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Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 9 No. 3 (2020), pp. 125-138.

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THE DISTANCE SPECTRUM OF TWO NEW OPERATIONS OF GRAPHS

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Communicated by Ebrahim Ghorbani

Abstract. Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix $D = D(G)$ of G is defined so that its (i, j)-entry is equal to the distance $d_G(v_i, v_j)$ between the vertices v_i and v_j of G. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $D(G)$ are the D-eigenvalues of G and form the distance spectrum or the D-spectrum of G, denoted by $Spec_D(G)$. In this paper, we introduce two new operations $G_1 \blacksquare_k G_2$ and $G_1 \blacklozenge_k G_2$ on graphs G_1 and G_2, and describe the distance spectra of $G_1 \blacksquare_k G_2$ and $G_1 \blacklozenge_k G_2$ of regular graphs G_1 and G_2 in terms of their adjacency spectra. By using these results, we obtain some new integral adjacency spectrum graphs, integral distance spectrum graphs and a number of families of sets of noncospectral graphs with equal distance energy.

1. Introduction

The characteristic polynomials and the associated spectra of various matrices related a graph have been the topics of many investigations in recent years. Adjacency matrix of a graph and its spectrum have arisen as a natural tool with which one can study graphs and its structural properties. The distance matrix and its spectrum have arisen independently from a data communication problem studied by Graham and Pollack [7] in 1971, in which the most important feature is the number of negative eigenvalues of the distance matrix. They are a natural generalization of an adjacency matrix. While the problem of computing the characteristic polynomial of the adjacency matrix of a graph and its spectrum has been solved for many families of graphs, the computation of the characteristic polynomial of the distance matrix has received much less attention, it was probably because the

MSC(2010): Primary: 05C50; Secondary: 05C12, 05C09, 05C92.
Keywords: Adjacency spectrum, Distance spectrum, Distance energy.
Received: 09 April 2019, Accepted: 13 February 2020
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Project supported by the Hunan Provincial Natural Science Foundation of China (2018JJ2249) and the Department of Education of Hunan Province (19A318).

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

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distance matrix is more complex than the adjacency matrix of a graph, and the computation of the characteristic polynomial of the distance matrix is much more intense problem and there are no simple analytical solutions except for a few trees [6]. However, due to numerous applications of the distance matrix of a graph in chemical sciences, music theory, ornithology, molecular biology, psychology, archeology etc (a survey see [2] and also the papers cited therein), the characteristic polynomial of the distance matrix and the corresponding spectra have been considered by many authors. D. Stevanović and G. Indulal [11] described the distance spectrum and energy of the join-based compositions of regular graphs in terms of their adjacency spectrum. These results showed that there exist a number of families of sets of noncospectral graphs with equal distance energy. G.Indulal and R.Balakrishnan [9] obtain the distance spectrum of the Indu-Bala product of graphs G_1 and G_2 in terms of the adjacency spectra of G_1 and G_2, and obtained a new class of distance equienergetic graphs of diameter 3. G. Indulal, I. Gutman and A. Vijayakumar [10] obtained bounds for the distance spectral radius and distance energy of graphs of diameter 2 and constructed pairs of equiregular distance-equienergetic graphs of diameter 2 on $3t+1$ vertices. In [5], some relations between the distance, the harmonic index and the largest signless Laplacian eigenvalue were determined. A. Heydari [8] computed the spectrum of the reduced distance matrix of regular monocentric dendrimers, the reduced distance matrices were used for modeling of amino acid sequences of proteins and of the genetic code. For some recent works on D-spectrum see a survey [1].

Let G be a connected graph with vertex set $V(G) = v_1, v_2, \ldots, v_n$. The distance matrix $D = D(G)$ of G is the square matrix of order n defined by setting its (i, j)-entry is equal to the distance $d_G(v_i, v_j)$, i.e., the length of a shortest path between the vertices v_i and v_j of G. The eigenvalues of $D(G)$ are called the D-eigenvalues of G and they form the D-spectrum of G, denoted by $Spec_D(G)$.

Let $G_{1i} = G_1$ and $G_{2i} = G_2 (1 \leq i \leq k)$ be k copies of graphs G_1 and G_2, respectively. The first product $G_1 \blacksquare_k G_2$ of G_1 and G_2 is obtained from the k joins $G_{1i} \vee G_{2i}$ $(i = 1, \ldots, k)$ by adding all edges between the corresponding vertices of G_{2m} and G_{2n} for $1 \leq m, n \leq k$ and $m \neq n$. The second product $G_1 \blacklozenge_k G_2$ of G_1 and G_2 is obtained from the k joins $G_{1i} \vee G_{2i}$ $(1 \leq i \leq k)$ by adding all edges between the corresponding vertices of G_{1m} and G_{1n} and all edges between the corresponding vertices of G_{2m} and G_{2n} for $1 \leq m, n \leq k$ and $m \neq n$. Clearly, $G_1 \blacklozenge_k G_2$ and $G_2 \blacklozenge_k G_1$ are isomorphic. $K_2 \blacksquare_3 P_3$ and $K_2 \blacklozenge_3 P_3$ are shown in Figure 1.

Note that the first product $G_1 \blacksquare_k G_2$ of G_1 and G_2 is a generalization of the join $G_1 \nabla G_2$ and the Indu-Bala product $G_1 \blacktriangledown G_2$ in [9]. If $k = 1$, then $G_1 \blacksquare_1 G_2$ is just the join $G_1 \nabla G_2$ two vertex-disjoint graphs G_1 and G_2, which is the graph obtained from the union $G_1 \blacksquare_k G_2$ of $G_1 \cup G_2$ by adding all possible edges between each vertex of G_1 and each vertex of G_2. If $k = 2$, then $G_1 \blacksquare_2 G_2$ is just the Indu-Bala product $G_1 \blacktriangledown G_2$ of G_1 and G_2, which is obtained from two disjoint copies of the join $G_1 \nabla G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2.

All graphs considered in this paper are simple and we follow [4] for graph theoretic terminology. We recall the following two definitions.

