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FURTHER RESULTS ON MAXIMAL RAINBOW DOMINATION NUMBER

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ABSTRACT. A *2-rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A *maximal 2-rainbow dominating function* of a graph G is a 2-rainbow dominating function f such that the set $\{w(G)|f(w) = \emptyset\}$ is not a dominating set of G. The *weight* of a maximal 2RDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *maximal 2-rainbow domination number* of a graph G, denoted by $\gamma_{m2r}(G)$, is the minimum weight of a maximal 2RDF of G. In this paper, we continue the study of maximal 2-rainbow domination number in graphs. Specially, we first characterize all graphs with large maximal 2-rainbow domination number. Finally, we determine the maximal 2-rainbow domination number in the sun and sunlet graphs.

1. Introduction

Domination is a very important part in graph theory. Two books by Haynes et al, [10, 11], provide a comprehensive treatment of the general results on domination in graphs. A *dominating set* of a graph G is a subset S of vertices of G such that every vertex not in S is adjacent to at least one vertex in S. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. There exist several variations of domination.

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We begin by establishing the basic terminology and notations which is used throughout the article. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg(v) = |N(v)|$. The *minimum* and *maximum* degrees of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. A *clique* of a graph G is a complete subgraph of G, and the clique of largest possible size is referred to as a *maximum clique*. The *clique number* of a graph G, denoted $\omega(G)$, is the number of vertices in a maximum clique of G. Equivalently, it is the size of a largest clique or maximal clique of G.

A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. We use C_n for a cycle of n vertices, P_n for a path of n vertices and K_n for a complete graph of n vertices. The diameter of G, denoted by $\text{diam}(G)$, is the maximum value among minimum distances between all pairs of vertices of G. If the graph is not connected, we define $\text{diam}(G) = \infty$. The *girth* of G, denoted by $g(G)$, is the minimum length of a cycle in G. A matching M of a graph G is a subset of the edges E, such that no two edges in M have a common vertex.

Let f be a function that assigns to each vertex of a graph G a subset of colors chosen from the set $\{1, 2, \ldots, k\}$. If for any vertex $v \in V$ with $f(v) = \emptyset$ we have $\bigcup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$, then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight*, $\omega(f)$, of a function f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *k-rainbow domination number* of a graph G, $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$-*function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Clearly $\gamma_{r1}(G)$ is the usual domination number $\gamma(G)$. The k-rainbow domination was introduced by Brešar et al. [7] and this concept coincides with the study of usual domination of the Cartesian product of any graph with the complete graph (see [8, 9]). Rainbow domination number in graphs is now very well studied (see for example [1, 2, 4, 5, 6, 12]).

A 2-rainbow dominating function (2RDF) $f : V \to \mathcal{P}(\{1, 2\})$ can be presented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ of the vertex set V, where $V_0 = \{v \in V | f(v) = \emptyset\}$, $V_1 = \{v \in V | f(v) = \{1\}\}$, $V_2 = \{v \in V | f(v) = \{2\}\}$ and $V_{1,2} = \{v \in V | f(v) = \{1, 2\}\}$. In this representation, the weight of f is $\omega(f) = |V_1| + |V_2| + |V_{1,2}|$. In this sense, a *maximal 2-rainbow dominating function* (M2RDF) of a graph G is a 2-rainbow dominating function such that V_0 is not a dominating set of G. The *maximal 2-rainbow domination number* of G, denoted by $\gamma_{m2r}(G)$, equals the minimum weight of an M2RDF of G. A $\gamma_{m2r}(G)$-*function* is a maximal 2-rainbow dominating function of G with weight $\gamma_{m2r}(G)$. Maximal 2-rainbow domination number was introduced in [3].

It was observed in [3] that for any graph G with order n,

$$\gamma_{2r}(G) \leq \gamma_{m2r}(G) \leq n. \tag{1.1}$$

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The authors of [3] showed that the decision problem, regarding the maximal 2-rainbow domination number, is NP-complete even when restricted to bipartite or chordal graphs. From this, determining the exact value of maximal 2-rainbow domination number of a graph is not so easy.

