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ON QUADRILATERALS IN THE SUBORBITAL GRAPHS OF THE NORMALIZER

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Abstract. In this paper, we investigate suborbital graphs formed by $N\big(\Gamma_0(N)\big)$-invariant equivalence relation induced on $\hat{\mathbb{Q}}$. Conditions for being an edge are obtained as a main tool, then necessary and sufficient conditions for the suborbital graphs to contain a circuit are investigated.

1. Introduction

1.1. **The normalizer.** $\Gamma_0(N) = \{g \in \Gamma : c \equiv 0 \pmod N\}$ is a well known congruence subgroup of the classical modular group Γ. The normalizer of $\Gamma_0(N)$ in PSL(2,$\mathbb{R}$) turns to be a very important group in the study of moonshine and for this reason has been studied by many authors [5, 6, 16]. It consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}, ade^2 - bcN/h^2 = e \tag{1.1}$$

where $e \parallel \frac{N}{h^2}$ and h is the largest divisor of 24 for which $h^2|N$ with understandings that the determinant e of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, \frac{s}{r}) = 1$ (r is called an exact divisor of s).We denote the normaliser by $N\big(\Gamma_0(N)\big)$.

If $h = 1$, then the group is called Atkin-Lehner Group. Also we denote the set of the elements of $N(\Gamma_0(N))$ with determinat 1 by $\Gamma_C(N)$. $\Gamma_C(N)$ is a subgroup of $N\big(\Gamma_0(N)\big)$. So we can easily verify that

$$\Gamma_C(N) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0\left(N/h^2\right) \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}.$$

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The standard classification of the elements of $\mathrm{PSL}(2,\mathbb{R})$, motivated by their geometric properties when acting on $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$, depends on the trace of the corresponding matrices. Let G be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Any point in the boundary of $\mathbb{H}$ fixed by a parabolic element of G is called *a parabolic point* or *cusp* of G. The set of the parabolic points of G is called the *cusp set* of G. Two points x_1, x_2 in the cusp set of G are called G-inequivalent if there is no $g \in G$ such that $gx_1 = x_2$. The number of G-inequivalent cusps of G, that is, the number of orbits in the action of G on its cusp set, is called the parabolic class number of G. It can be seen easily that the cusp set of Γ is $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. The cusp set of $\Gamma_0(N)$ is also $\hat{\mathbb{Q}}$ because they are finite index subgroups of Γ.

Since $\Gamma_0(N)$ is a Fuchsian group whose fundamental domain has finite area, the same is true of $N\big(\Gamma_0(N)\big)$. So it has a signature consisting of the geometric invariants

$$(g; m_1, \ldots, m_r; s)$$

where g is the genus of the compactified quotient space, $m_1, \ldots, m_r$ are the periods of the elliptic elements and s is the parabolic class number.

The signature of a discrete Fuchsian group is a very interesting problem in group theory and arithmetic-algebraic geometry. The signature of $N\big(\Gamma_0(N)\big)$ was determined for square-free N by Maclachlan in [16]. Clearly, the general statement seems to be very difficult. A necessary and sufficient condition of transitive action of $N\big(\Gamma_0(N)\big)$ on its cusp set was given by Akbaş and Singerman in [2].

1.2. **Motivation.** The modular group acts transitively on $\hat{\mathbb{Q}}$ and in a paper of Jones, Singerman, Wicks, the suborbital graphs were studied and the most basic one turn out to be the well-known Farey graph [10].

In a series of papers, suborbital graphs of the normalizer were studied by same idea. All circuits in the suborbital graph were found when N is a square-free positive integer [3, 13] and when N satisfies the condition of transitive action [14]. Then, non-transitive cases have been examined to reach the general statement [7, 9, 11, 12]. An interesting contribution of these studies was that the action of normalizer offers solutions for some congruence equations deal with the sizes of circuits in the suborbital graph [8]. In this paper, we continue to examine some new cases. Verification of these congruences would be interesting because one immediate result is that we come close to obtain the suborbital graphs of $N\big(\Gamma_0(N)\big)$ for arbitrary-N. To avoid repetition, we also refer to introduction part of [8] for the aim of this work.

2. The Action of $N\big(\Gamma_0(N)\big)$ on $\hat{\mathbb{Q}}$

Every element of the extended set of rationals $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\dfrac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. ∞ is represented as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\dfrac{x}{y}$ is

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy}.$$

If the determinant of the matrix is 1 and $(x,y)=1$, then $(ax+by, cx+dy)=1$.

