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ON CLIQUE VALUES IDENTITIES AND MANTEL-TYPE THEOREMS

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Abstract. In this paper, we first extend the weighted handshaking lemma, using a generalization of the concept of the degree of vertices to the values of graphs. This edge-version of the weighted handshaking lemma yields an immediate generalization of the Mantel's classical result which asks for the maximum number of edges in triangle-free graphs to the class of K_4-free graphs. Then, by defining the concept of value for cliques (complete subgraphs) of higher orders, we also extend the classical result of Mantel for any graph G. We finally conclude our paper with a discussion about the possible future works.

1. **Introduction**

The idea of counting subgraphs in a given graph to obtain some key invariants of the graph originates from *reconstruction* problems. The original reconstruction problem conjectured by Kelly and Ullam [2] simply states that any graph G with at least three vertices can be reconstructed form its *vertex-deck* $\{G-v\}_{v\in V}$. Unfortunately, it seems that finding a complete solution to this open problem is a very challenging issue and only some partial solutions have found. Therefore, it is reasonable to think about some alternative methods to tackle the problem. One such an alternative approach is to prove that some key invariants (parameters) of graphs are reconstructible from the vertex-deck rather than the graph itself.

In this paper, we choose this second approach by finding some subgraph-counting identities of particular type. Our main concentration will be on the class of those subgraphs which are isomorphic to *complete* subgraphs which we will call them *cliques*. These identities are also called the *clique value*

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identities. The main goal here is to obtain Mantel-type results which indicates a relationship between the number of the largest clique appear in a given graph G with the number of smaller cliques in G.

The paper will be organized, as follows. In the next section, we provide a review of the weighted handshaking lemma and the Mantel's theorem [3] for triangle-free graphs. Then, in section three, we first obtain a generalization of Mantel's result for the class of K_4-free graphs using the idea of generalizing the concept of the degree of a vertex to the value of an edge. We furthermore extend the Mantel's theorem for any given graph by introducing the new idea of the value of cliques of higher order. In the last section, we conclude the paper with a discussion about possible future research works.

2. Basic Definitions and Notations

Throughout this paper, we will assume that our graphs are simple, finite and undirected. For the terminology not given here, we invite the interested reader to consult the book [1].

For a given graph $G = (V, E)$ and a vertex $v \in V(G)$, the set of vertices adjacent to v is called the *open neighborhood* of v in G and will be denoted by $N_G(v)$.

The cardinality of $N_G(v)$ is called the *degree* of v and is denoted by $\deg_G(v)$. A complete subgraph of G is called a *clique* of G. A clique on k vertices is called a k-clique. A clique on three vertices is called a *triangle*. We will denote the set of triangles of G by $T(G)$. We denote the set of all k-cliques of G by $\Delta_k(G)$.

The number of k-cliques of G is denoted by $c_k(G)$. We also recall that the well-known *Cauchy-Schwarz* inequality [5] is the following inequality.

Lemma 2.1 (Cauchy-Schwarz Inequality)**.** *For any two real sequences $\{a_k\}_{k\geq 1}$ and $\{b_k\}_{k\geq 1}$, we have*

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \leq \Big(\sum_{k=1}^{n} a_k^2\Big)\Big(\sum_{k=1}^{n} b_k^2\Big) \tag{2.1}$$

In particular, we have

$$\frac{\Big(\sum_{k=1}^{n} a_k\Big)^2}{n} \leq \sum_{k=1}^{n} a_k^2. \tag{2.2}$$

In what follows, we quickly review the weighted-version of the well-known handshaking lemma and one of it's consequences which is known as Mantel's theorem for triangle-free graphs [3].

The *weighted-version* of the well-known *handshaking lemma* [6] can be read, as follows. From now on, we will denote the set of real numbers with $\mathbb{R}$.

Lemma 2.2 (Weighted Handshaking Lemma)**.** *Let $G = (V, E)$ be a graph and $f : V(G) \mapsto \mathbb{R}$ be a* weight *function on vertices of G. Then, we have*

(2.3)
$$\sum_{v\in V(G)} f(v)\deg_G(v) = \sum_{e=uv\in E(G)} \Big(f(u)+f(v)\Big).$$

In particular, we have

(2.4)
$$\sum_{v\in V(G)} \deg_G^2(v) = \sum_{e=uv\in E(G)} \Big(\deg_G(u)+\deg_G(v)\Big).$$

There are many interesting proofs of the following classical result of mantel in the literature [7] that can be regarded as one of the first result in *extremal graph* theory.

