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EDGE-GROUP CHOOSABILITY OF OUTERPLANAR AND NEAR-OUTERPLANAR GRAPHS

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ABSTRACT. Let $\chi_{gl}(G)$ be the *group choice number* of G. A graph G is called *edge-k-group choosable* if its line graph is k-group choosable. The *group-choice index* of G, $\chi'_{gl}(G)$, is the smallest k such that G is edge-k-group choosable, that is, $\chi'_{gl}(G)$ is the group chice number of the line graph of G, $\chi_{gl}(\ell(G))$. It is proved that, if G is an outerplanar graph with maximum degree $\Delta < 5$, or if G is a $({K_2}^c + (K_1 \cup K_2))$-minor-free graph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$. As a straightforward consequence, every $K_{2,3}$-minor-free graph G or every K_4-minor-free graph G is edge-$(\Delta(G) + 1)$-group choosable. Moreover, it is proved that if G is an outerplanar graph with maximum degree $\Delta \geq 5$, then $\chi'_{gl}(G) \leq \Delta$.

1. Introduction

We consider only simple graphs in this paper unless otherwise stated. For a graph G, we denote its vertex set, edge set, minimum degree, and maximum degree by $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$, respectively. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. We denote the set of faces of a plane graph G by $F(G)$. For a plane graph G and $f \in F(G)$, we write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of f enumerated clockwise. The *degree of a face* is the number of edge-steps in the boundary walk. Let $d_G(x)$, or simply $d(x)$, denote the degree of a vertex (or face) x in G. A vertex (or face) of degree k is called a *k-vertex* (or *k-face*). A 3-face $f = [u_1u_2u_3]$ is called an *(i, j, k)-face* if $(d(u_1), d(u_2), d(u_3)) = (i, j, k)$. For $v \in V(G)$, $N_G(v)$ is the set of all vertices of G that are adjacent to v in G.

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A *k-coloring* of a graph G is a mapping ϕ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge xy. A graph G is *k-colorable* if it has a k-coloring. The *chromatic number* $\chi(G)$ is the smallest integer k such that G is k-colorable. A mapping L is said to be a *list assignment* for G if it supplies a list $L(v)$ of possible colors to each vertex v. A *k-list assignment* of G is a list assignment L with $|L(v)| = k$ for each vertex $v \in V(G)$. If G has some k-coloring ϕ such that $\phi(v) \in L(v)$ for each vertex v, then G is *L-colorable* or ϕ is an *L-coloring* of G. We say that G is *k-choosable* if it is L-colorable for every k-list assignment L. The *choice number* or *list chromatic number* $\chi_l(G)$ is the smallest k such that G is k-choosable. To distinguish the objects by different notions we denote the line graph of a graph G by $\ell(G)$. By considering colorings for $E(G)$, we can define analogous notions such as *edge-k-colorability*, *edge-k-choosability*, the *chromatic index* $\chi'(G)$, the *choice index* $\chi'_l(G)$, etc. Clearly, we have $\chi'(G) = \chi(\ell(G))$ and $\chi'_l(G) = \chi_l(\ell(G))$. The notion of list coloring of graphs has been introduced by Erdős, Rubin, and Taylor [8] and Vizing [17]. The following conjecture, which first appeared in [1], is well-known as the List Edge Coloring Conjecture.

Conjecture 1.1. *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

Although Conjecture 1.1 has been proved for a few special cases such as bipartite multigraphs [9], complete graphs of odd order [10], multicircuits [19], graphs with $\Delta(G) \geq 12$ that can be embedded in a surface of non-negative characteristic [2], and outerplanar graphs [18], it is regarded as very difficult. Vizing proposed the following weaker conjecture (see [13]).