In this paper, we continue the study of maximal 2-rainbow domination number in graphs. This paper is organized as follows: in Section 2, we characterize all graphs with large maximal 2-rainbow domination number. Finally, in section 3, we determine the maximal 2- rainbow domination number in the sun and sunlet graphs.

We make use of the following results in this paper. The proof of the following results can be found in [3].

Observation 1.1. For $n \geq 2$, $\gamma_{m2r}(P_n) = \lceil \frac{n+1}{2} \rceil$ if n is even and $\gamma_{m2r}(P_n) = \lceil \frac{n+1}{2} \rceil + 1$ if n is odd.

Observation 1.2. For $n \geq 3$, $\gamma_{m2r}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$ if $n \equiv 2 \pmod 4$ and $\gamma_{m2r}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor + 1$ if $n \equiv 0, 1, 3 \pmod 4$.

Observation 1.3. For $n \geq 4$ and any non-empty matching M of K_n, $\gamma_{m2r}(K_n - M) = n - 1$.

Observation 1.4. Let G be a connected graph G of order $n \geq 2$. Then $\gamma_{m2r}(G) = n$ if and only if $G = K_2, P_3, C_3$ or $G = K_n$.

2. Classifying all graphs G for which $\gamma_{m2r}(G) = |V(G)| - 1$

It is shown in [3] that for any connected graph G of order $n \geq 2$, $\gamma_{m2r}(G) = n$ if and only if $G = K_n$, P_2, P_3, or C_3. In this section we characterize all graphs G of order n for which $\gamma_{m2r}(G) = n - 1$. We start with a simple but sharp bound.

Lemma 2.1. Let G be a connected graph of order $n \geq 4$. Then $\gamma_{m2r}(G) \leq n - \left\lfloor \frac{\mathrm{diam}(G)-1}{2} \right\rfloor$.

Proof. Let $\mathrm{diam}(G) = d$ and $P := x_0 x_1 \cdots x_d$ be a diametral path in G. Define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(x_i) = \emptyset$ for $i \equiv 1 \pmod 2$ and $1 \leq i \leq d - 2$, $f(x_i) = \{2\}$ for $i \equiv 2 \pmod 4$, and $f(x) = 1$ otherwise. It is easy to verify that f is a M2RDF of G and hence $\gamma_{m2r}(G) \leq n - \left\lfloor \frac{\mathrm{diam}(G)-1}{2} \right\rfloor$. $\square$

Lemma 2.2. Let G be a connected graph of order $n \geq 4$ different from C_n. If $g(G) \geq 4$, then $\gamma_{m2r}(G) \leq n - 2$.

Proof. Let $C := (x_1 x_2 \cdots x_g)$ be a cycle of G of length $g(G)$. Since $G \neq C_n$, without loss of generality, we may assume that $\deg(x_1) \geq 3$. Define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(x_1) = \{1\}, f(x_2) = f(x_g) = \emptyset$ and $f(x) = 2$ otherwise. Clearly, f is a M2RDF of G of weight at most $n - 2$ and this implies that $\gamma_{m2r}(G) \leq n - 2$. $\square$

Theorem 2.3. Let T be a tree of order $n \geq 4$. Then $\gamma_{m2r}(T) = n - 1$ if and only if $T \in \{P_4, P_5, K_{1,3}, S(2, 1), S(2, 2)\}$.

