edges.*

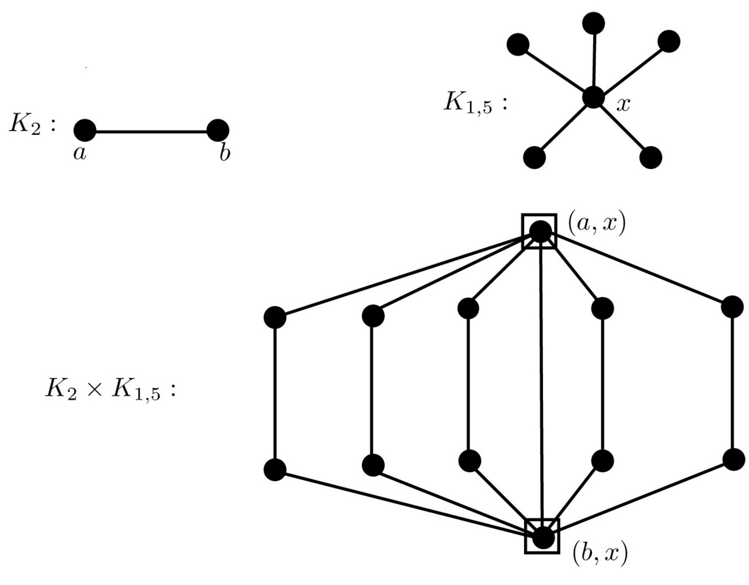

Figure 1. The graphs $K_2, K_{1,5}$ and $K_2 \times K_{1,5}$.

Theorem 3.2. *Let $K_2 \times K_{1,n}$ be the product of two graphs K_2 and $K_{1,n} (n \geq 2)$ with $2(n+1)$ vertices. Then*

$$I(K_2 \times K_{1,n}) = 4.$$

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Proof. Let $V(K_2) = \{a, b\}$ and $V(K_{1,n}) = \{x, v_1, v_2, \ldots, v_n\}$, where the vertex x of degree n in $K_{1,n}$ is called the central vertex. If we consider $S = \{(a, x), (b, x)\} \subset V(K_2 \times K_{1,n})$ with $|S| = 2$, then $K_2 \times K_{1,n} - S$ has n components of K_2. Thus, $I(K_2 \times K_{1,n}) = 2 + 2 = 4$. An example is shown in Figure 1. □

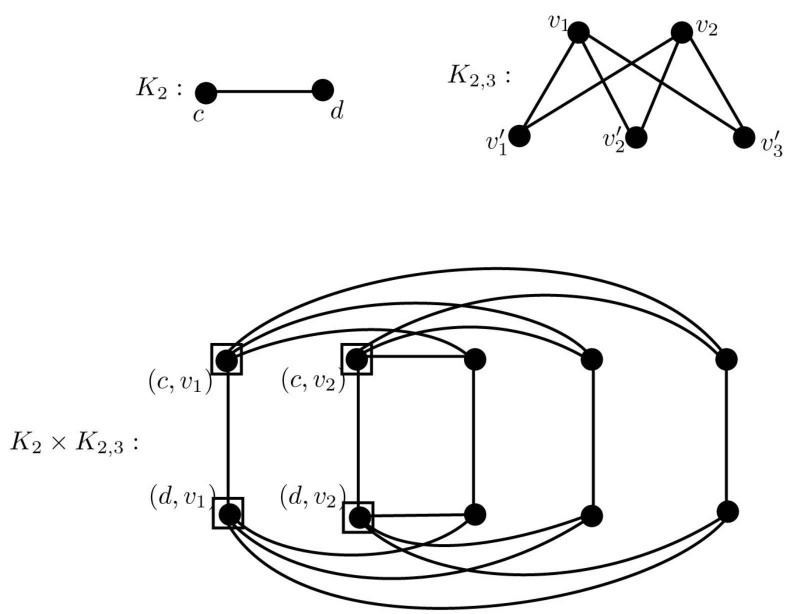

FIGURE 2. The graphs $K_2, K_{2,3}$ and $K_2 \times K_{2,3}$.