Conjecture 1.2. *Every graph G is edge-$(\Delta(G) + 1)$-choosable.*

Assume A is an Abelian group and $F(G, A)$ denotes the set of all functions $f : E(G) \longrightarrow A$. Consider an arbitrary orientation of G. The graph G is *A-colorable* if for every $f \in F(G, A)$, there is a vertex coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for each directed edge from x to y. The *group chromatic number* of G, $\chi_g(G)$, is the minimum k such that G is A-colorable for any Abelian group A of order at least k. The notion of group coloring of graphs was first introduced by Jaeger et al. [12].

The concept of *group choosability* was introduced by Král and Nejedlý in [14] and some first results in this area were obtained in [4, 5, 15]. Let A be an Abelian group of order at least k and $L : V(G) \longrightarrow 2^A$ be a list assignment of G. Here 2^A is the set of subsets of A. For $f \in F(G, A)$, an *(A, L, f)-coloring* under an orientation D of G is an L-coloring $c : V(G) \longrightarrow A$ such that $c(x) - c(y) \neq f(xy)$ for every edge $e = xy$ which is directed from x to y. If for each $f \in F(G, A)$ there exists an (A, L, f)-coloring for G, then we say that G is *(A, L)-colorable*. The graph G is k-group choosable if G is (A, L)-colorable for each Abelian group A of order at least k and any k-list assignment $L : V(G) \longrightarrow \binom{A}{k}$. Here $\binom{A}{k}$ is the set of k-lists. The minimum k for which G is k-group choosable is called the *group choice number* of G and is denoted by $\chi_{gl}(G)$. It is clear that the concept of group choosability is independent of the orientation on G.

Graph G is called *edge-k-group choosable* if its line graph is k-group choosable. The *group-choice index* of G, $\chi'_{gl}(G)$, is the smallest k such that G is edge-k-group choosable, i.e. $\chi'_{gl}(G) = \chi_{gl}(\ell(G))$.

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It is easily seen that every even cycle is not edge-2-group choosable. This example shows that $\chi'_{gl}(G)$ is not generally equal to $\chi'(G)$. But we can extend the Vizing Conjecture as follows.

Conjecture 1.3. *If G is a multigraph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$.*

Since $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, as a sufficient condition, we have the following weaker conjecture.

Conjecture 1.4. *If G is a multigraph, then $\chi'_{gl}(G) \leq \chi'(G) + 1$.*

A planar graph is called *outerplanar* if it has a drawing on the plane such that each vertex lies on the boundary of the outer face. It is well-known that a graph is outerplanar if it contains neither K_4 nor $K_{2,3}$ as a minor (see for example [6]). In this paper, we see that Conjecture 1.3 is true for outerplanar graphs and *near-outerplanar* graphs, i.e. graphs that are either K_4-minor- free or $K_{2,3}$-minor-free.

2. Main Results

First, we have the following lemmas.

Lemma 2.1. *Let l be a natural number, v be a vertex of degree at most two of G and e be an edge incident with v. If $\chi'_{gl}(G-e) \leq \Delta(G) + l$, then $\chi'_{gl}(G) \leq \Delta(G) + l$.*

Proof. Let $\Delta = \Delta(G)$, D be an orientation of $\ell(G)$, A be an Abelian group of order at least $\Delta + l$, $L : V(\ell(G)) \longrightarrow \binom{A}{\Delta+l}$ be a $(\Delta + l)$-list assignment and $f \in F(\ell(G), A)$. Suppose that $G' = G - e$. Then $\ell(G') = \ell(G) - e$ and since $\chi'_{gl}(G') \leq \Delta + l$, there exists an (A, L, f)-coloring $c : V(\ell(G')) \longrightarrow A$. For each $e' \in N_{\ell(G)}(e)$ we can consider, without loss of generality, ee' to be directed from e to e'. Since v is a vertex of degree at most two, $d_{\ell(G)}(e) \leq \Delta$. This together with $|L(e)| = \Delta + l$ implies that $|L(e) - \{c(e') + f(ee') : e' \in N_{\ell(G)}(e)\}| \geq 1$ and so there is now at least one color available for e. Thus we can color all edges of G. This completes the proof of lemma. $\square$

An argument similar to the proof of Lemma 2.1 gives the following lemma.