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

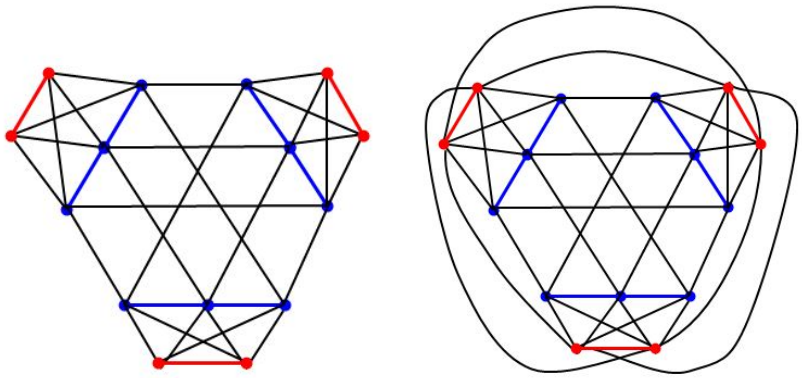

FIGURE 1. The two type products $K_2\blacksquare_3 P_3$ and $K_2\blacklozenge_3 P_3$.

Definition 1.1. [10] *The distance energy $ED(G)$ of a graph G is the sum of absolute values of the eigenvalues of the distance matrix $D(G)$ of G.*

Definition 1.2. [10] *Two graphs of the same order are said to be distance equienergetic if they have the same distance energy.*

This work is motivated by [9] on the distance spectrum of product of graphs. They obtained the distance spectrum of a new produce in terms of the adjacency spectra of G_1 and G_2 and a new class of distance equienergetic graphs of diameter 3. In this paper, we describe the distance spectra of $G_1\blacksquare_k G_2$ and $G_1\blacklozenge_k G_2$ of regular graphs G_1 and G_2 in terms of their adjacency spectra. These results are used to show that there exist a number of families of sets of noncospectral graphs with equal distance energy.

2. The distance spectra of $G_1\blacksquare_k G_2$ and $G_1\blacklozenge_k G_2$

In this section, we will describe the distance spectra of $G_1\blacksquare_k G_2$ and $G_1\blacklozenge_k G_2$ of regular graphs G_1 and G_2 in terms of their adjacency spectra.

Theorem 2.1. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,3} \geq \cdots \geq \lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i} of G_i. Then the D-spectrum of $G_1\blacksquare_k G_2$ is the set consisting of the numbers $-(\lambda_{1,j} + 2)$ for $j = 2, 3, \ldots, n_1$, each with multiplicity k, $(k - 2t)(\lambda_{2,j} + 2)$ $(1 \leq t \leq k)$ for $j = 2, 3, \ldots, n_2$, and $2k$ more eigenvalues which are the roots of the characteristic polynomial of a $2k \times 2k$ real and symmetric matrix M_1, where*

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

$$M_1 = \begin{bmatrix} w_1 & 3n_1 & \dots & 3n_1 & n_2 & 2n_2 & \dots & 2n_2 \\ 3n_1 & w_1 & \dots & 3n_1 & 2n_2 & n_2 & \dots & 2n_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n_1 & 3n_1 & \dots & w_1 & 2n_2 & 2n_2 & \dots & n_2 \\ n_1 & 2n_1 & \dots & 2n_1 & w_3 & w_4 & \dots & w_4 \\ 2n_1 & n_1 & \dots & 2n_1 & w_4 & w_3 & \dots & w_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n_1 & 2n_1 & \dots & n_1 & w_4 & w_4 & \dots & w_3 \end{bmatrix}$$

and $w_1 = 2n_1 - 2 - r_1$, $w_3 = 2n_2 - 2 - r_2$ *and* $w_4 = 3n_2 - 2 - r_2$.

Proof. By a proper labeling of the vertices of $G_1 \blacksquare_k G_2$, its distance matrix D can be written in the following form.

$$D = \begin{bmatrix} A_1 & 3J_{n_1\times n_1} & \dots & 3J_{n_1\times n_1} & J_{n_1\times n_2} & 2J_{n_1\times n_2} & \dots & 2J_{n_1\times n_2} \\ 3J_{n_1\times n_1} & A_1 & \dots & 3J_{n_1\times n_1} & 2J_{n_1\times n_2} & J_{n_1\times n_2} & \dots & 2J_{n_1\times n_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3J_{n_1\times n_1} & 3J_{n_1\times n_1} & \dots & A_1 & 2J_{n_1\times n_2} & 2J_{n_1\times n_2} & \dots & J_{n_1\times n_2} \\ J_{n_2\times n_1} & 2J_{n_2\times n_1} & \dots & 2J_{n_2\times n_1} & A_2 & A_3 & \dots & A_3 \\ 2J_{n_2\times n_1} & J_{n_2\times n_1} & \dots & 2J_{n_2\times n_1} & A_3 & A_2 & \dots & A_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2J_{n_2\times n_1} & 2J_{n_2\times n_1} & \dots & J_{n_2\times n_1} & A_3 & A_3 & \dots & A_2 \end{bmatrix}$$

where J stands for the all-one matrix, and $A_1 = 2(J-I) - A_{G_1}$, $A_2 = 2(J-I) - A_{G_2}$ and $A_3 = 3J - 2I - A_{G_2}$.

As a regular graph, G_1 has the all-one vector $\mathbf{1}$ as an eigenvector corresponding to the eigenvalue r_1, while all the other eigenvectors are orthogonal to $\mathbf{1}$. Note that as G_1 may not be connected, r_1 need not be a simple eigenvalue of G_1. Let $\lambda \neq r_1$ be an arbitrary eigenvalue of the adjacency matrix A_{G_1} of G_1 with corresponding eigenvector X, such that $\mathbf{1}^T X = 0$. Then $[X^T, \mathbf{0}_{1\times(kn_1+kn_2-n_1)}]^T$ is an eigenvector of D corresponding to the eigenvalue $-(\lambda+2)$. This is because

$$\begin{bmatrix} A_1 & 3J_{n_1\times n_1} & \dots & 3J_{n_1\times n_1} & J_{n_1\times n_2} & 2J_{n_1\times n_2} & \dots & 2J_{n_1\times n_2} \\ 3J_{n_1\times n_1} & A_1 & \dots & 3J_{n_1\times n_1} & 2J_{n_1\times n_2} & J_{n_1\times n_2} & \dots & 2J_{n_1\times n_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3J_{n_1\times n_1} & 3J_{n_1\times n_1} & \dots & A_1 & 2J_{n_1\times n_2} & 2J_{n_1\times n_2} & \dots & J_{n_1\times n_2} \\ J_{n_2\times n_1} & 2J_{n_2\times n_1} & \dots & 2J_{n_2\times n_1} & A_2 & A_3 & \dots & A_3 \\ 2J_{n_2\times n_1} & J_{n_2\times n_1} & \dots & 2J_{n_2\times n_1} & A_3 & A_2 & \dots & A_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2J_{n_2\times n_1} & 2J_{n_2\times n_1} & \dots & J_{n_2\times n_1} & A_3 & A_3 & \dots & A_2 \end{bmatrix} \begin{bmatrix} X \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

$$= \begin{bmatrix} (2(J-I)-A_{G_1})X \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} = -(\lambda+2) \begin{bmatrix} X \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