Proof. One side is clear. To prove the other side, let $v \in V(T)$ be a vertex of maximum degree and let $N(v) = \{v_1, v_2, \ldots, v_k\}$. If $k \geq 4$, then the function $f : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f(v) = \{1,2\}, f(x_i) = \emptyset$ for $1 \leq i \leq k-1$ and $f(x) = \{1\}$ otherwise, is clearly a M2RDF of T of weight at most $n-2$ which is a contradiction. Thus $k \leq 3$. If $k = 2$, then T is a path and we conclude from Observation 1.1 that $T = P_4$ or P_5. Assume that $k = 3$. Suppose without loss of generality that $\deg(x_3) \geq \deg(x_2) \geq \deg(x_1)$. If $\deg(x_3) \geq \deg(x_2) \geq 2$, then the function $f : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f(v) = \{1\}, f(x_2) = f(x_3) = \emptyset$ and $f(x) = \{2\}$ otherwise, is obviously a M2RDF of T of weight $n-2$, a contradiction again. Thus $\deg(x_2) = \deg(x_1) = 1$. If $\deg(x_3) = 1$, then $T = K_{1,3}$. Now let $\deg(x_3) \geq 2$. Since $\Delta(T) \leq 3$, we have $\deg(x_3) = 2$ or 3. If there is a leaf at distance at least three from v, then the function $f : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f(v) = \{1,2\}, f(x_1) = f(x_2) = f(x_3) = \emptyset$ and $f(x) = \{1\}$ otherwise, is clearly a M2RDF of T of weight at most $n-2$, a contradiction. Thus all leaves are at distance at most two from v. It follows that $T = S(2,1)$ or $T = S(2,2)$ and the proof is complete. □

Theorem 2.4. Let G be a connected graph G of order $n \geq 4$ with clique number at least four. Then $\gamma_{m2r}(G) = n-1$ if and only if $G = K_n - M$ where M is a non-empty matching of G.

Proof. By Observation 1.3, we only need to prove necessity. Let $\gamma_{m2r}(G) = n-1$ and let $\{x_1, x_2, \ldots, x_\omega\}$ be the vertex set of a maximum clique in G. Since $\gamma_{m2r}(K_n) = n$, we have $\omega < n$. If there is a vertex $x \in V(G)$ which is not adjacent to at least two vertices in $\{x_1, x_2, \ldots, x_\omega\}$, say $x_\omega, x_{\omega-1}$, then the function $f : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}, f(x_\omega) = f(x_{\omega-1}) = \emptyset$ and $f(u) = \{2\}$ otherwise, is obviously a M2RDF of T of weight at most $n-2$ which is a contradiction. Henceforth, every vertex in $V(G) - \{x_1, x_2, \ldots, x_\omega\}$ is adjacent to all vertices in $\{x_1, x_2, \ldots, x_\omega\}$ but one. If there are two vertices $x, y \in V(G) - \{x_1, x_2, \ldots, x_\omega\}$ which are not adjacent to some x_i, say x_n, then the function $f : V(T) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}, f(x) = f(y) = \emptyset$ and $f(u) = \{2\}$ otherwise, is clearly a M2RDF of T of weight at most $n-2$ which is a contradiction again. This implies that each x_i is adjacent to all vertices of G but at most one for $i = 1, 2, \ldots, \omega$. Now, if there are two non-adjacent vertices $y, z \in V(G) - \{x_1, x_2, \ldots, x_\omega\}$, then let $x_\omega y \notin E(G)$, $x_{\omega-1}z \notin E(G)$ and define $f : V(T) \to \mathcal{P}(\{1,2\})$ by $f(x_1) = \{1\}, f(x_\omega) = f(z) = \emptyset$ and $f(u) = \{2\}$ otherwise. It is easy to see that f is a M2RDF of T of weight at most $n-2$ which is a contradiction. Thus the subgraph induced by $V(G) - \{x_1, x_2, \ldots, x_\omega\}$ is a complete graph and hence each vertex in $V(G) - \{x_1, x_2, \ldots, x_\omega\}$ is adjacent to all vertices of G but one. All in all, we conclude that $G = K_n - M$ where M is a non-empty matching of G. □