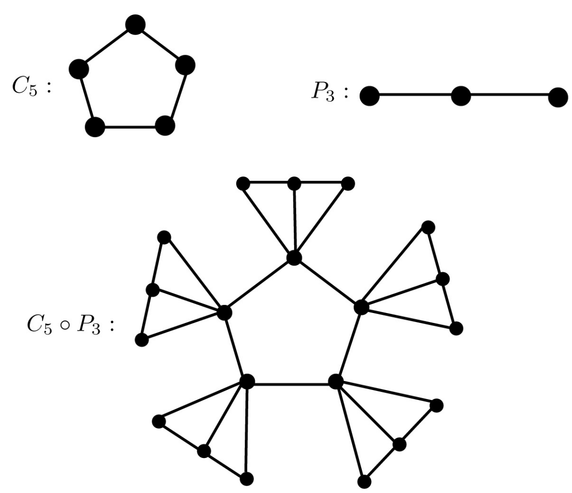

FIGURE 3. The graphs $K_2, W_{1,6}$ and $K_2 \times W_{1,6}$.

Theorem 3.3. *Let $K_2 \times K_{a,b}$ be the product of two graphs K_2 and $K_{a,b}(a \leq b, a, b \geq 2)$ with $2(a+b)$ vertices. Then*

$$I(K_2 \times K_{a,b}) = 2\,(a+1)\,.$$

Proof. Let $V(K_2) = \{c, d\}$ and $V(K_{a,b}) = \{v_1, v_2, \ldots, v_a, v'_1, v'_2, \ldots, v'_b\}$, where the vertices v_i's are in one partite set while the vertices v'_i's are in another partite set of $K_{a,b}$. If we consider $S = \{(c, v_i), (d, v_i) : i = 1, 2, \ldots, a\} \subset V(K_2 \times K_{a,b})$ with $|S| = 2a$, then $K_2 \times K_{a,b} - S$ has b components of K_2. Thus, $I(K_2 \times K_{a,b}) = 2a + 2 = 2\,(a+1)\,$. An example is shown in Figure 2. □

Theorem 3.4. *Let $K_2 \times W_{1,n}$ be the product of two graphs K_2 and $W_{1,n}(n \geq 3)$ with $2(n+1)$ vertices. Then*

$$I(K_2 \times W_{1,n}) = 2 + I(K_2 \times C_n).$$

Proof. Let $V(K_2) = \{a, b\}$ and $V(W_{1,n}) = \{x, v_1, v_2, \ldots, v_n\}$, where the vertex x of degree n in $W_{1,n}$ is called the central vertex. If we consider $S = \{(a, x), (b, x)\} \subset V(K_2 \times W_{1,n})$ with $|S| = 2$, then $K_2 \times W_{1,n} - S$ has only one component $K_2 \times C_n$. Thus, $I(K_2 \times W_{1,n}) = 2 + I(K_2 \times C_n)$. An example is shown in Figure 3. □

Definition 3.5. *The corona [14] $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has n_1 vertices, m_1 edges) and n_1 copies of G_2 (which has n_2 vertices, m_2 edges) and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. It follows from the definition of corona that $G_1 \circ G_2$ has $n_1(1+n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.*

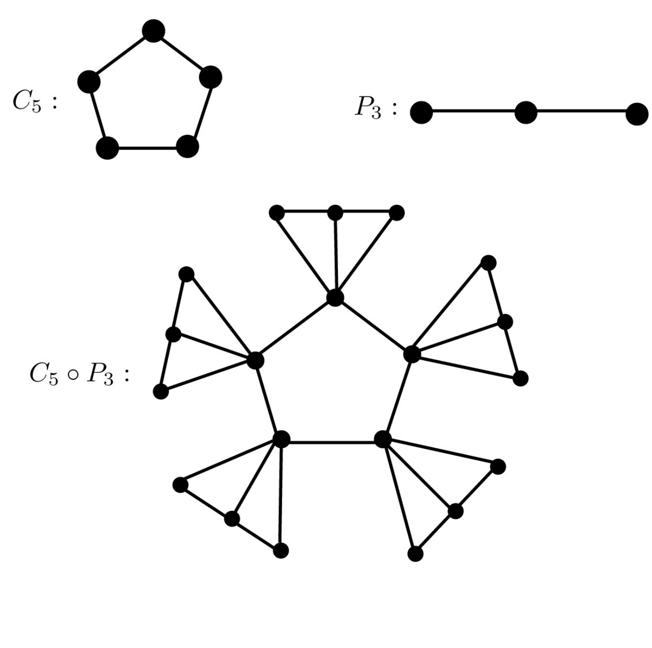

FIGURE 4. The graphs C_5, P_3 and $C_5 \circ P_3$.