Theorem 2.3. [6, Mantel's theorem for K_3-free graphs] *Let $G=(V,E)$ be a triangle-free graph. Then, we have*

(2.5)
$$|E(G)| \le \frac{|V(G)|^2}{4}.$$

3. Clique Value Identities and Mantel-Type Results

In this section, we aim to obtain an extension of the classical result of Mantel based on a generalization of the concept of the *degree* of a vertex.

Definition 3.1. *Let $G=(V,E)$ be a graph and $e=uv$ be an edge of G. Then, we define the* edge value *of e, denoted by $val_G(e)$, as follows*

(3.1)
$$val_G(e) = |N_G(e)| = |N_G(u)\cap N_G(v)|.$$

Here, $N_G(e)$ denotes the set of common neighbors of end vertices of the edge e.

Remark 3.2. *It is worth to note that the similar notion of the value of a clique has been already appeared in the literature under the name of the co-degree of two given vertices (not necessarily adjacent).*

Next, we generalize the weighted handshaking lemma for values of edges of a given graph.

Lemma 3.3 (Weighted Edge Handshaking Lemma)**.** *Let $G=(V,E)$ be a graph and $g:\ E(G)\mapsto\mathbb{R}$ be a* weight *function on edges of G. Then, we have*

(3.2)
$$\sum_{e\in E(G)} g(e)val_G(e) = \sum_{\delta=e_1e_2e_3\in T(G)} \Big(g(e_1)+g(e_2)+g(e_3)\Big).$$

In particular, we have

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$$\sum_{e\in E(G)} val_G^2(e) = \sum_{\delta=e_1e_2e_3\in T(G)} \Big(val_G(e_1)+val_G(e_2)+val_G(e_3)\Big). \tag{3.3}$$

Now, using the above identity and a similar proof technique, one can prove the following interesting result. We recall that the notation $T(G)$ denotes the set of *triangles* of a given graph G.

Theorem 3.4 (Edge-Mantel's Theorem)**.** *Let $G=(V,E)$ be a K_4-free graph. Then, we have*

$$|T(G)| \leq \frac{|V(G)||E(G)|}{9}. \tag{3.4}$$

Proof. For any triangle $\delta = e_1e_2e_3 \in T(G)$, since G is a K_4-free graph we conclude that $N_G(e_i)$'s $(i=1,2,3)$ are clearly *pairwise disjoint*. Hence, by inclusion-exclusion principle, we get

$$\begin{aligned} val_G(e_1)+val_G(e_2)+val_G(e_3) &= |N_G(e_1)|+|N_G(e_2)|+|N_G(e_3)| \\ &= |N_G(e_1)\cup N_G(e_2)\cup N_G(e_3)| \\ &\leq |V(G)|. \end{aligned} \tag{3.5}$$

Thus, by considering inequality (3.5) and Cauchy-Schwarz inequality (2.2), we get

$$\begin{aligned} \frac{(3|T(G)|)^2}{|E(G)|} &= \frac{\Big(\sum_{e\in E(G)} val_G(e)\Big)^2}{|E(G)|} \\ &\leq \sum_{e\in E(G)} val_G^2(e) \\ &= \sum_{\delta=e_1e_2e_3\in T(G)} \Big(val_G(e_1)+val_G(e_2)+val_G(e_3)\Big) \\ &\leq \sum_{\delta=e_1e_2e_3\in T(G)} |V(G)| \\ &= |T(G)||V(G)|, \end{aligned}$$

which is equivalent to $|T(G)| \leq \frac{|V(G)||E(G)|}{9}$, as required. □

Recall that the well-known theorem of Turán [4] in extremal graph theory states the maximum number of edges in a graph G on n vertices with clique number $\omega(G)=\omega$ is equal to $\frac{n^2}{2}\Big(1-\frac{1}{\omega}\Big)$.