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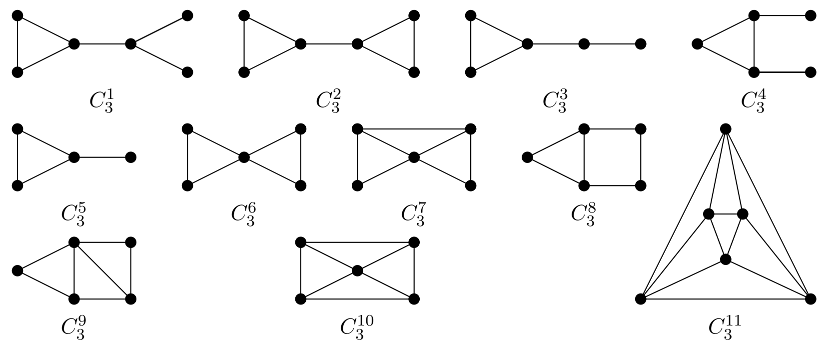

FIGURE 1. Family $\mathcal{F}_3$

Let $\mathcal{F}_1 = \{K_n - M \mid n \geq 4$ and M is a non − empty matching of $G\}$, let $\mathcal{F}_2 = \{C_5, K_{1,3}, P_4, P_5, S(1,2), S(2,2)\}$ and let $\mathcal{F}_3$ be the family of graphs illustrated in Figure 1. An *universal vertex* is a vertex adjacent to every vertex of the graph.

Theorem 2.5. Let G be a connected graph G of order $n \geq 4$ with clique number at most three. Then $\gamma_{m2r}(G) = n - 1$ if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Proof. If $G \in \mathcal{F}_1$, then $\gamma_{m2r}(G) = n - 1$ by Observation 1.3, and if $G \in \mathcal{F}_2$, then $\gamma_{m2r}(G) = n - 1$ by Lemma 2.3 and Observation 1.2. Let $G \in \mathcal{F}_3$. If $G \in \mathcal{F}_3 - \{C_3^5, C_3^6, C_3^7, C_3^9, C_3^{10}\}$, then clearly G has no universal vertex, and this implies $\gamma_{m2r}(G) \geq n - 1$. It follows from Observation 1.4 that $\gamma_{m2r}(G) = n - 1$. Finally let $G \in \{C_3^5, C_3^6, C_3^7, C_3^9, C_3^{10}\}$. Using Observation 1.4, we have that $\gamma_{m2r}(G) \leq n - 1$. Clearly G has a universal vertex, say x, thus x will be assigned by $\{1\}, \{2\}$ or $\{1, 2\}$. On the other hand, $G - x \in \{K_2 \cup K_1, K_2 \cup K_2, P_4, P_4, C_4\}$, thus $\gamma_{m2r}(G - x) \geq n - 2$. By this assumption, we conclude that $\gamma_{m2r}(G) \geq n - 1$. Therefore $\gamma_{m2r}(G) = n - 1$.

Conversely, let $\gamma_{m2r}(G) = n - 1$. If G is acyclic, then the result follows from Theorem 2.3. Assume that G has a cycle. If $G = C_n$, then we have $G = C_4$ or C_5 by Observation 1.2. Let $G \neq C_n$ and let $C_g := (x_1 x_2 \ldots x_g)$ be a cycle of length $g(G)$ in G. If $g(G) \geq 4$, then we have $G = K_n - M$ for some non-empty matching M of G by Lemma 2.2. Assume that $g(G) = 3$. If there is a vertex x with $d(x, V(C_g)) \geq 3$ and $x_1 z_1 z_2 \ldots z_k = x$ is a shortest $(x, V(C_g))$-path in G, then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(x_1) = \{1, 2\}$, $f(x_2) = f(x_3) = f(z_1) = \emptyset$ and $f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most $n - 2$ which is a contradiction. Hence, $d(x, V(C_g)) \leq 2$ for each $x \in V(G)$. We consider two cases.