Theorem 3.6. *For any graphs G and H with n_1 and n_2 vertices, respectively. Then*

$$I(G \circ H) \leq \min\{\alpha_0(G) + n_2 + 1, I(H) + (n_1 - 1)|S_I| + n_1\},$$

where S_I is the I-set of H such that $|S_I| \geq |S|$, for all I-sets S of H.

Proof. Let G and H be two graphs with n_1 and n_2 vertices, respectively. If V_α is the point cover of G and $|V_\alpha| = \alpha_0(G)$, then by removing the members of V_α from $G \circ H$, we get

$$G \circ H - V_\alpha = \alpha_0(G)H \cup (n_1 - \alpha_0(G))(K_1 + H).$$

Thus, by the definition of integrity of a graph, we have

$$I(G \circ H) \leq \min\{\alpha_0(G) + n_2 + 1, I(H) + (n_1 - 1)|S_I| + n_1\},$$

where S_I is the I-set of H such that $|S_I| \geq |S|$, for all I-sets S of H. □

If we choose the appropriate subset S_I in the above Theorem 3.6, then we get the equality. Now, by choosing the appropriate subset S_I for different graph families, we have the following corollaries.

Corollary 3.7. *Let G be any graph with n_1 vertices and P_n be a path with n vertices. Then*

$$I(G \circ P_n) = \min\left\{\alpha_0(G) + n + 1, \lceil 2\sqrt{n+1}\rceil - 2 + (n_1 - 1)\left\lfloor \frac{n}{4} \right\rfloor + n_1\right\}.$$

Corollary 3.8. *Let G be any graph with n_1 vertices and C_n be a cycle with n vertices. Then*

$$I(G \circ C_n) = \min\left\{\alpha_0(G) + n + 1, \lceil 2\sqrt{n}\rceil - 1 + (n_1 - 1)\left\lceil \frac{n}{3} \right\rceil + n_1\right\}.$$

Corollary 3.9. *Let G be any graph with n_1 vertices and K_n be a complete graph with n vertices. Then*

$$I(G \circ K_n) = \alpha_0(G) + n + 1.$$

Corollary 3.10. *Let G be any graph with n_1 vertices and $K_{1,n}$ be a star graph with $n+1$ vertices. Then*

$$I(G \circ K_{1,n}) = \min\left\{\alpha_0(G) + n + 2, 2(n_1 + 1) - 1\right\}.$$

Corollary 3.11. *Let G be any graph with n_1 vertices and $K_{a,b}$ be a complete bipartite graph with $a+b$ vertices. Then*

$$I(G \circ K_{a,b}) = \min\left\{\alpha_0(G) + a + b + 1, n_1(\min\{a, b\} + 1) + 1\right\}.$$

Corollary 3.12. *Let G be any graph with n_1 vertices and $W_{1,n}$ be a wheel graph with $n+1$ vertices. Then*

$$I(G \circ W_{1,n}) = \min\left\{\alpha_0(G) + n + 2, \lceil 2\sqrt{n}\rceil + (n_1 - 1)\left(\left\lceil \frac{n}{3} \right\rceil + 1\right) + n_1\right\}.$$

Definition 3.13. *The graph G^{+k} is a graph obtained from the graph G by attaching k ($k \geq 1$) pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n + nk$ and size $m + nk$.*

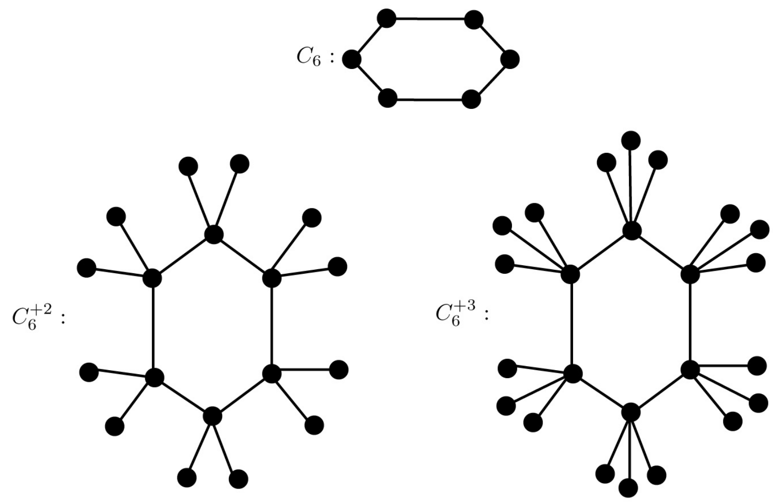

FIGURE 5. The graphs C_6, C_6^{+2} and C_6^{+3}.