Lemma 2.2. *Let G be a graph with $\chi'_{gl}(G-e) < \chi'_{gl}(G)$ for each $e \in E(G)$. Then $\delta(\ell(G)) \geq \chi'_{gl}(G)-1$.*

Lemma 2.3. [3] *If G is an outerplanar graph, then at least one of the following four cases holds.*

(1) *$\delta(G) = 1$,*
(2) *There exists an edge uv such that $d(u) = d(v) = 2$,*
(3) *There exists a 3-face $[uxy]$ such that $d(u) = 2$, $d(x) = 3$,*
(4) *G contains the graph G_1 consisting of two 3-faces $[xu_1v_1]$ and $[xu_2v_2]$ such that $d(u_1) = d(u_2) = 2$ and $d(x) = 4$ and these five vertices are all distinct.*

Using Lemma 2.3, we can prove that Conjecture 1.3 is true for outerplanar graphs. In fact, we have the following stronger result.

Theorem 2.4. *Let G be an outerplanar graph with maximum degree Δ. Then*

(a) *If $\Delta < 5$, then $\chi'_{gl}(G) \leq \Delta + 1$,*
(b) *If $\Delta \geq 5$, then $\chi'_{gl}(G) \leq \Delta$.*

Proof. We prove the second part of the theorem and leave the first part to the reader. Let G be a minimal counterexample to the assertion. Then there is an Abelian group A with $|A| \geq \Delta$, a Δ-assignment $L : V(\ell(G)) \longrightarrow \binom{A}{\Delta}$ and $f \in F(\ell(G), A)$ such that $\ell(G)$ is not (A, L, f)-colorable. By Lemma 2.3, we consider four cases as follows.

Case 1. $\delta(G) = 1$. Then G contains an edge uv such that $d(u) = 1$ and $d(v) \leq \Delta$. Hence $\ell(G)$ contains a vertex $e = uv$ with $d_{\ell(G)}(e) \leq \Delta - 1$, a contradiction by the first part of this theorem and Lemma 2.2.

Case 2. There exists an edge uv such that $d(u) = d(v) = 2$. Then $\ell(G)$ contains a vertex e with $d_{\ell(G)}(e) = 2$, a contradiction by the first part of this theorem and Lemma 2.2.

Case 3. There exists a 3-face $C : [uxy]$ such that $d(u) = 2$, $d(x) = 3$. Remove the vertices of $\ell(C)$ from $\ell(G)$ and color the remaining vertices of $\ell(G)$ from their lists. There are now at least $1, 2, 4$ colors available for the vertices of $\ell(C)$, respectively. It is easily seen that we can color these vertices. Thus G is not a counterexample, which is a contradiction.

Case 4. G contains the subgraph G_1 of Lemma 2.3 (see Figure 1). Remove the vertices of $\ell(G_1)$ from

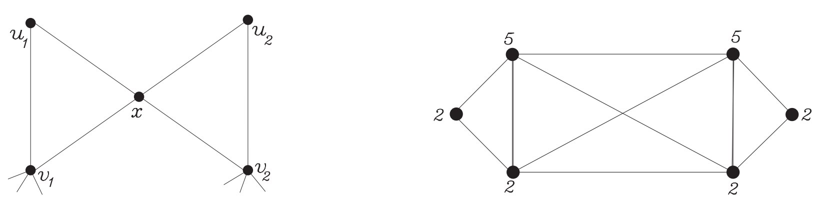

FIGURE 1. The graph G_1 and its line graph.

$\ell(G)$ and color the remaining vertices of $\ell(G)$ from their lists, which is possible by the minimality of G as a counterexample. There are now at least $2, 2, 2, 2, 5, 5$ colors available for the vertices of $\ell(G_1)$. In Figure 1, the label of each vertex is the number of available colors for it. First, we color the four vertices of $\ell(G_1)$ labeled 2 with two different available colors, alternatively. Then we color two vertices labeled 5 with two new distinct colors. Thus G is not a counterexample, which is a contradiction. $\square$

A graph is called *series-parallel* if it has no subgraph isomorphic to a subdivision of K_4. It is well-known that every simple series-parallel graph has a vertex of degree at most two [7]. Thus using Lemma 2.1, we have the following corollary.