Similarly, the $k-1$ vectors $[\mathbf{0}_{1\times n_1}, X^T, \mathbf{0}_{1\times(kn_1+kn_2-2n_1)}]^T$, $[\mathbf{0}_{1\times 2n_1},\ X^T,\ \mathbf{0}_{1\times(kn_1+kn_2-3n_1)}]^T$, $\ldots$, $[\mathbf{0}_{1\times(kn_1-n_1)},\ X^T,\ \mathbf{0}_{1\times kn_2}]^T$ are the eigenvectors of D corresponding to the eigenvalue $-(\lambda+2)$.

Let $\mu \neq r_2$ be an arbitrary eigenvalue of the adjacency matrix A_{G_2} of G_2 with corresponding eigenvector Y, such that $\mathbf{1}^T Y = 0$. Then by a similar argument we see that the k vectors $[\mathbf{0}_{1\times kn_1}, \overbrace{Y^T, \cdots, Y^T}^{t}, \overbrace{-Y^T, \cdots, -Y^T}^{k-t}]^T$ are eigenvectors of D with corresponding eigenvalues $(k-2t)(\mu+2)$ for $1 \le t \le k$.

In this way, we obtain eigenvectors of the forms $[X^T, \mathbf{0}_{1\times(kn_1+kn_2-n_1)}]^T$, $[\mathbf{0}_{1\times n_1}, X^T, \mathbf{0}_{1\times(kn_1+kn_2-2n_1)}]^T$, $\ldots$, $[\mathbf{0}_{1\times(kn_1-n_1)}, X^T, \mathbf{0}_{1\times kn_2}]^T$ and $[\mathbf{0}_{1\times kn_1}, \overbrace{Y^T, \ldots, Y^T}^{t}, \overbrace{-Y^T, \ldots, -Y^T}^{k-t}]^T$ $(1 \le t \le k)$, and these account for a total of $k(n_1-1)+k(n_2-1) = k(n_1+n_2)-2k$ eigenvectors. All these eigenvectors are orthogonal to the $2k$ vectors $[\mathbf{0}_{1\times(t-1)n_1}, \mathbf{1}_{1\times n_1}, \mathbf{0}_{1\times(kn_1-tn_1+kn_2)}]^T$ and $[\mathbf{0}_{1\times(kn_1+(t-1)n_2)}, \mathbf{1}_{1\times n_2}, \mathbf{0}_{1\times(k-t)n_2)}]^T$, $1 \le t \le k$. This means that these $2k$ vectors span the space spanned by the remaining $2k$ eigenvectors of D. Thus the remaining $2k$ eigenvectors of D are of the form $[a_1\mathbf{1}_{1\times n_1}, a_2\mathbf{1}_{1\times n_1}, \ldots, a_k\mathbf{1}_{1\times n_1}, b_1\mathbf{1}_{1\times n_2}, b_2\mathbf{1}_{1\times n_2}, \ldots, b_k\mathbf{1}_{1\times n_2}]^T$ for some $[a_1, a_2, \ldots, a_k,\ b_1, b_2, \ldots, b_k] \neq \mathbf{0}_{1\times 2k}$.

If ν is an eigenvalue of D with an eigenvector $[a_1\mathbf{1}_{1\times n_1}, \ldots, a_k\mathbf{1}_{1\times n_1}, b_1\mathbf{1}_{1\times n_2}, \ldots, b_k\mathbf{1}_{1\times n_2}]^T$, then

$$\begin{aligned} &D[a_1\mathbf{1}_{1\times n_1}, \ldots, a_k\mathbf{1}_{1\times n_1}, b_1\mathbf{1}_{1\times n_2}, \ldots, b_k\mathbf{1}_{1\times n_2}]^T \\ &= \nu[a_1\mathbf{1}_{1\times n_1}, \ldots, a_k\mathbf{1}_{1\times n_1}, b_1\mathbf{1}_{1\times n_2}, \ldots, b_k\mathbf{1}_{1\times n_2}]^T \end{aligned}$$

and $A_{G_1}\mathbf{1}^T_{1\times n_1} = r_1\mathbf{1}^T_{1\times n_1}$, $A_{G_2}\mathbf{1}^T_{1\times n_2} = r_2\mathbf{1}^T_{1\times n_2}$. We get the system of $2k$ equations:

$$\begin{cases} (2n_1-2-r_1)a_i + 3n_1\sum\limits_{j\neq i}^{k} a_j + 2n_2\sum\limits_{j=1}^{k} b_j - n_2 b_i = \nu a_i, & i = 1,2,\ldots,k \\ 2n_1\sum\limits_{j=1}^{k} a_j - n_1 a_i + (2n_2-2-r_2)b_i + (3n_2-2-r_2)\sum\limits_{j\neq i}^{k} b_j = \nu b_i, & i = 1,2,\ldots,k. \end{cases}$$

This shows that ν is also an eigenvalue of the matrix M_1 since $[a_1, a_2, \ldots, a_k,\ b_1, b_2, \ldots, b_k] \neq \mathbf{0}_{1\times 2k}$, where

$$M_1 = \begin{bmatrix} w_1 & 3n_1 & \ldots & 3n_1 & n_2 & 2n_2 & \ldots & 2n_2 \\ 3n_1 & w_1 & \ldots & 3n_1 & 2n_2 & n_2 & \ldots & 2n_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n_1 & 3n_1 & \ldots & w_1 & 2n_2 & 2n_2 & \ldots & n_2 \\ n_1 & 2n_1 & \ldots & 2n_1 & w_3 & w_4 & \ldots & w_4 \\ 2n_1 & n_1 & \ldots & 2n_1 & w_4 & w_3 & \ldots & w_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n_1 & 2n_1 & \ldots & n_1 & w_4 & w_4 & \ldots & w_3 \end{bmatrix}$$

and $w_1 = 2n_1 - 2 - r_1$, $w_3 = 2n_2 - 2 - r_2$ and $w_4 = 3n_2 - 2 - r_2$. □

Generally, it is difficult to find all eigenvalues of the $2k \times 2k$ matrix M. But for $k = 1, 2, 3$, we have