Case 1. There exists a vertex $x \in V(G)$ with $d(x, V(C_g)) = 2$.
Let $x_1 z_1 z_2 = x$ is a shortest $(x, V(C_g))$-path in G. If $\deg(x_i) \geq 3$ for some $i = 2, 3$, say $i = 2$, then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(x_3) = \{2\}$, $f(x_1) = f(x_2) = \emptyset$ and $f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most $n - 2$ which is a contradiction. Therefore $\deg(x_2) = \deg(x_3) = 2$. We claim that $\deg(x_1) = 3$. Assume to the contrary that $\deg(x_1) \geq 4$ and let $y \in N(x_1) - \{x_2, x_3, z_1\}$. If $yz_2 \in E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(x_1) = \{2\}$, $f(y) = f(z_1) = \emptyset$ and

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$f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most $n-2$, a contradiction. If $yz_2 \notin E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1,2\}$, $f(y) = f(x_2) = f(x_3) = \emptyset$ and $f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most $n-2$, a contradiction. Thus $\deg(x_1) = 3$. If $\deg(z_1) \geq 4$ and $y_1, y_2 \in N(z_1) - \{x_1, z_2\}$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(z_1) = \{1,2\}$, $f(y_1) = f(y_2) = f(z_2) = \emptyset$ and $f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most $n-2$, a contradiction. Hence $\deg(z_1) \leq 3$. This implies that $G \in \{C_3^1, C_3^2, C_3^3\}$.

Case 2. $d(x, V(C_g)) = 1$ for each $x \in V(G) - \{x_1, x_2, x_3\}$.
Since the clique number of G is at most three, every vertex in $V(G) - \{x_1, x_2, x_3\}$ has at most two neighbors in $\{x_1, x_2, x_3\}$. Consider two subcases.

Subcase 2.1. There exists a vertex $x \in V(G) - V(C_g)$ with exactly one neighbor in $V(C_g)$.
Assume without loss of generality that $xx_1 \in V(G)$. If $\deg(x_2), \deg(x_3) \geq 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}$, $f(x_2) = f(x_3) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$ which is a contradiction. Hence, we may assume that $\deg(x_2) = 2$. If there is a vertex $y \in N(x_1) - \{x, x_2, x_3\}$ with $yx \notin E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1,2\}$, $f(y) = f(x_2) = f(x_3) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$ which is a contradiction again. Thus each neighbor of x_1 not in $\{x, x_2, x_3\}$ is adjacent to x. If $\deg(x_1) \geq 5$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}$, $f(y) = \emptyset$ for $y \in N(x_1) - \{x, x_2, x_3\}$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$, a contradiction. Therefore $\deg(x_1) \leq 4$.

First let $\deg(x) = 1$. It follows that $\deg(x_1) = 3$. If $\deg(x_3) \geq 4$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_3) = \{1,2\}$, $f(y) = \emptyset$ for $y \in N(x_3) - \{x_1\}$ and $f(u) = \{1\}$ otherwise, is a M2RDF of G of weight at most at most $n-2$ which is a contradiction. Thus $2 \leq \deg(x_3) \leq 3$ and so $G = G_3^4$ or G_3^5.

Now let $\deg(x) \geq 2$. If $\deg(x) \geq 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x) = \{1\}$, $f(y) = \emptyset$ for $y \in N(x) - \{x_1\}$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most at most $n-2$ which is a contradiction. Assume that $\deg(x) = 2$ and let $xz \in E(G)$. Since $d(z, V(C_g)) = 1$, we have $|N(z) \cap \{x_1, x_3\}| \geq 1$. We distinguish the following.

(a): $zx_1 \in E(G)$.
Then we have $\deg(x_1) = 4$. If $n = 5$, then clearly $G = G_3^6$ or G_3^7. Let $n \geq 6$ and let $u \in V(G) - \{x_1, x_2, x_3, x, z\}$. Considering Case 2, we may assume that $d(u, \{z, x, x_1\}) = 1$. This implies that $uz, ux_3 \in E(G)$. But then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}$, $f(x) = f(x_2) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$ which leads to a contradiction.

(b): $zx_1 \notin E(G)$.
Then we must have $zx_3 \in E(G)$. If $\deg(z) \geq 3$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(z) = \{1\}$, $f(y) = \emptyset$ for $y \in N(z) - \{x_3\}$ and $f(u) = \{2\}$ otherwise, is a M2RDF

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of G of weight at most $n-2$ which is a contradiction. Hence $\deg(z) = 2$. As above, we can see that $\deg(x_3) = 3$ and so $G = G_3^8$.