Lemma 2.1. [2, Corollary 2] *Let N have the prime power decomposition as $2^{\alpha_1}\cdot 3^{\alpha_2}\cdot p_3^{\alpha_3}\cdots p_r^{\alpha_r}$. Then $N\big(\Gamma_0(N)\big)$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7$, $\alpha_2 \le 3$ and $\alpha_i \le 1$ for $i = 3,\dots,r$.*

In this paper, we will take N as $2\cdot 3^2p^2$, where p is prime number greater than 3 and $p \equiv 1 \pmod 4$. From (1.1), $h=3$ and $e = 1, 2, p^2$ or $2p^2$. So we get the elements of $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ of the form

$$T_1 = \begin{pmatrix} a & \frac{b}{3} \\ 2\cdot 3p^2c & d \end{pmatrix},\ det\ T_1 = 1, \quad T_2 = \begin{pmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 2d \end{pmatrix},\ det\ T_2 = 2,$$

$$T_3 = \begin{pmatrix} ap^2 & \frac{b}{3} \\ 2\cdot 3p^2c & dp^2 \end{pmatrix},\ det\ T_3 = p^2, \quad T_4 = \begin{pmatrix} 2ap^2 & \frac{b}{3} \\ 2\cdot 3p^2c & 2dp^2 \end{pmatrix},\ det\ T_4 = 2p^2.$$

Hence, by Lemma 2.1 the following result is obvious.

Corollary 2.2. *The action of the normalizer $N(\Gamma_0(2\cdot 3^2p^2))$ is not transitive on $\hat{\mathbb{Q}}$.* □

Now, we will find a maximal subset of $\hat{\mathbb{Q}}$ on which the normalizer acts transitively. From this action, we will investigate some properties of suborbital graphs for $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$.

Lemma 2.3. [7, Corollary 2.4] *Let $d|N$. Then the orbit $\begin{pmatrix} a \\ d \end{pmatrix}$ of a/d with $(a,d)=1$ under $\Gamma_0(N)$ is the set $\Big\{x/y \in \hat{\mathbb{Q}} : (N,y) = d, a \equiv x\frac{y}{d} \pmod{(d, N/d)}\Big\}$. Furthermore the number of orbits $\begin{pmatrix} a \\ d \end{pmatrix}$ with $d|N$ under $\Gamma_0(N)$ is just $\varphi(d, N/d)$, where $\varphi(n)$ is Euler's totient function, which is the number of positive integers less than or equal to n that are coprime to n.* □

In the view of the above lemma, we can give the following

Corollary 2.4. *The orbits of the action of $\Gamma_0(2\cdot 3^2p^2)$ on $\hat{\mathbb{Q}}$ are*

$$\begin{pmatrix}1\\1\end{pmatrix};\begin{pmatrix}1\\2\end{pmatrix};\begin{pmatrix}1\\3\end{pmatrix};\begin{pmatrix}2\\3\end{pmatrix};\begin{pmatrix}1\\2\cdot 3\end{pmatrix};\begin{pmatrix}5\\2\cdot 3\end{pmatrix};\begin{pmatrix}1\\3^2\end{pmatrix};\begin{pmatrix}1\\2\cdot 3^2\end{pmatrix};\begin{pmatrix}1\\p^2\end{pmatrix};$$

$$\begin{pmatrix}1\\2p^2\end{pmatrix};\begin{pmatrix}1\\3p^2\end{pmatrix};\begin{pmatrix}2\\3p^2\end{pmatrix};\begin{pmatrix}1\\2\cdot 3p^2\end{pmatrix};\begin{pmatrix}11\\2\cdot 3p^2\end{pmatrix};\begin{pmatrix}1\\3^2p^2\end{pmatrix};\begin{pmatrix}1\\2\cdot 3^2p^2\end{pmatrix};$$

$$\begin{pmatrix}1\\p\end{pmatrix},\begin{pmatrix}2\\p\end{pmatrix},\dots,\begin{pmatrix}p-1\\p\end{pmatrix};\quad \begin{pmatrix}1\\2p\end{pmatrix},\begin{pmatrix}p+2\\2p\end{pmatrix},\begin{pmatrix}3\\2p\end{pmatrix},\dots,\begin{pmatrix}2p-1\\2p\end{pmatrix};$$

$$\begin{pmatrix}a_1\\3p\end{pmatrix},\begin{pmatrix}a_2\\3p\end{pmatrix},\dots,\begin{pmatrix}a_{2p-2}\\3p\end{pmatrix};\quad \begin{pmatrix}b_1\\3^2p\end{pmatrix},\begin{pmatrix}b_2\\3^2p\end{pmatrix},\dots,\begin{pmatrix}b_{p-1}\\3^2p\end{pmatrix};$$

$$\begin{pmatrix}c_1\\2\cdot 3p\end{pmatrix},\begin{pmatrix}c_2\\2\cdot 3p\end{pmatrix},\dots,\begin{pmatrix}c_{2p-2}\\2\cdot 3p\end{pmatrix};\quad \begin{pmatrix}d_1\\2\cdot 3^2p\end{pmatrix},\begin{pmatrix}d_2\\2\cdot 3^2p\end{pmatrix},\dots,\begin{pmatrix}d_{p-1}\\2\cdot 3^2p\end{pmatrix},$$