Theorem 3.14. *Let G be a graph with n vertices. Then for a positive integer $k \geq 1$,*

$$I(G^{+k}) = \begin{cases} n+1 & if \;\; G = K_n, \\ \alpha_0(G) + k + 1 & otherwise. \end{cases}$$

Proof. If $G = K_n$, then $V(G)$ should be taken as S. Clearly, $K_n^{+k} - S$ has nk components of K_1. Thus, $I(K_n^{+k}) = n + 1$.

Suppose $G \neq K_n$ and $I(G) = |S| + m(G - S)$ for an I-set S of G. Let V_α be the point cover of G and $|V_\alpha| = \alpha_0(G)$. Then $G^{+k} - V_\alpha$ has at least n components. Clearly, $m(G^{+k} - V_\alpha) = k + 1$. Thus, $I(G^{+k}) = \alpha_0(G) + k + 1$. □

3.2. **Integrity of some graph operators.**

Definition 3.15. *The subdivision graph* [14] *$S(G)$ of a graph G is a graph with the vertex set $V(S(G)) = V(G) \cup E(G)$ and two vertices of $S(G)$ are adjacent whenever they are incident in G.*

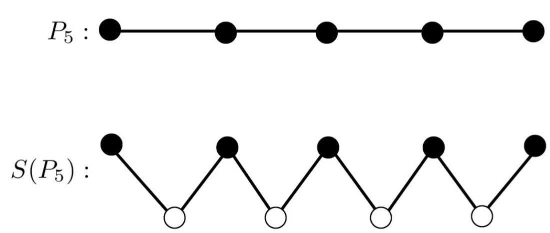

FIGURE 6. The path P_5 and its subdivision graph $S(P_5)$.

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Theorem 3.16. *If G is a graph with n vertices and m edges, then*

$$I(S(G)) \leq min\{n+1, m+1\}.$$

Proof. The vertex set of $S(G)$ is the union of vertex set of G and edge set of G.
Case 1. Suppose we consider $S_1 = V(G)$. Then by the definition of the subdivision of a graph, $S(G) - S_1 = |E(G)|K_1$. Thus, $|S_1| + m(S(G) - S_1) \leq n+1$. Therefore, $I(S(G)) \leq n+1$.
Case 2. Suppose we consider $S_2 = E(G)$. Then by the definition of the subdivision of a graph, $S(G) - S_2 = |V(G)|K_1$. Thus, $|S_2| + m(S(G) - S_2) \leq m+1$. Therefore, $I(S(G)) \leq m+1$. The minimum value among $n+1$ and $m+1$ gives integrity of subdivision graph. Therefore, $I(S(G)) \leq min\{n+1, m+1\}$. □

Theorem 3.17. *Let P_n be a path with n vertices. Then*

$$I(S(P_n)) = \left\lceil 2\sqrt{2n} \right\rceil - 2.$$

Proof. The subdivision graph of a path P_n of order n is again a path of order $2n$. Therefore, $I(S(P_n)) = \lceil 2\sqrt{2n} \rceil - 2$. □

Theorem 3.18. *Let C_n be a cycle with n vertices. Then*

$$I(S(C_n)) = \left\lceil 2\sqrt{2n} \right\rceil - 1.$$

Proof. The subdivision graph of a cycle C_n of order n is again a cycle of order $2n$. Therefore, $I(S(C_n)) = \lceil 2\sqrt{2n} \rceil - 1$. □

Theorem 3.19. *Let $K_n (n \geq 3)$ be a complete graph with n vertices. Then*

$$I(S(K_n)) = n+1.$$

Proof. Suppose we consider $S = V(K_n)$. Then $S(K_n) - S = |E(K_n)|K_1$. Thus, $I(S(G)) = n+1$. □

Theorem 3.20. *Let $K_{1,n} (n \geq 2)$ be a star graph with $n+1$ vertices. Then*

$$I(S(K_{1,n})) = 3.$$

Proof. If we consider $S = \{x\}$, where x is a vertex of degree n in $S(K_{1,n})$, then $S(K_{1,n}) - S$ has at most n components of K_2. Thus, $I(S(K_{1,n})) = 3$. □

Theorem 3.21. *Let $K_{a,b} (a \leq b, a, b \geq 2)$ be a complete bipartite graph with $a+b$ vertices. Then*

$$I(S(K_{a,b})) = 2a+1.$$

Proof. Let $V(K_{a,b}) = \{u_1, u_2, \ldots, u_a, v_1, v_2, \ldots, v_b\}$. If we consider $S = \{u_1, u_2, \ldots, u_a\}$, then $S(K_{a,b}) - S$ has at most b components of $K_{1,a}$. Thus, $I(S(K_{a,b})) = 2a+1$. □