Corollary 2.2. [11] *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,3} \geq \cdots \geq \lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i}. Then the D-spectrum of $G_1 \blacksquare_1 G_2$ is the set consisting of the numbers $-(\lambda_{1,j} + 2)$ for $j = 2, 3, \ldots, n_1$, $-(\lambda_{2,j} + 2)$ for $j = 2, 3, \ldots, n_2$, and two more eigenvalues of the form*

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{(n_1 - n_2 - \frac{r_1 - r_2}{2})^2 + n_1 n_2}.$$

Proof. From Theorem 2.1, the D-spectrum of $G_1 \blacksquare_1 G_2$ is the set consisting of the numbers $-(\lambda_{1,j} + 2)$ for $j = 2, 3, \ldots, n_1$, $-(\lambda_{2,j} + 2)$ for $j = 2, 3, \ldots, n_2$, and two more eigenvalues of matrix

$$M_1 = \begin{pmatrix} 2n_1 - 2 - r_1 & n_2 \\ n_1 & 2n_2 - 2 - r_2 \end{pmatrix}$$

It is easy to see that two more eigenvalues of M_1 are

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{(n_1 - n_2 - \frac{r_1 - r_2}{2})^2 + n_1 n_2}.$$

□

Corollary 2.3. [9] *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,3} \geq \cdots \geq \lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i}. Then the D-spectrum of $G_1 \blacksquare_2 G_2$ is the set consisting of the numbers $-(\lambda_{1,j} + 2)$ for $j = 2, 3, \ldots, n_1$, each with multiplicity 2, $-2(\lambda_{2,j} + 2)$ for $j = 2, 3, \ldots, n_2$, 0 with multiplicity $(n_2 - 1)$ and four more eigenvalues which are the roots of the biquadratic equation:*

$$\begin{aligned} &[x^2 + (6 - 5(n_1 + n_2) + r_1 + 2r_2)x + 2r_1 r_2 + 4(r_1 + r_2) - 10n_1 r_2 - 5n_2 r_1 \\ &-10(2n_1 + n_2) + 16n_1 n_2 + 8][x^2 + (2 + n_1 + n_2 + r_1)x + n_2(r_1 + 2)] = 0. \end{aligned}$$

Proof. By Theorem 2.1, the D-spectrum of $G_1 \blacksquare_2 G_2$ is the set consisting of the numbers $-(\lambda_{1,j}+2)$ for $j=2,3,\ldots,n_1$, each with multiplicity 2, $(2-2t)(\lambda_{2,j}+2)$ $(t=1,2)$ for $j=2,3,\ldots,n_2$, each with multiplicity 2, and 4 more eigenvalues of matrix M_1:

$$M_1 = \begin{pmatrix} 2n_1-2-r_1 & 3n_1 & n_2 & 2n_2 \\ 3n_1 & 2n_1-2-r_1 & 2n_2 & n_2 \\ n_1 & 2n_1 & 2n_2-2-r_2 & 3n_2-2-r_2 \\ 2n_1 & n_1 & 3n_2-2-r_2 & 2n_2-2-r_2 \end{pmatrix}$$

and the characteristic polynomial of the above matrix is

$$[x^2+(6-5(n_1+n_2)+r_1+2r_2)x+2r_1r_2+4(r_1+r_2)-10n_1r_2-5n_2r_1$$
$$-10(2n_1+n_2)+16n_1n_2+8][x^2+(2+n_1+n_2+r_1)x+n_2(r_1+2)].$$

□

Corollary 2.4. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1}=r_i\geq\lambda_{i,2}\geq\lambda_{i,3}\geq\cdots\geq\lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i}. Then the D-spectrum of $G_1\blacksquare_3G_2$ is the set consisting of the numbers $-(\lambda_{1,j}+2)$ for $j=2,3,\ldots,n_1$, each with multiplicity 3, $(3-2t)(\lambda_{2,j}+2)(1\leq t\leq 3)$ for $j=2,3,\ldots,n_2$, each with multiplicity 3, and six eigenvalues which are the roots of the 6-degree equation:*

$$[x^2+(r_1+3r_2-8n_1-8n_2+8)x+3r_1r_2-48n_1-16n_2+39n_1n_2+6r_1$$
$$-8n_2r_1+6r_2-24n_1r_2+12][x^2+(n_1+n_2+r_1+2)x+2n_2+n_2r_1]^2=0. \tag{2.1}$$

Proof. From Theorem 2.1, the D-spectrum of $G_1\blacksquare_3G_2$ is the set consisting of the numbers $-(\lambda_{1,j}+2)$ for $j=2,3,\ldots,n_1$, each with multiplicity 3, $(3-2t)(\lambda_{2,j}+2)(1\leq t\leq 3)$ for $j=2,3,\ldots,n_2$, each with multiplicity 3, and six more eigenvalues of matrix

$$M_1 = \begin{pmatrix} w_1 & 3n_1 & 3n_1 & n_2 & 2n_2 & 2n_2 \\ 3n_1 & w_1 & 3n_1 & 2n_2 & n_2 & 2n_2 \\ 3n_1 & 3n_1 & w_1 & 2n_2 & 2n_2 & n_2 \\ n_1 & 2n_1 & 2n_1 & w_3 & w_4 & w_4 \\ 2n_1 & n_1 & 2n_1 & w_4 & w_3 & w_4 \\ 2n_1 & 2n_1 & n_1 & w_4 & w_4 & w_3 \end{pmatrix}$$

where $w_1=2n_1-2-r_1, w_3=2n_2-2-r_2$ and $w_4=3n_2-2-r_2$, and the characteristic polynomial of M_1 is

$$[x^2+(r_1+3r_2-8n_1-8n_2+8)x+3r_1r_2-48n_1-16n_2+39n_1n_2+6r_1-8n_2r_1$$
$$+6r_2-24n_1r_2+12][x^2+(n_1+n_2+r_1+2)x+2n_2+n_2r_1]^2.$$

□

In the following, we will consider the distance spectrum of $G_1\blacklozenge_kG_2$ of regular graphs G_1 and G_2 in terms of their adjacency spectra.