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where $a_i \not\equiv a_j \pmod{3p}, b_i \not\equiv b_j \pmod{p}, c_i \not\equiv c_j \pmod{3p}$ *and* $d_i \not\equiv d_j \pmod{p}$ *and the number of orbits is* $8p+8$.

Proof. According to Lemma 2.3, the possible values of d for the orbit $\begin{pmatrix} a \\ d \end{pmatrix}$ are $1, 2, 3, 23, 3^2, 2\cdot 3^2, p, 2p, 3p, 2\cdot 3p, 3^2p, 2\cdot 3^2p, p^2, 2p^2, 3p^2, 2\cdot 3p^2, 3^2p^2, 2\cdot 3^2p^2$. Hence, the number of non-conjugate classes of these orbits with Euler's formula are found as follows;

- 1 for $1, 2, 3^2, 2\cdot 3^2, p^2, 2p^2, 3^2p^2, 2\cdot 3^2p^2$
- 2 for $3, 2\cdot 3, 3p^2, 2\cdot 3p^2$,
- $p-1$ for $p, 2p, 3^2p, 2\cdot 3^2p$,
- $2p-2$ for $3p, 2\cdot 3p$.

□

Theorem 2.5. *The set*

$$\hat{\mathbb{Q}}(2\cdot 3^2p^2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2\cdot 3 \end{pmatrix} \cup \begin{pmatrix} 5 \\ 2\cdot 3 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2\cdot 3^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2\cdot 3p^2 \end{pmatrix} \cup \begin{pmatrix} 11 \\ 2\cdot 3p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 2\cdot 3^2p^2 \end{pmatrix},$$

is an orbit of $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ *on* $\hat{\mathbb{Q}}$.

Proof. Let's just look at the action of $T_4 = \begin{pmatrix} 2ap^2 & \frac{b}{3} \\ 2\cdot 3p^2c & 2dp^2 \end{pmatrix} \in N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ on the orbit $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Since *det* $T_4 = 2p^2$, we have $2adp^2 - bc = 1$. So b and c must be odd. If we use Lemma 2.3 for

$$T_4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2ap^2 & \frac{b}{3} \\ 2\cdot 3p^2c & 2dp^2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6ap^2+b \\ 2\cdot 3p^2(3c+d) \end{pmatrix} \approx \begin{pmatrix} x \\ y \end{pmatrix},$$

then the following cases are obtained.

(i) If $3|b$ and $3\nmid d$, then $T_4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2ap^2+b_0 \\ 2p^2(3c+d) \end{pmatrix} \in \begin{pmatrix} 1 \\ 2p^2 \end{pmatrix}$.

(ii) If $3\nmid b$ and $3|d$, then $T_4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 2\cdot 3^2p^2 \end{pmatrix}$.

(iii) If $3\nmid b$ and $3\nmid d$, then $y = \big(2\cdot 3^2p^2, 2\cdot 3p^2(3c+d)\big) = 2\cdot 3p^2$ and $\big(y, \frac{2\cdot 3^2p^2}{y}\big) = 3$.

Therefore, $x \equiv \big(6ap^2+b\big)\cdot\big(3c+d\big) \pmod 3$, that is, $x \equiv bd \pmod 3$. Since $b = 6k\pm 1$, $d = 6\ell \pm 1$ or $d = 6\ell \pm 2$ for some integers k and ℓ, we get $x \equiv 1 \pmod 3$ or $x \equiv 2 \pmod 3$. As $p \equiv 1 \pmod 4$, we obtain $T_4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \begin{pmatrix} 1 \\ 2\cdot 3p^2 \end{pmatrix}$ or $\begin{pmatrix} 11 \\ 2\cdot 3p^2 \end{pmatrix}$.