Theorem 3.22. *Let $W_{1,n} (n \geq 3)$ be a star graph with $n+1$ vertices. Then*

$$I(S(W_{1,n})) = n+2.$$

Proof. Suppose we consider $S = V(W_{1,n})$. Then $S(W_{1,n}) - S = 2nK_1$. Thus, $I(S(W_{1,n})) = n+2$. □

Definition 3.23. *The complement [14] $\overline{G}$ of a graph G is a graph with the vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent in $\overline{G}$ if and only if they are not adjacent in G.*

Example 1. *Let $\overline{P_5}$ be the complement graph of a path P_5. Then $I(\overline{P_5}) = 4$.*

Illustration: Let $S \subset V(\overline{P_5})$. There are three cases to choose the set S (as shown in Figure 7).
Case 1. In Figure 7 (a), if we choose the set $S = \{a_1, a_2, a_3\}$, then we have $m(\overline{P_5} - S) = 2$. So $I(\overline{P_5}) = 3 + 2 = 5$.
Case 2. In Figure 7 (b), if we choose the set $S = \{a_1, a_2\}$, then we have $m(\overline{P_5} - S) = 3$. So $I(\overline{P_5}) = 2 + 3 = 5$.
Case 3. In Figure 7 (c), if we choose the set $S = \{a_1, a_2, a_3\}$, then we have $m(\overline{P_5} - S) = 1$. So $I(\overline{P_5}) = 3 + 1 = 4$.

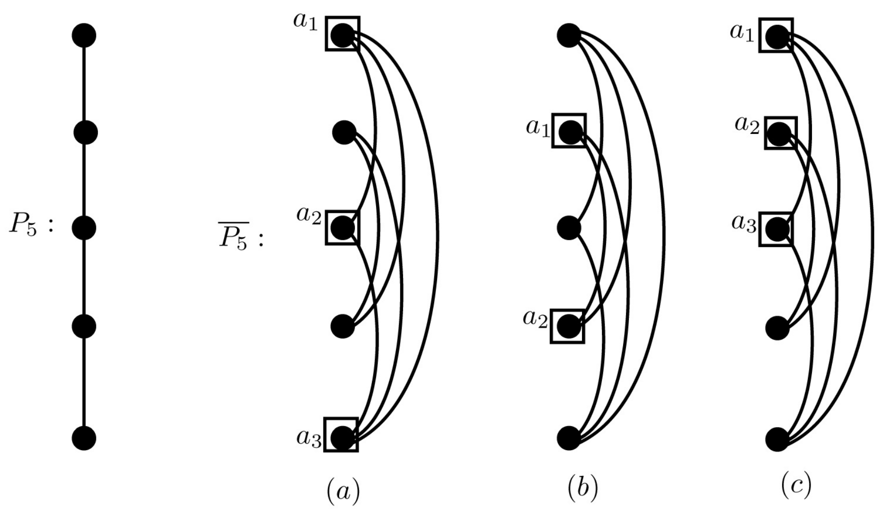

FIGURE 7. The path P_5 and its complement.

Theorem 3.24. *Let P_n be a path with n vertices. Then*

$$I(\overline{P_n}) = n - 1.$$

Proof. If the proof is done for n vertices according to the above example, then we choose $(n-2)$ vertices of P_n which form a path of length $(n-2)$ in P_n. Suppose we remove these $(n-2)$ vertices from $\overline{P_n}$, then we get two components of K_1. Thus, $I(\overline{P_n}) = n - 1$. □

Example 2. *Let $\overline{C_6}$ be the complement graph of a cycle C_6. Then $I(\overline{C_6}) = 5$.*

Illustration: Let $S \subset V(\overline{C_6})$. There are three cases to choose the set S (as shown in Figure 8).
Case 1. In Figure 8 (a), if we choose the set $S = \{a_1, a_2\}$, then we have $m(\overline{C_6} - S) = 4$. So

$I(\overline{C_6}) = 2 + 4 = 6$.

Case 2. In Figure 8 (b), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(\overline{C_6} - S) = 2$. So $I(\overline{C_6}) = 4 + 2 = 6$.

Case 3. In Figure 8 (c), if we choose the set $S = \{a_1, a_2, a_3, a_4\}$, then we have $m(\overline{C_6} - S) = 1$. So $I(\overline{C_6}) = 4 + 1 = 5$.