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

Theorem 2.5. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,3} \geq \cdots \geq \lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i}. Then the D-spectrum of $G_1 \blacklozenge_k G_2$ is the set consisting of the numbers $(k-2t)(\lambda_{1,j}+2)$ for $j = 2, 3, \ldots, n_1$, each with multiplicity k, $(k-2t)(\lambda_{2,j}+2)$ $(1 \leq t \leq k)$ for $j = 2, 3, \ldots, n_2$, each with multiplicity k, and $2k$ more eigenvalues of matrix M_2*

$$M_2 = \begin{bmatrix} w_1 & w_2 & \ldots & w_2 & n_2 & 2n_2 & \ldots & 2n_2 \\ w_2 & w_1 & \ldots & w_2 & 2n_2 & n_2 & \ldots & 2n_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_2 & w_2 & \ldots & w_1 & 2n_2 & 2n_2 & \ldots & n_2 \\ n_1 & 2n_1 & \ldots & 2n_1 & w_3 & w_4 & \ldots & w_4 \\ 2n_1 & n_1 & \ldots & 2n_1 & w_4 & w_3 & \ldots & w_4 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n_1 & 2n_1 & \ldots & n_1 & w_4 & w_4 & \ldots & w_3 \end{bmatrix}$$

where $w_1 = 2n_1 - 2 - r_1$, $w_2 = 3n_1 - 2 - r_1$, $w_3 = 2n_2 - 2 - r_2$ and $w_4 = 3n_2 - 2 - r_2$.

Proof. By a proper labeling of the vertices of $G_1 \blacklozenge_k G_2$, its distance matrix D can be written in the form

$$D = \begin{bmatrix} A_1 & A_4 & \ldots & A_4 & J_{n_1\times n_2} & 2J_{n_1\times n_2} & \ldots & 2J_{n_1\times n_2} \\ A_4 & A_1 & \ldots & A_4 & 2J_{n_1\times n_2} & J_{n_1\times n_2} & \ldots & 2J_{n_1\times n_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_4 & A_4 & \ldots & A_1 & 2J_{n_1\times n_2} & 2J_{n_1\times n_2} & \ldots & J_{n_1\times n_2} \\ J_{n_2\times n_1} & 2J_{n_2\times n_1} & \ldots & 2J_{n_2\times n_1} & A_2 & A_3 & \ldots & A_3 \\ 2J_{n_2\times n_1} & J_{n_2\times n_1} & \ldots & 2J_{n_2\times n_1} & A_3 & A_2 & \ldots & A_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2J_{n_2\times n_1} & 2J_{n_2\times n_1} & \ldots & J_{n_2\times n_1} & A_3 & A_3 & \ldots & A_2 \end{bmatrix}$$

where J stands for the all-one matrix, $A_1 = 2(J-I) - A_{G_1}$, $A_2 = 2(J-I) - A_{G_2}$, $A_3 = 3J - 2I - A_{G_2}$ and $A_4 = 3J - 2I - A_{G_1}$.

As a regular graph, G_1 has the all-one vector $\mathbf{1}$ as an eigenvector corresponding to the eigenvalue r_1, while all the other eigenvectors are orthogonal to $\mathbf{1}$. Note that as G_1 may not be connected, r_1 need not be a simple eigenvalue of G_1. Let $\lambda \neq r_1$ be an arbitrary eigenvalue of the adjacency matrix of G_1 with corresponding eigenvector X, such that $\mathbf{1}^T X = 0$. Then we can see that the k vectors $[\overbrace{X^T, \ldots, X^T}^{t}, \overbrace{-X^T, \ldots, -X^T}^{k-t}, \mathbf{0}_{1\times kn_2}]^T$ $(1 \leq t \leq k)$ are eigenvectors of D with corresponding eigenvalues $(k-2t)(\lambda+2)$.

Let $\mu \neq r_2$ be an arbitrary eigenvalue of the adjacency matrix of G_2 with corresponding eigenvector Y, such that $\mathbf{1}^T Y = 0$. Then by a similar argument, we can see that the k vectors $[\mathbf{0}_{1\times kn_1}, \overbrace{Y^T, \ldots, Y^T}^{t}, \overbrace{-Y^T, \ldots, -Y^T}^{k-t}]^T$ $(1 \leq t \leq k)$ are eigenvectors of D with corresponding eigenvalues $(k-2t)(\mu+2)$.

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

In this way, we obtain $k(n_1-1)+k(n_2-1)=k(n_1+n_2)-2k$ eigenvectors, and all these eigenvectors are orthogonal to the $2k$ vectors $[\mathbf{0}_{1\times(t-1)n_1},\mathbf{1}_{1\times n_1},\mathbf{0}_{1\times(kn_1-tn_1+kn_2)}]^T$ and $[\mathbf{0}_{1\times(kn_1+(t-1)n_2)},\mathbf{1}_{1\times n_2}, \mathbf{0}_{1\times(k-t)n_2)}]^T$, $1\le t\le k$. This means that these $2k$ vectors span the space spanned by the remaining $2k$ eigenvectors of D. Thus the remaining $2k$ eigenvectors of D are of the form $[a_1\mathbf{1}_{1\times n_1},a_2\mathbf{1}_{1\times n_1},\ldots,a_k\mathbf{1}_{1\times n_1},b_1\mathbf{1}_{1\times n_2},b_2\mathbf{1}$ for some $[a_1,a_2,\ldots,a_k,b_1,b_2,\ldots,b_k]\neq\mathbf{0}_{1\times 2k}$.