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Furthermore, by the similar calculations, we get the orbits $\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\3^2\end{pmatrix}$ for T_1, $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\2\cdot 3\end{pmatrix}, \begin{pmatrix}5\\2\cdot 3\end{pmatrix}, \begin{pmatrix}1\\2\cdot 3^2\end{pmatrix}$ for T_2 and $\begin{pmatrix}1\\p^2\end{pmatrix}, \begin{pmatrix}1\\3p^2\end{pmatrix}, \begin{pmatrix}2\\3p^2\end{pmatrix}, \begin{pmatrix}1\\3^2p^2\end{pmatrix}$ for T_3. □

Therefore the set $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$ is one on which $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ acts transitively. We now consider the imprimitivity of the action of $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ on $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$, beginning with a general discussion of primitivity of permutation groups.

Let (G,Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called *G-invariant* if, whenever $\alpha,\beta\in\Omega$ satisfy $\alpha\approx\beta$, then $g(\alpha)\approx g(\beta)$ for all $g\in G$. The equivalence classes are called blocks, and the block containing α is denoted by $[\alpha]$.

We call (G,Ω) *imprimitive* if Ω admits some G-invariant equivalence relation different from

- **(i)** the *identity relation*, $\alpha\approx\beta$ if and only if $\alpha=\beta$;
- **(ii)** the *universal relation*, $\alpha\approx\beta$ for all $\alpha,\beta\in\Omega$.

Otherwise (G,Ω) is called *primitive*. These two relations are supposed to be trivial relations. Clearly, a primitive group must be transitive, for if not the orbits would form a system of blocks. The converse is false. Now we give the useful result.

Lemma 2.6. [4, Theorem 1.6.5] *Let (G,Δ) be a transitive permutation group. (G,Δ) is primitive if and only if G_α,the stabilizer of $\alpha\in\Delta$, is a maximal subgroup of G for each $\alpha\in\Delta$.* □

From Lemma 2.6, we see that whenever, for some α, $G_\alpha \lneq H \lneq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g\in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g_1(\alpha)\approx g_2(\alpha) \text{ if and only if } g_2\in g_1H. \tag{2.1}$$

The number of blocks is the index $|G:H|$ and the block containing α is just the orbit $H(\alpha)$.

Lemma 2.7. *The stabilizer of a point in $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$ is an infinite cyclic group.*

Proof. By transitive action, the stabilizers of any two points are conjugate. Hence, it is enough to consider the stabilizers of ∞ in $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$. As

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}ae & \frac{b}{3}\\ 2\cdot 3p^2c & de\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}ae\\ 2\cdot 3p^2c\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},$$

we have $c=0$ and $e=1$. Also, by determinant, $ad=1$. So we get $T=\begin{pmatrix}1 & \frac{b}{3}\\ 0 & 1\end{pmatrix}$. This shows that $\Big(N(2\cdot 3^2p^2)\Big)_\infty=\left\langle\begin{pmatrix}1 & \frac{1}{3}\\ 0 & 1\end{pmatrix}\right\rangle$. □

Now, if we take $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$, $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$, $\big(N(2\cdot 3^2p^2)\big)_\infty$ and

$$N_0(2\cdot 3^2p^2) := \left\langle \Gamma_C\Big(2\cdot 3^2p^2\Big), T_2 = \begin{pmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 2d \end{pmatrix} \right\rangle$$

instead of G, Ω, G_α and H in Lemma 2.6, respectively, then it is easily seen that

$$\big(N(2\cdot 3^2p^2)\big)_\infty < N_0(2\cdot 3^2p^2) < N\big(\Gamma_0(2\cdot 3^2p^2)\big).$$

So $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ acts imprimitively on $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$. In this study, we will investigate the structure of normalizer by approximating to normalizer with the elements of $N_0\big(2\cdot 3^2p^2\big)$.

Lemma 2.8. [1, Proposition 2] *The index of* $\Gamma_0(N)$ *in* $\Gamma_C(N)$ *is given by*

$$\big|\Gamma_C(N) : \Gamma_0(N)\big| = h^2\tau,$$

where

$$\tau = (\frac{3}{2})^{\varepsilon_1}(\frac{4}{3})^{\varepsilon_2},\ \varepsilon_1 = \begin{cases} 1, & \text{if } 2^2, 2^4, 2^6 \parallel N; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \varepsilon_2 = \begin{cases} 1, & \text{if } 9 \parallel N; \\ 0, & \text{otherwise.} \end{cases}$$

□

Corollary 2.9. [1, Corollary 2] $\big|N\big(\Gamma_0(N)\big) : \Gamma_0(N)\big| = 2^\rho h^2\tau$, *where* τ *as in above and* ρ *is the number of prime factors of* $\frac{N}{h^2}$. □