Subcase 2.2. Every vertex $V(G) - V(C_g)$ has exactly two neighbors in $V(C_g)$.
If x_i, x_j $(i \neq j)$ have two common neighbors, say y, z, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_i) = \{1\}$, $f(y) = f(z) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$ which is a contradiction. Thus we may assume that x_i, x_j have at most one common neighbor if $i \neq j$. This implies that $n \leq 6$ and $\deg(x_i) = 4$ for $i = 1, 2, 3$. If $n = 4$, then clearly $G = K_4 - e$ and if $n = 5$, then obviously $G = G_3^9$ or G_3^{10}. Let $n = 6$. Then x_i, x_j $(i \neq j)$ have exactly one common neighbor. Assume that y_i is the common neighbor of x_i, x_{i+1} where the sum are taken modulo 3. If the induced subgraph $G[\{y_1, y_2, y_3\}]$ has an isolated vertex, say y_3, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_2) = \{1\}$, $f(y_1) = f(y_2) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$ which is a contradiction. Thus we may assume that $y_1y_2, y_2y_3 \in E(G)$. If $y_1y_3 \notin E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1\}$, $f(y_1) = f(x_2) = \emptyset$ and $f(u) = \{2\}$ otherwise, is a M2RDF of G of weight at most $n-2$, a contradiction. Thus $y_1y_3 \in E(G)$ and hence $G = G_3^{11}$. This completes the proof. □

3. The sun and the sunlet graphs

A *sun* (or *trampoline*) S_n is a chordal graph on $2n$ vertices, where $n \geq 3$, whose vertex set can be partitioned into two sets $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, such that Y is an independent set and x_i is adjacent to y_j if and only if $i = j$ or $i = j+1$, $i, j \in \{0, 1, \ldots, n-1\}$. Notice that, according to the chordality property, there are several other edges between vertices in X. For instance, each x_i is adjacent to x_{i+1}. Moreover, if X induces a complete graph, then the graph is called the *complete sun*. All the operations on the subscripts of vertices in S_n are taken modulo n.

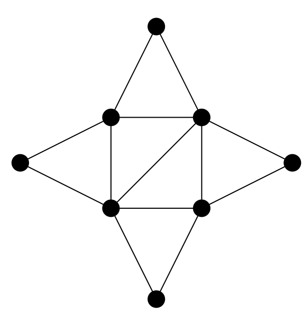

FIGURE 2. Example of a sun graph

In the case of the 2-rainbow domination number of sun graphs, Brešar et al. proved the following in [8].

Theorem 3.1. [8] For any sun graph S_n of order $2n$, $\gamma_{r2}(S_n) = n$.

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From Theorem 3.1 and equation (1.1) we know that the maximal 2-rainbow domination number of sun graph S_n is at least n. Next result shows that the value of the maximal 2-rainbow domination number of S_n is more than n, that is $n+1$.

Theorem 3.2. For any sun graph S_n of order $2n$, $\gamma_{m2r}(S_n) = n+1$.

Proof. Define $f : V(S_n) \to \mathcal{P}(\{1,2\})$ by $f(y_0) = \{1\}$, $f(x_i) = \{1\}$ if $i \equiv 0 \pmod 2$, $f(x_i) = \{2\}$ if $i \equiv 1 \pmod 2$, and $f(x) = \emptyset$ otherwise. Clearly, f is a M2RDF of S_n of weight $n-1$ and so $\gamma_{m2r}(S_n) \le n+1$. Using Theorem 3.1 and equation (1.1) we obtain $n \le \gamma_{m2r}(S_n) \le n+1$.