If ν is an eigenvalue of D with an eigenvector $[a_1\mathbf{1}_{1\times n_1},\ldots,a_k\mathbf{1}_{1\times n_1},b_1\mathbf{1}_{1\times n_2},\ldots,b_k\mathbf{1}_{1\times n_2}]^T$, then

$$\begin{aligned}&D[a_1\mathbf{1}_{1\times n_1},\ldots,a_k\mathbf{1}_{1\times n_1},b_1\mathbf{1}_{1\times n_2},\ldots,b_k\mathbf{1}_{1\times n_2}]^T\\&=\nu[a_1\mathbf{1}_{1\times n_1},\ldots,a_k\mathbf{1}_{1\times n_1},b_1\mathbf{1}_{1\times n_2},\ldots,b_k\mathbf{1}_{1\times n_2}]^T\end{aligned}$$

and $A_{G_1}\mathbf{1}^T_{1\times n_1}=r_1\mathbf{1}^T_{1\times n_1}$, $A_{G_2}\mathbf{1}^T_{1\times n_2}=r_2\mathbf{1}^T_{1\times n_2}$. We get the system of $2k$ equations:

$$\begin{cases}(2n_1-2-r_1)a_i+(3n_1-2-r_1)\sum\limits_{j\neq i}^{k}a_j+2n_2\sum\limits_{j=1}^{k}b_j-n_2b_i=\nu a_i, & i=1,2,\ldots,k;\\ 2n_1\sum\limits_{j=1}^{k}a_j-n_1a_i+(2n_2-2-r_2)b_i+(3n_2-2-r_2)\sum\limits_{j\neq i}^{k}b_j=\nu b_i, & i=1,2,\ldots,k.\end{cases}$$

This shows that ν is also an eigenvalue of the matrix M_2 since $[a_1,a_2,\ldots,a_k,\ b_1,b_2,\ldots,b_k]\neq\mathbf{0}_{1\times 2k}$, where

$$M_2=\begin{bmatrix} w_1 & w_2 & \ldots & w_2 & n_2 & 2n_2 & \ldots & 2n_2\\ w_2 & w_1 & \ldots & w_2 & 2n_2 & n_2 & \ldots & 2n_2\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ w_2 & w_2 & \ldots & w_1 & 2n_2 & 2n_2 & \ldots & n_2\\ n_1 & 2n_1 & \ldots & 2n_1 & w_3 & w_4 & \ldots & w_4\\ 2n_1 & n_1 & \ldots & 2n_1 & w_4 & w_3 & \ldots & w_4\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2n_1 & 2n_1 & \ldots & n_1 & w_4 & w_4 & \ldots & w_3\end{bmatrix}$$

where $w_1=2n_1-2-r_1$, $w_2=3n_1-2-r_1$, $w_3=2n_2-2-r_2$ and $w_4=3n_2-2-r_2$. □

For $k=1$, $G_1\blacklozenge_1G_2$ is just the join $G_1\nabla G_2$ and Corollary 2.2 can be also obtained from Theorem 2.5. For $k=2$, we can get the following result.

Corollary 2.6. *For $i=1,2$, let G_i be an r_i-regular graph with n_i vertices and let $\lambda_{i,1}=r_i\ge\lambda_{i,2}\ge\lambda_{i,3}\ge\cdots\ge\lambda_{i,n_i}$ be the eigenvalues of the adjacency matrix A_{G_i}. Then the D-spectrum of $G_1\blacklozenge_2G_2$ is the set consisting of the numbers $-2(\lambda_{1,j}+2)$ for $j=2,3,\ldots,n_1$, $-2(\lambda_{2,j}+2)$ for $j=2,3,\ldots,n_2$, 0 with multiplicity n_1+n_2-1, $-(n_1+n_2)$ and two eigenvalues which are the roots of the quadratic equation:*

$$\begin{aligned}&x^2+(2r_1+2r_2-5n_1-5n_2+8)x+4r_1r_2+8(r_1+8r_2)\\&-10(n_2r_1+n_1r_2)+16n_1n_2-20(n_1+n_2)+16=0.\end{aligned}$$

Proof. By Theorem 2.5, the D-spectrum of $G_1\blacklozenge_2G_2$ is the set consisting of the numbers $-2(\lambda_{1,j}+2)$ for $j=2,3,\ldots,n_1$, each with multiplicity 1, $-2(\lambda_{2,j}+2)$ for $j=2,3,ldots,n_2$, each with multiplicity

1, 0 with multiplicity n_1+n_2-2, and four more eigenvalues which are the eigenvalues of matrix M_2

$$M_2 = \begin{pmatrix} 2n_1-2-r_1 & 3n_1-2-r_1 & n_2 & 2n_2 \\ 3n_1-2-r_1 & 2n_1-2-r_1 & 2n_2 & n_2 \\ n_1 & 2n_1 & 2n_2-2r_2 & 3n_2-2-r_2 \\ 2n_1 & n_1 & 3n_2-2r_2 & 2n_2-2-r_2 \end{pmatrix}$$

and the characteristic polynomial of M_2 is

$$x(x+n_1+n_2)[x^2+(2r_1+2r_2-5n_1-5n_2+8)x$$
$$+4r_1r_2+8(r_1+8r_2)-10(n_2r_1+n_1r_2)+16n_1n_2-20(n_1+n_2)+16].$$

So, we prove the conclusion. □

3. Some new integral adjacency spectrum and integral distance spectrum graphs

It was showed in [10] that an r-regular graph G of diameter 2 is integral with respect to its adjacency matrix if and only if it is integral with respect to its distance matrix. However, a regular graph with integral adjacency spectrum and integral distance spectrum does not need to have the diameter 2. An infinite family of regular graphs of diameter 3 all of which having integral adjacency spectrum and integral distance spectrum was found and an open problem was given in [9]: Characterize graphs for which both the adjacency spectrum and distance spectrum are integral. A weaker problem would be: Find new families of graphs for which the adjacency spectrum and the distance spectrum are both integral.

The following result gives rise to a new families of graphs for which the adjacency spectrum and the distance spectrum are both integral.