The next result is immediate based on the *Turán's graph theorem.* It is well-known as the *Zykov's theorem.*

Corollary 3.5. *Let G be a K_4-free graph with n vertices and t triangles. Then, we have*

$$t \leq \left(\frac{n}{3}\right)^3. \tag{3.6}$$

Remark 3.6. *It is worthy to note that Fisher [8] used more analytical methods to obtain an upper bound for the number of triangles in K_4-free graphs which only depends on the number of edges. Also Eckhoff [9] carried out a more careful analysis using structural properties of K_4-free graphs to maximize the number of triangles in terms of the number of edges over all K_4-free graphs. Even better results were obtained using more sophisticated tools like flag algebra [12]. But in contrast with them, our arguments are more combinatorial and hence much more easier to digest.*

At the end of this section, we attempt to find a more generalized version of mantel's theorem for any graph G. To this end, we first need to present an extension of the definition of the degree of a vertex to the value of a triangle or any clique of higher order.

Definition 3.7. *Let $G = (V, E)$ be a graph and $q_k \in \Delta_k(G)$ be a k-clique in G. Then, we define the value of the clique $q_k = v_{i_1} \cdots v_{i_k}$ denoted by $val_G(q_k)$, as follows*

$$val_G(q) = \Big| \bigcap_{i=1}^{k} N_G(v_i) \Big|. \tag{3.7}$$

Note that any k-clique $q_k = v_{i_1} \cdots v_{i_k} \in \Delta_k(G)$ in G can also be represented (uniquely) by $q_k = q_{k-1,1} \cdots q_{k-1,k}$ where for each $i = 1, \ldots, k$ the symbol $q_{k-1,i}$ denotes a $(k-1)$-clique subgraph of q_k. We will use this fact in our next key lemma.

Lemma 3.8 (Weighted Clique Handshaking Lemma)**.** *Let $G = (V, E)$ be a graph and let $h : \Delta_k(G) \mapsto \mathbb{R}$ $(k \geq 2)$ be a* weight *function on k-cliques of G. Then, we have*

$$\sum_{q_k \in \Delta_k(G)} h(q_k) val_G(q_k) = \sum_{q_{k+1} = q_{k,1} \cdots q_{k,k+1} \in \Delta_{k+1}(G)} \Big(h(q_{k,1}) + \cdots + h(q_{k,k+1}) \Big). \tag{3.8}$$

In particular, we have

$$\sum_{q_k \in \Delta_k(G)} val_G^2(q_k) = \sum_{q_{k+1} = q_{k,1} \cdots q_{k,k+1} \in \Delta_{k+1}(G)} \Big(val_G(q_{k,1}) + \cdots + val_G(q_{k,k+1}) \Big). \tag{3.9}$$

Proof. We proceed by defining the weighted *subclique-superclique* matrix $I_{f,k}(G)$ of order k, as follows

$$\big(I_{f,k}(G)\big)_{q_k, q_{k+1}} = \begin{cases} h(q_k) & \text{if } q_k \text{ is a subgraph of } q_{k+1}, \\ 0 & \text{otherwise}. \end{cases}$$

Next, we note that in the matrix $I_{f,k}(G)$ each row corresponding to the clique q_k has $val_G(q_k)$ non-zero entries. Hence, the resulting row-sum equals to $h(q_k) val_G(q_k)$. On the other hand, each column corresponding to the clique $q_{k+1} = q_{k,1} \cdots q_{k,k+1}$ has the column-sum $h(q_{k,1}) + \cdots + h(q_{k,k+1})$.

Thus, by summing over all rows and columns and equating them we get the desired result. □

Now, we are at the position to state the main result of this section. We recall that the number of k-cliques of G is denoted by $c_k(G)$.

Theorem 3.9. *Let $G = (V, E)$ be a $K_{\omega(G)+1}$-free graph. Then, we have*

$$c_{\omega(G)} \leq \frac{|V(G)|c_{\omega(G)-1}(G)}{\omega^2(G)}. \tag{3.10}$$

Proof. Put $\omega = \omega(G)$. Now let $q_\omega = q_{\omega-1,1} \cdots q_{\omega-1,\omega}$ be an ω-clique in G. Since G is a $K_{\omega+1}$-free graph, we conclude that $N_G(q_{\omega-1,i})$'s $(i = 1, 2, \ldots, \omega)$ are pairwise disjoint. Here by $N_G(q)$, we mean the set of common neighbors of the vertices of the clique q. Hence, by the inclusion-exclusion principle, we obtain

$$\begin{aligned} \sum_{i=1}^{\omega} val_G(q_{\omega-1,i}) &= \sum_{i=1}^{\omega} \left|N_G(q_{\omega-1,i})\right| \\ &= \left|\bigcup_{i=1}^{\omega} N_G(q_{\omega-1,i})\right| \\ &\leq |V(G)|. \end{aligned} \tag{3.11}$$