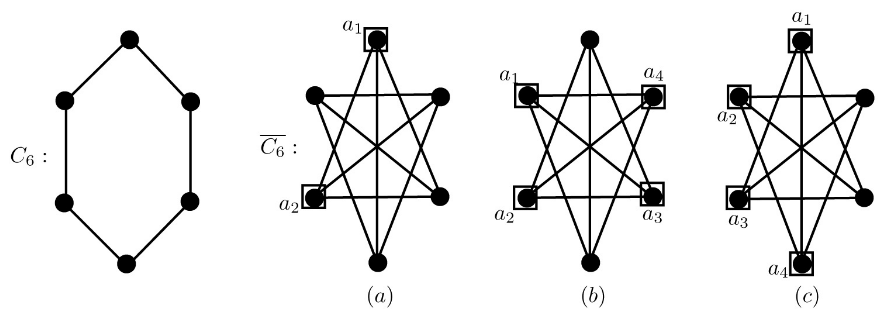

FIGURE 8. The cycle C_6 and its complement.

Theorem 3.25. *Let C_n be a cycle with n vertices. Then*

$$I(\overline{C_n}) = \begin{cases} 1 & if \quad n = 3, \\ 2 & if \quad n = 4, \\ n-1 & otherwise. \end{cases}$$

Proof. Suppose $n = 3$. Then $\overline{C_3}$ is same as $3K_1$ and the result follows from Theorem 2.1 (b). Suppose $n = 4$. Then $\overline{C_4}$ is same as $2K_2$ and by the definition of integrity, we have $I(\overline{C_4}) = 2$.

If the proof is done for $n(n \geq 5)$ vertices according to the above example, then we choose $(n-2)$ vertices of C_n which form a path of length $(n-2)$ in C_n. Suppose we remove these $(n-2)$ vertices from $\overline{C_n}$, then we get two components of K_1. Thus, $I(\overline{C_n}) = n - 1$. □

Theorem 3.26. *Let $K_{1,n}$ be a star graph with $n+1$ vertices. Then*

$$I(\overline{K_{1,n}}) = n.$$

Proof. The complement of a star graph $K_{1,n}$ is same as the graph $K_n \cup K_1$. Therefore, by Theorem 2.1 (a), we have $I(\overline{K_{1,n}}) = n$. □

Theorem 3.27. *Let $K_{a,b}(a, b \geq 2)$ be a complete bipartite graph with $a + b$ vertices. Then*

$$I(\overline{K_{a,b}}) = \max\{a, b\}.$$

Proof. The complement of a complete bipartite graph $K_{a,b}$ is same as the graph $K_a \cup K_b$. Therefore, $I(\overline{K_{a,b}}) = \max\{a, b\}$. □

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Theorem 3.28. *Let $W_{1,n}(n \geq 3)$ be a star graph with $n+1$ vertices. Then*

$$I(\overline{W_{1,n}}) = I(\overline{C_n}).$$

Proof. The complement of a wheel graph $W_{1,n}$ is same as the graph $\overline{C_n} \cup K_1$. Therefore, by Theorem 3.25, we have $I(\overline{W_{1,n}}) = I(\overline{C_n})$. □

The neighbourhood of a vertex $v \in V(G)$ is the set $N_G(v)$ consisting of all vertices u which are adjacent to v in G and the closed neighbourhood of a vertex $v \in V(G)$ is the set $N_G(v) \cup \{v\}$.

Definition 3.29. *The splitting graph [18] $S'(G)$ of a graph G is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent to every vertex that is adjacent to v.*

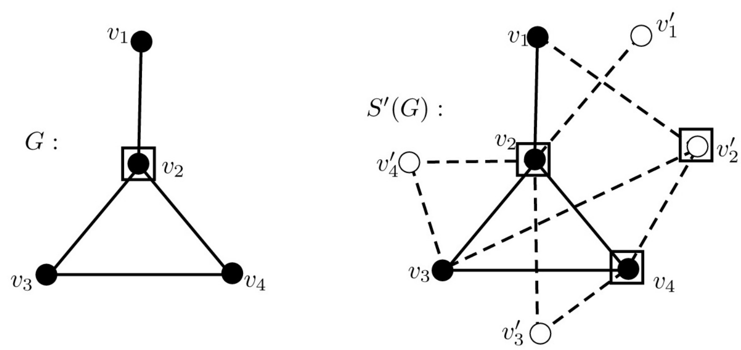

FIGURE 9. The graph G and its splitting graph $S'(G)$.