Now, let $g' = (V'_0, V'_1, V'_2, V'_{1,2})$ be a $\gamma_{m2r}(S_n)$-function. So, there exists x_i or y_j, $i,j \in \{0,1,\ldots,n-1\}$, which is not dominated by V'_0. If x_i is such a vertex, then we have $\{x_i, x_{i-1}, x_{i+1}, y_{i-1}, y_i\} \cap V'_0 = \emptyset$, which implies that also y_{i-1} and y_i are not dominated by V'_0. Moreover $|g'(x_i)| = 1$, otherwise $(V'_0 \cup \{y_{i-1}\}, V'_1 - \{y_{i-1}\}, V'_2 - \{y_{i-1}\}, V'_{1,2})$ is a M2RDF of S_n with weight less than $\gamma_{m2r}(S_n)$, a contradiction. Consider a function $g = (V_0, V_1, V_2, V_{1,2})$ of S_n defined as follows:

$$g(v) = \begin{cases} \emptyset & \text{if } v = y_{i-1} \\ \{1,2\} & \text{if } v = x_i \\ g'(v) & \text{otherwise.} \end{cases}$$

Notice that g is a M2RDF of S_n with weight $\omega(g) = \omega(g')$, so g is also a $\gamma_{m2r}(S_n)$-function. In this sense, from now on we investigate a $\gamma_{m2r}(S_n)$-function g that every x_i, $i \in \{0,1,\ldots,n-1\}$, is dominated by V_0^g.

Suppose that $\gamma_{m2r}(S_n) = n$. Let $Y = Y' \cup Y''$, where Y' consists of vertices y_i such that $g(y_i) = \emptyset$ and Y'' the remaining vertices. If $Y = Y'$, then it is a contradiction with the fact that there exists a vertex $y_j \in Y$ which is not dominated by V_0^g. If $Y = Y''$, then $\gamma_{m2r}(S_n) \ge |Y''| + \sum_{x\in X} |g(x)| > n$ and we have contradiction again. Hence, $Y' \neq \emptyset$ and $Y'' \neq \emptyset$. Notice that for every $y \in Y'$ we have $g(N(y)) = \{1,2\}$, and moreover $|N(y_i) \cap N(y_j)| \le 1$, where $i \neq j$ and $i,j \in \{0,1,\ldots,n-1\}$. This implies that $\sum_{x\in X} |g(x)| \ge |Y'|$. In addition, since $n = \gamma_{m2r}(S_n) \ge |Y''| + \sum_{x\in X} |g(x)| \ge |Y''| + |Y'| = n$, we get $\sum_{x\in X} |g(x)| = |Y'|$.

We now consider a sequence of consecutive vertices $y_k, y_{k+1}, \ldots, y_l$, $l \ge k$, belonging to Y', such that $y_{k-1}, y_{l+1} \in Y''$. From the definition of sun graphs and the fact $\sum_{x\in X} |g(x)| = |Y'|$ we derive $\sum_{i=k}^{l+1} |g(x_i)| = l-k+1$. Otherwise, there exists $y \in Y'$ such that $g(N(y)) \neq \{1,2\}$. Observe that, if $l-k$ is an even number, then $\sum_{i=k}^{l+1} |g(x_i)| = l-k+2$, which is a contradiction. If $l-k$ is an odd number, then $g(x_{k+1}) = g(x_{k+3}) = \cdots = g(x_l) = \{1,2\}$ and $g(x_k) = g(x_{k+2}) = \cdots = g(x_{l+1}) = \emptyset$, which is the only one way to obtain that $g(N(y_j)) = \{1,2\}$, for every $j \in \{k, k+1 \ldots, l\}$, and $\sum_{i=k}^{l+1} |g(x_i)| = l-k+1$.

On the other hand, let $y_j \in Y$ be a vertex which is not dominated by V_0^g. Since g is a $\gamma_{m2r}(S_n)$-function so that every $x_i \in X$ is dominated by V_0^g, we get $y_{j-1}, y_{j+1} \in Y'$. This implies $g(x_j) = g(x_{j+1}) = \emptyset$, which is a contradiction with the fact that y_j is not dominated by V_0^g. Therefore, $\gamma_{m2r}(S_n) = n+1$. □

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Let G be a graph of order $n \geq 3$ and let H be any graph. The *corona product* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. One special corona graph is $C_n \odot K_1$, that is, a graph on $2n$ vertices obtained by attaching one pendant vertex to each vertex of a cycle C_n. This graph is known as a *sunlet graph* L_n.