Thus, considering Cauchy-Schwarz inequality (2.2), identity (3.9) and inequality (3.11), we get

$$\begin{aligned} \frac{(\omega c_\omega)^2}{c_{\omega-1}} &= \frac{\left(\sum_{q_{\omega-1}\in\Delta_{\omega-1}(G)} val_G(q_{\omega-1})\right)^2}{c_{\omega-1}} \\ &\leq \sum_{q_{\omega-1}\in\Delta_{\omega-1}(G)} val_G^2(q_{\omega-1}) \\ &= \sum_{q_\omega\in\Delta_\omega(G)} \left(val_G(q_{\omega-1,1}) + \cdots + val_G(q_{\omega-1,\omega})\right) \\ &\leq \sum_{q_\omega\in\Delta_\omega(G)} |V(G)| \\ &= c_\omega|V(G)|, \end{aligned}$$

which is clearly equivalent to $c_\omega \leq \frac{|V(G)|c_{\omega-1}}{\omega^2}$, as required. □

Remark 3.10. *It is important to note that Fisher and Ryan [10] obtained an upper-bound for the number of complete subgraphs using some analytical tools and structural graph analysis. Also Nikiforov [11] found an interesting lower-bound for the number of cliques of a graph of given order and size, using constraint minimization of certain multilinear forms. As mentioned by Razborov in [12], all those proofs including his arguments have obtained using various analytical methods. In contrast, our arguments are only based on counting tools and Cauchy-Schwarz inequality and therefore are more comprehensive.*

4. **Conclusion and Future Works**

In this paper, we mainly concentrate on the problem of identities related to cliques. The main ingredients of the proof were weighted-version of clique handshaking lemma, the property of being k-clique free (for some given natural number k) and the Cauchy-Schwarz inequality.

It seems that one possible direction for extending the result of the research in this paper is to generalize the idea of clique handshaking identity to other classes of subgraphs. To do this, we first introduce a new family of subgraphs that we call them the *Join* family of graphs, as follows. Recall that for two given graphs G_1 and G_2, the join of them denoted by $G_1 \vee G_2$ is defined as a new graph obtained from G_1 and G_2 by adding any vertex of G_1 to G_2 and vice versa.

Definition 4.1. *The family* $\mathbf{J} = \{J_k\}_{k \geq 1}$ *of graphs is called a* join *family if for each* $k > 1$ *we have*

$$J_k = J_{k-1} \vee \{v\}. \tag{4.1}$$

Indeed, we define this family recursively by starting from a given graph J_1 *and then we add a new vertex and connect it to all vertices of the previously constructed member of the family.*

We note that in the above definition if we choose the initial graph J_1 to be the graph K_1, then we clearly obtain the family of k-cliques graphs $(k \geq 1)$.

In the next step, for any graph G, we define the value of any member $J_k \in \mathbf{J}$ of the join family of graphs $\mathbf{J}$ in G in analogy with the value of any $k - clique$ by

$$val_G(J_k) = \Big| \bigcap_{v \in V(J_k)} N_G(v) \Big|, \tag{4.2}$$

where $V(J_k)$ denotes the vertex-set of J_k. Finally, we need to find a weighted-version of subgraph-counting identity for this new class of graphs $\mathbf{J}$ which we will call it *weighted join value handshaking* lemma based on the idea of double-counting technique. Having this identity in hand, one can obtain a Mantel-type theorem for the class of $\mathbf{J}$-free graphs (join-free graphs). Here by the class of join-free graphs, we mean that any graph G in this class has the following property:

> There exists an integer $k > 1$ for which J_k is a subgraph of G but J_{k+1} is not the subgraph of G. We will call the number k the *join number* of G and we denote it by $join(G)$.

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