Theorem 3.30. *If G is any connected graph with n vertices, then*

$$I(S'(G)) \leq n+1.$$

Proof. The vertex set of $S'(G) = V(G) \cup V'(G)$. Suppose we consider $S = V(G)$. Then by the definition of the splitting graph of a graph, $S'(G) - S = |V(G)|K_1$. Thus, $|S| + m(S(G) - S) = n+1$. Therefore, $I(S'(G)) \leq n+1$. □

Proposition 3.31. *If G is any connected graph with n vertices, then*

$$I(S'(G)) = \begin{cases} n & \text{if } G \text{ is a path } P_n, \\ n+1 & \text{if } G \text{ is a cycle } C_n, \\ \min\{2\alpha_0(G)+1, 2I(G), n+1\} & \text{otherwise.} \end{cases}$$

Definition 3.32. *The total closed neighbourhood graph [15] $N_{tc}(G)$ of a graph G is the graph having vertex set $V(G) \cup V'(G)$ and two vertices are adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex v_i' of $V'(G)$ and the other to a vertex w_j of G and w_j is in $N_G[v_i]$.*

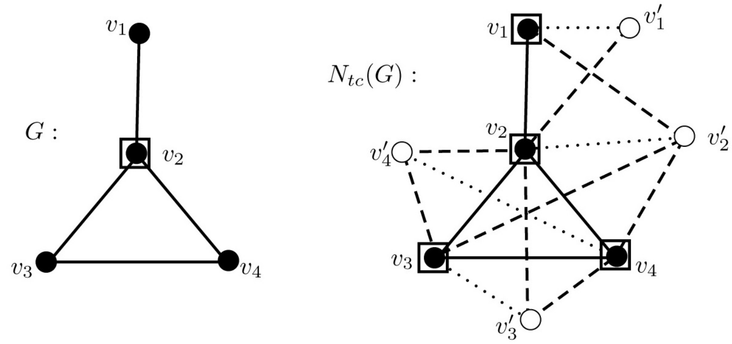

FIGURE 10. The graph G and its total closed neighbourhood graph $N_{tc}(G)$.

Theorem 3.33. *If G is any connected graph with n vertices, then*

$$I(N_{tc}(G)) \leq n+1.$$

Proof. The vertex set of $N_{tc}(G) = V(G) \cup V'(G)$. Suppose we consider $S = V(G)$. Then by the definition of the total closed neighbouhood graph of a graph, $N_{tc}(G) - S = |V(G)|K_1$. Thus, $|S| + m(S(G) - S) = n+1$. Therefore, $I(N_{tc}(G)) \leq n+1$. □

Proposition 3.34. *If G is any connected graph with n vertices, then*

$$I(N_{tc}(G)) = \begin{cases} n+1 & \text{if } G \text{ is a path } P_n \text{ or a cycle } C_n, \\ \min\{2(\alpha_0(G)+1), 2I(G), n+1\} & \text{otherwise.} \end{cases}$$

In search of triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [17] introduced the transformation graph as follows.

Definition 3.35. *Let G be a graph with vertex set $V = \{v_i : 1 \leq i \leq n\}$. The Mycielskian $\mu(G)$ of G is the graph obtained from G by adding $n+1$ new vertices $V' = \{v'_i : 1 \leq i \leq n\}$ and u, then for $1 \leq i \leq n$, joining v'_i to the neighbours of v_i and to u. v_i and v'_i are known as twins vertices, and V and V' are known as twin sets in $\mu(G)$. The vertex u is called the root of $\mu(G)$. Clearly, $V(\mu(G)) = V \cup V' \cup \{u\}$.*

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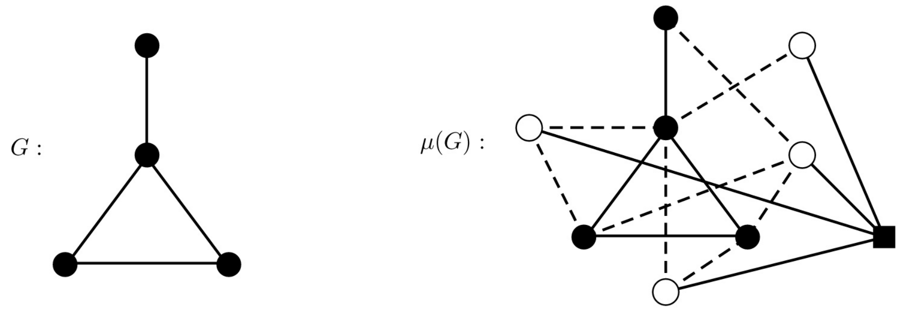

FIGURE 11. The graph G and the Mycielskian of G.