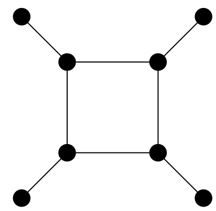

FIGURE 3. The sunlet graph L_4

Next result refers to the (maximal) 2-rainbow domination number of sunlet graph L_n. In order to present it, we need to partition the vertex set of L_n into two sets $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{n-1}\}$, such that each x_i is adjacent to x_{i-1}, x_{i+1} and y_i, where $i \in \{0, 1, \dots, n-1\}$ and all the operations on the subscripts of vertices in L_n are expressed modulo n.

Theorem 3.3. For any sunlet graph L_n of order $2n$, $\gamma_{m2r}(L_n) = \gamma_{r2}(L_n) = n + \lceil \frac{n}{3} \rceil$.

Proof. Let $g' = (V'_0, V'_1, V'_2, V'_{1,2})$ be a $\gamma_{r2}(L_n)$-function. From the definition of sunlet graphs ($g'(x_i) \neq \emptyset$ or $g'(y_i) \neq \emptyset$) and $1 \leq |g'(x_i)| + |g'(y_i)| \leq 2$, for any $i \in \{0, 1, \dots, n-1\}$. Moreover, if there exists a vertex $y_j \in Y$ such that $g'(y_j) = \{1, 2\}$, then we consider a function $g'' = (V'_0, V'_1 \cup \{x_j\}, V'_2 \cup \{y_j\}, V'_{1,2} - \{y_j\})$ which is readily a $\gamma_{r2}(S_n)$-function. Now, if there exists a vertex $x_k \in X$ such that $g''(x_k) = \{1, 2\}$, then $g''(y_k) = \emptyset$. Also, if $g''(x_{k-1}) = \emptyset$ or $g''(x_{k+1}) = \emptyset$, then $|g''(y_{k-1})| = 1$ or $|g''(y_{k+1})| = 1$, respectively. Hence, we study a function $g = (V_0, V_1, V_2, V_{1,2})$ of L_n defined in the following way.

$$g(v) = \begin{cases} \{1\} & \text{if } v = x_k \\ \{2\}, & \text{if } v = y_k \text{ or } (v = y_{k-1} \text{ and } g''(x_{k-1}) = \emptyset) \text{ or } (v = y_{k+1} \text{ and } g''(x_{k+1}) = \emptyset) \\ g''(v), & \text{otherwise.} \end{cases}$$

It is then easy to derive that g is a 2RDF of S_n with weight $\omega(g) = \omega(g'')$, so g is also a $\gamma_{r2}(S_n)$-function. So, from now on we consider a $\gamma_{r2}(L_n)$-function such that any vertex in L_n has not assigned $\{1, 2\}$ and moreover any $y_i \in Y$ has not assigned $\emptyset$. According to this and the definition of sunlet graphs we have that for every $x_j \in X$ with $f(x_j) = \emptyset$, $f(x_{j-1}) \neq \emptyset$ or $f(x_{j+1}) \neq \emptyset$. This implies the following $\gamma_{r2}(L_n) \geq |Y| + \sum_{x \in X} |g(x)| \geq |Y| + \gamma(C_n) = n + \gamma(C_n)$. It is well known the domination number of a cycle graph C_n, $n \geq 3$, is equal to $\lceil \frac{n}{3} \rceil$. Thus, we get $\gamma_{r2}(L_n) \geq n + \lceil \frac{n}{3} \rceil$.

On the other hand, let f be a function of L_n such that $f(y_i) = \{2\}$ for $i \in \{0, 1, \dots, n-1\}$, $f(x_j) = \{1\}$ for $i \equiv 0 \pmod 3$, and $\emptyset$ to each other vertex. It is readily seen that the function f is not only a 2RDF of L_n but also a M2RDF of L_n. Thus, $\gamma_{r2}(L_n) \leq \gamma_{m2r}(L_n) \leq n + \lceil \frac{n}{3} \rceil$. This completes the proof. □

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