Corollary 2.5. *If G is a simple series-parallel graph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$. In particular, every K_4-minor-free graph G is edge-$(\Delta(G) + 1)$-group choosable.*

In the following, we will prove that Conjecture 1.3 holds for $({K_2}^c \wedge (K_1 \cup K_2))$-minor-free graphs, where ${K_2}^c \wedge (K_1 \cup K_2)$ is the graph obtained from the union of K_1 and K_2 and joining them to ${K_2}^c$;

equivalently, it is the graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree 2. This implies that Conjecture 1.3 is true for $K_{2,3}$-minor-free graphs.

Lemma 2.6. [11] *Let G be a $({K_2}^c \wedge (K_1 \cup K_2))$-minor-free graph. Then each block of G is either K_4-minor-free or isomorphic to K_4.*

A graph G is called *D-group choosable* if it is (A, L)-colorable for every Abelian group A with $|A| \geq \Delta(G)$ and every list assignment $L : V(G) \longrightarrow 2^A$ with $|L(v)| = d(v)$ for each vertex v. There is a characterization of all D-group choosable graphs in [5] as follows.

Theorem 2.7. *A connected graph G is not D-group choosable if and only if every block of G is either complete or a cycle.*

By Theorem 2.7, it is easily seen that $\ell(K_4)$ is D-group choosable and so $\chi'_{gl}(K_4) \leq 4$. Moreover, it is well-known that a K_4-minor-free graph G with $|V(G)| \geq 4$ has at least two non-adjacent vertices with degree at most two [16]. By modifying some proofs in [11], we have the following result.

Theorem 2.8. *If G is a $({K_2}^c \wedge (K_1 \cup K_2))$-minor-free graph, then $\chi'_{gl}(G) \leq \Delta(G) + 1$. In particular, every $K_{2,3}$-minor-free graph G is edge-$(\Delta(G) + 1)$-group choosable.*

Proof. Let G be a minimal counterexample to Theorem 2.8. Then there is an Abelian group A with $|A| \geq \Delta(G)+1$, a $(\Delta(G)+1)$-assignment $L : V(\ell(G)) \longrightarrow \binom{A}{\Delta(G)+1}$ and $f \in F(\ell(G), A)$ such that $\ell(G)$ is not (A, L, f)-colorable. Clearly G is connected, $G \neq K_4$ and by Lemma 2.1, $\delta(G) \geq 3$. If $\Delta(G) = 3$, then G is 3-regular and not complete, a contradiction follows since $\ell(G)$ is 4-regular and so it is D-group choosable by Theorem 2.7. So we may assume that $\Delta(G) \geq 4$. By Lemma 2.6, every block of G is either K_4-minor-free or isomorphic to K_4. We first show that G is not 2-connected. Suppose on the contrary that G is 2-connected. Then G is K_4-minor free, by Lemma 2.6, since $\Delta(G) \geq 4$ and so by Corollary 2.5, $\chi'_{gl}(G) \leq \Delta(G) + 1$, a contradiction. Thus G is not 2-connected. Let B be an end-block of G with cut-vertex z_0. Clearly $B \ncong K_2$. Moreover, if $B \cong K_4$, it is easily seen that an (A, L, f)-coloring of $\ell(G) - V(\ell(B))$ can be extended to an (A, L, f)-coloring of $\ell(G)$. This contradiction shows that $B \ncong K_4$. Hence using Lemma 2.6, B is K_4-minor-free, and so it contains at least two vertices of degree at most 2. Hence G has at least one vertex of degree at most 2, this contradiction completes the proof of the theorem. $\square$

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