Theorem 3.36. *If G is any connected graph with n vertices, then*

$$I(\mu(G)) \leq n+2.$$

Proof. The vertex set of $\mu(G) = V(G) \cup V'(G) \cup \{u\}$. Suppose we consider $S = V(G) \cup \{u\}$. Then by the definition of the Mycielskian of a graph, $\mu(G) - S = |V(G)|K_1$. Thus, $|S| + m(S(G) - S) = n+2$. Therefore, $I(\mu(G)) \leq n+2$. □

Proposition 3.37. *If G is any connected graph with n vertices, then*

$$I(\mu(G)) = \begin{cases} n+1 & \text{if } G \text{ is a path } P_n, \\ n+2 & \text{if } G \text{ is a cycle } C_n, \\ \min\{2(\alpha_0(G)+1), 2I(G), n+2\} & \text{otherwise.} \end{cases}$$

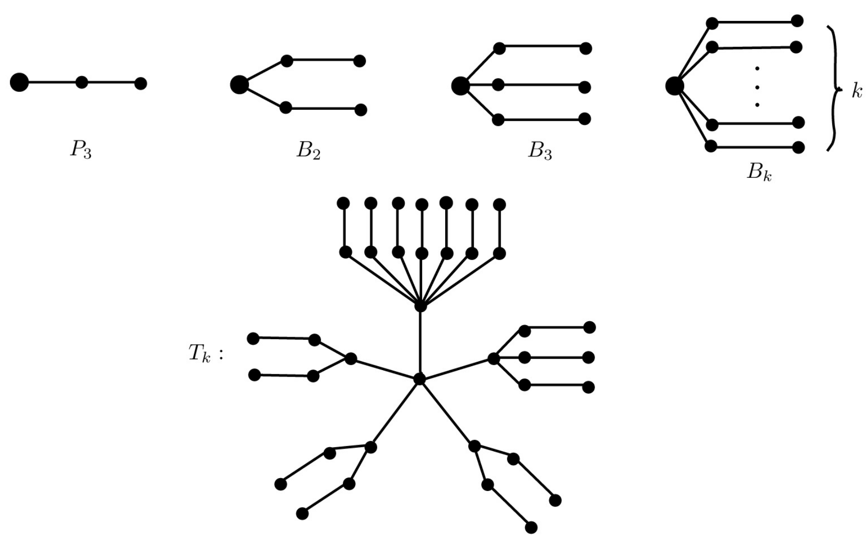

FIGURE 12. The rooted trees B_k's and the Kragujevac tree T_k.

Definition 3.38. [13] *Let P_3 be the 3-vertex tree rooted at one its terminal vertices. See Figure 12. For $k = 2, 3, \dots$ construct the rooted tree B_k by identifying the roots of k copies of P_3. The vertex obtained by identifying the roots of P_3-trees is the root of B_k. The illustrative structure of the rooted tree B_k is depicted in Figure 12.*

Definition 3.39. [13] *Let d be an integer and $\beta_1, \beta_2, \dots, \beta_d$ be rooted trees as specified in Definition 3.38, i.e., $\beta_1, \beta_2, \dots, \beta_d \in \{B_2, B_3, \dots\}$. A Kragujevac tree T_k is a tree possessing a vertex of degree d, adjacent to the roots of $\beta_1, \beta_2, \dots, \beta_d$. This vertex is said to be the central vertex of T_k. The subgraphs $\beta_1, \beta_2, \dots, \beta_d$ are the branches of T_k. Note that, some (or all) branches of T_k may be mutually isomorphic.*

Theorem 3.40. *Let T_k be the Kragujevac tree with $\beta_1, \beta_2, \dots, \beta_d$ branches. Then*

$$I(T_k) = d + 3.$$

Proof. If we consider the set S to be the collection of central vertex of T_k and the roots of all (i.e., d) the branches, then $T_k - S$ has all its components isomorphic to K_2. Thus, $I(T_k) = d+1+2 = d+3$. □

4. Conclusion

The integrity is one of the measures of graph vulnerability. The values of vulnerability helps the researchers to construct such a communication network which remains stable after some of its nodes or communication links are get defected. In this paper, we obtained the integrity values of some graph operations and some special graphs.

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