Theorem 3.1. *The adjacency spectrum and distance spectrum of $\overline{K_t}\blacklozenge_2\overline{K_t}$ are both integral.*

Proof. The adjacency matrix of $\overline{K_t}\blacklozenge_2\overline{K_t}$ can be written in the form

$$\begin{pmatrix} 0 & J_{t\times t} & I_{t\times t} & 0 \\ J_{t\times t} & 0 & 0 & I_{t\times t} \\ I_{t\times t} & 0 & 0 & J_{t\times t} \\ 0 & I_{t\times t} & J_{t\times t} & 0 \end{pmatrix}$$

and its adjacency spectrum is

$$\begin{pmatrix} t+1 & t-1 & 1 & -1 & -t+1 & -t-1 \\ 1 & 1 & 2t-2 & 2t-2 & 1 & 1 \end{pmatrix}.$$

From Corollary 2.6, we know the distance spectrum of $\overline{K_t}\blacklozenge_2\overline{K_t}$ is

$$\begin{pmatrix} 8t-4 & 2t-4 & 0 & -4 & -2t \\ 1 & 1 & 2t-1 & 2t-2 & 1 \end{pmatrix}.$$

So, $\overline{K_t}\blacklozenge_2\overline{K_t}$ is adjacency-integral and distance-integral. □

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

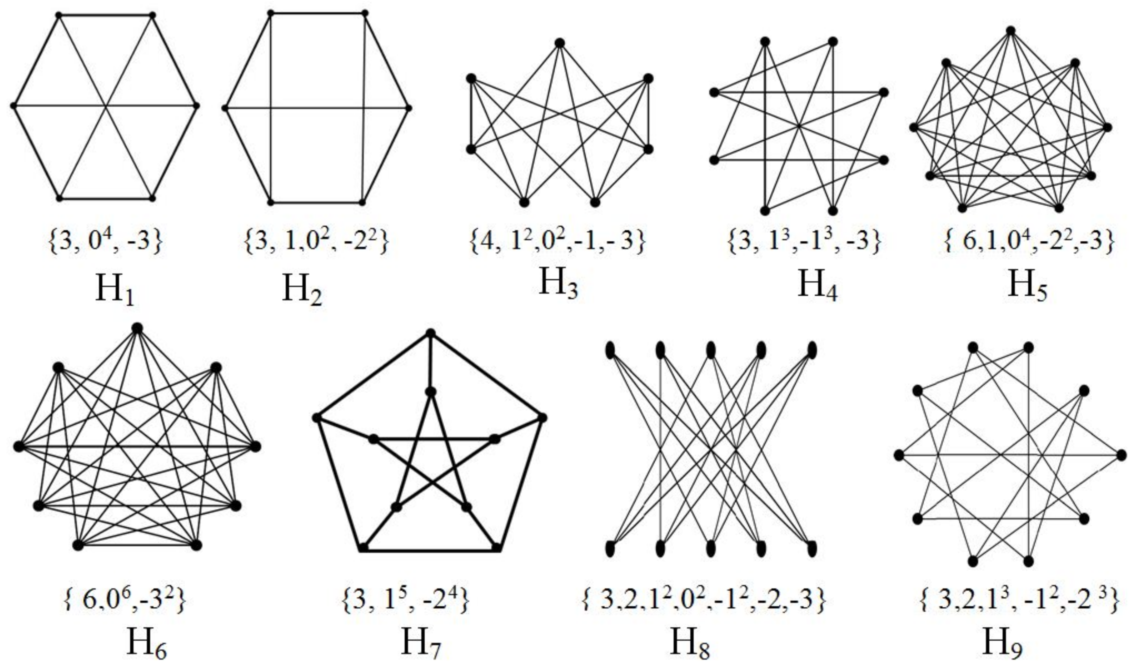

FIGURE 2. Some regular graphs and their adjacency spectra.

Based on Corollary 2.4, in order to make that $G_1 \blacksquare_k G_2$ is an integral distance spectrum graph for r_i-regular and integral graphs G_i with n_i vertices $(i = 1, 2)$, we only need that six roots of the 6-degree equation (2.1) are integral for the corresponding group $\{n_1, n_2, r_1, r_2\}$.

Clearly, C_4, K_n, $\overline{K_n}$ and $K_{n,n}$ are regular adjacency integral graphs, and from [3, 12] we know that H_i are also regular adjacency integral graphs, where H_i $(1 \leq i \leq 9)$ are depicted in Figure 2. By using **Matlab 7.0**, we find that there are 14 groups $\{n_1, n_2, r_1, r_2\}$ in Table 1 such that six roots of the 6-degree equation (2.1) are integral. So, we can get 21 new integral distance spectrum graphs as follows (see Table 2).

TABLE 1. 14 groups $\{n_1, n_2, r_1, r_2\}$ and the roots of the 6-degree equation (2.1).

♯	$\{n_1, n_2, r_1, r_2\}$	The roots of the 6-degree equation (2.1)	♯	$\{n_1, n_2, r_1, r_2\}$	The roots of the 6-degree equation (2.1)
1	$\{2, 6, 1, 3\}$	$(-9)^2, (-2)^2, 3, 43$	2	$\{3, 4, 0, 3\}$	$(-8)^2, (-1)^2, 2, 37$
3	$\{4, 2, 3, 0\}$	$(-10)^2, (-1)^2, 2, 35$	4	$\{4, 6, 2, 5\}$	$(-12)^2, (-2)^2, 3, 52$
5	$\{4, 10, 1, 5\}$	$(-15)^2, (-2)^2, 9, 79$	6	$\{5, 6, 0, 3\}$	$(-12)^2, (-1)^2, 8, 63$
7	$\{5, 8, 4, 3\}$	$(-16)^2, (-3)^2, 9, 74$	8	$\{6, 4, 1, 2\}$	$(-12)^2, (-1)^2, 5, 60$
9	$\{7, 8, 0, 3\}$	$(-16)^2, (-1)^2, 14, 89$	10	$\{7, 10, 4, 3\}$	$(-0)^2, (-3)^2, 15, 100$
11	$\{8, 7, 7, 0\}$	$(-21)^2, (-3)^2, 15, 90$	12	$\{8, 9, 0, 6\}$	$(-18)^2, (-1)^2, 12, 98$
13	$\{9, 10, 0, 3\}$	$(-20)^2, (-1)^2, 20, 115$	14	$\{10, 6, 1, 0\}$	$(-18)^2, (-1)^2, 17, 102$

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

TABLE 2. twenty-one graphs $G_1\blacksquare_3 G_2$ with integral distance spectrum.

♯	G_1	G_2	The adjacency spectrum of G_1	The adjacency spectrum of G_2	The roots of the 6-degree equation (2.1)
1	K_2	H_1	$1, -1$	$3, (0)^4, -3$	$(-9)^2, (-2)^2, 3, 43$
2	K_2	H_2	$1, -1$	$3, 1, (0)^2, (-2)^2$	$(-9)^2, (-2)^2, 3, 43$
3	$\overline{K_3}$	K_4	$(0)^3$	$3, (-1)^3$	$(-8)^2, (-1)^2, 2, 37$
4	K_4	$\overline{K_2}$	$3, (-1)^3$	$(0)^2$	$(-10)^2, (-1)^2, 2, 35$
5	C_4	K_6	$2, (0)^2, -2$	$5, (0)^5$	$(-12)^2, (-2)^2, 3, 52$
6	$2K_2$	$K_{5,5}$	$(1)^2, (-1)^2$	$5, (0)^8, -5$	$(-15)^2, (-2)^2, 9, 79$
7	$\overline{K_5}$	H_1	$(0)^5$	$3, (0)^4, -3$	$(-12)^2, (-1)^2, 8, 63$
8	$\overline{K_5}$	H_2	$(0)^5$	$3, 1, (0)^2, (-2)^2$	$(-12)^2, (-1)^2, 8, 63$
9	K_5	H_4	$4, (-1)^4$	$3, (1)^3, (-1)^3, -3$	$(-16)^2, (-3)^2, 9, 74$
10	$3K_2$	C_4	$(1)^3, (-1)^3$	$2, (0)^2, -2$	$(-12)^2, (-1)^2, 5, 60$
11	$\overline{K_7}$	H_4	$(0)^7$	$3, (1)^3, (-1)^3, -3$	$(-16)^2, (-1)^2, 14, 89$
12	H_3	H_7	$4, 1, (0)^2, (-1)^2, -3$	$3, (1)^5, (-2)^4$	$(-0)^2, (-3)^2, 15, 100$
13	H_3	H_8	$4, 1, (0)^2, (-1)^2, -3$	$3, 2, (1)^2, (0)^2, (-1)^2, -2, -3$	$(-0)^2, (-3)^2, 15, 100$
14	H_3	H_9	$4, 1, (0)^2, (-1)^2, -3$	$3, 2, (1)^3, (-1)^2, (-2)^3$	$(-0)^2, (-3)^2, 15, 100$
15	K_8	$\overline{K_7}$	$7, (-1)^7$	$(0)^7$	$(-21)^2, (-3)^2, 15, 90$
16	$\overline{K_8}$	H_5	$(0)^8$	$6, 1, (0)^4, (-2)^2, -3$	$(-18)^2, (-1)^2, 12, 98$
17	$\overline{K_8}$	H_6	$(0)^8$	$6, (0)^6, (-3)^2$	$(-18)^2, (-1)^2, 12, 98$
18	$\overline{K_9}$	H_7	$(0)^9$	$3, (1)^5, (-2)^4$	$(-20)^2, (-1)^2, 20, 115$
19	$\overline{K_9}$	H_8	$(0)^9$	$3, 2, (1)^2, (0)^2, (-1)^2, -2, -3$	$(-20)^2, (-1)^2, 20, 115$
20	$\overline{K_9}$	H_9	$(0)^9$	$3, 2, (1)^3, (-1)^2, (-2)^3$	$(-20)^2, (-1)^2, 20, 115$
21	$5K_2$	$\overline{K_6}$	$(1)^5, (-1)^5$	$(0)^6$	$(-18)^2, (-1)^2, 17, 102$

Theorem 3.2. *$K_2\blacksquare_3 H_1$, $K_2\blacksquare_3 H_2$, $\overline{K_3}\blacksquare_3 K_4$, $K_4\blacksquare_3\overline{K_2}$, $C_4\blacksquare_3 K_6$, $2K_2\blacksquare_3 K_{5,5}$, $\overline{K_5}\blacksquare_3 H_1$, $\overline{K_5}\blacksquare_3 H_2$, $K_5\blacksquare_3 H_4$, $3K_2\blacksquare_3 C_4$, $\overline{K_7}\blacksquare_3 H_4$, $H_3\blacksquare_3 H_7$, $H_3\blacksquare_3 H_8$, $H_3\blacksquare_3 H_9$, $K_8\blacksquare_3\overline{K_7}$, $\overline{K_8}\blacksquare_3 H_5$, $\overline{K_8}\blacksquare_3 H_6$, $\overline{K_9}\blacksquare_3 H_7$, $\overline{K_9}\blacksquare_3 H_8$, $\overline{K_9}\blacksquare_3 H_9$, and $5K_2\blacksquare_3\overline{K_6}$ are integral distance spectrum graphs, where H_i ($1 \le i \le 9$) are depicted in Figure 2.*

4. Two new infinite family of distance equienergetic graphs

In this section, we give two pairs of distance equienergetic graphs on $p = k(9+t)$ for positive integers k and t.

DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634

Theorem 4.1. *For every integer $n \geq 10$, there exists pairs of distance equienrgetic graphs with n vertices.*

Proof. Let H_1 and H_2 be the graphs of Figure 2, H_1 and H_2 are both 3-regular graphs on 6 vertices with adjacency spectra $\begin{pmatrix} 3 & 0 & -3 \\ 1 & 4 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$, respectively. Let G_1 and G_2 denote respectively their line graphs. Then G_1 and G_2 are both 4-regular on 9 vertices. From [4], the adjacency spectra of G_1 and G_2 are $\begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 2 & 1 & -1 & -2 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}$, respectively. Thus, G_1 and G_2 are not cospectral.

Since H_1 and H_2 are non-isomorphic, $\overline{K_t}\blacksquare_k G_1$ and $\overline{K_t}\blacksquare_k G_2$ are non-isomorphic, and $\overline{K_t}\blacklozenge_k G_1$ and $\overline{K_t}\blacklozenge_k G_2$ are also non-isomorphic. And each of these graphs has $k(9+t)$ vertices. By Theorems 2.1 and 2.5, we have $ED(\overline{K_t}\blacksquare_k G_1) = ED(\overline{K_t}\blacksquare_k G_2)$ and $ED(\overline{K_t}\blacklozenge_k G_1) = ED(\overline{K_t}\blacklozenge_k G_2)$. □

Acknowledgments

The authors wish to thank the reviewers for useful comments and suggestions. Zikai Tang and Hanyuan Deng were supported by the Department of Education of Hunan Province (19A318). Renfang Wu and Hanlin Chen were supported by the Hunan Provincial Natural Science Foundation of China (2018JJ2249).

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DOI: http://dx.doi.org/10.22108/toc.2020.116372.1634