Theorem 2.10. *The blocks arising from the imprimitive action of normalizer is as follows:*

$$[0] := \begin{pmatrix}1\\1\end{pmatrix} \cup \begin{pmatrix}1\\2\end{pmatrix} \cup \begin{pmatrix}1\\3\end{pmatrix} \cup \begin{pmatrix}2\\3\end{pmatrix} \cup \begin{pmatrix}1\\2\cdot 3\end{pmatrix} \cup \begin{pmatrix}5\\2\cdot 3\end{pmatrix} \cup \begin{pmatrix}1\\3^2\end{pmatrix} \cup \begin{pmatrix}1\\2\cdot 3^2.\end{pmatrix},$$

$$[\infty] := \begin{pmatrix}1\\p^2\end{pmatrix} \cup \begin{pmatrix}1\\2p^2\end{pmatrix} \cup \begin{pmatrix}1\\3p^2\end{pmatrix} \cup \begin{pmatrix}2\\3p^2\end{pmatrix} \cup \begin{pmatrix}1\\2\cdot 3p^2\end{pmatrix} \cup \begin{pmatrix}11\\2\cdot 3p^2\end{pmatrix} \cup \begin{pmatrix}1\\3^2p^2\end{pmatrix} \cup \begin{pmatrix}1\\2\cdot 3^2p^2\end{pmatrix}.$$

Proof. Since $h = 3, \tau = \frac{4}{3}$ and $\rho = 2$ for $N = 2\cdot 3^2p^2$, we get

$$\big|\Gamma_C\big(2\cdot 3^2p^2\big) : \Gamma_0\big(2\cdot 3^2p^2\big)\big| = 12 \text{ and } \big|N\big(\Gamma_0(2\cdot 3^2p^2)\big) : \Gamma_0\big(2\cdot 3^2p^2\big)\big| = 48,$$

by Lemma 2.8 and Corollary 2.9. Also, since $T_2^2 \in \Gamma_C\big(2\cdot 3^2p^2\big) \Leftrightarrow a+d = \pm 1$, we have $\big|N_0\big(2\cdot 3^2p^2\big) : \Gamma_0\big(2\cdot 3^2p^2\big)\big| = 24$. Using the equation

$$\big|N\big(\Gamma_0(2\cdot 3^2p^2)\big) : N_0\big(2\cdot 3^2p^2\big)\big| = \frac{\big|N\big(\Gamma_0(2\cdot 3^2p^2)\big) : \Gamma_0\big(2\cdot 3^2p^2\big)\big|}{\big|N_0\big(2\cdot 3^2p^2\big) : \Gamma_0\big(2\cdot 3^2p^2\big)\big|},$$

we obtain $\big|N\big(\Gamma_0(2\cdot 3^2p^2)\big) : N_0\big(2\cdot 3^2p^2\big)\big| = 2$.

Clearly $N\big(\Gamma_0(2\cdot 3^2p^2)\big) = N_0\big(2\cdot 3^2p^2\big) \cup \begin{pmatrix} ap^2 & \frac{b}{3} \\ 2\cdot 3p^2c & dp^2 \end{pmatrix} N_0\big(2\cdot 3^2p^2\big)$. So the number of blocks arising from the imprimitive action is 2 and these blocks are determined as desired by the relation (2.1). □

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3. Suborbital Graphs for $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$

Let (G,Ω) be transitive permutation group. Then G acts on $\Omega\times\Omega$ by $g(\alpha,\beta)=(g(\alpha),g(\beta))$, $(g\in G,\alpha,\beta\in\Omega)$. The orbits of this action are called *suborbitals* of G. The orbit containing (α,β) is denoted by $O(\alpha,\beta)$. From $O(\alpha,\beta)$ we can form a *suborbital graph* $G(\alpha,\beta)$: its vertices are the elements of Ω, and there is a directed edge from γ to δ if $(\gamma,\delta)\in O(\alpha,\beta)$. A directed edge from γ to δ is denoted by $\gamma\to\delta$. That is, there exists an edge $\gamma\to\delta$ in $G(\alpha,\beta)$ if and only if there is $g\in G$ such that $g(\alpha)=\gamma$ and $g(\beta)=\delta$. We can draw this edge as a hyperbolic geodesic in the upper half-plane $\mathbb{H}$.

If $O(\alpha,\beta)\neq O(\beta,\alpha)$, then $G(\beta,\alpha)$ is just $G(\alpha,\beta)$ with arrows reversed and we call $G(\alpha,\beta)$ and $G(\beta,\alpha)$ *paired suborbital graphs*. If $O(\alpha,\beta)=O(\beta,\alpha)$, then $G(\alpha,\beta)$ can be considered as an undirected graph and $G(\alpha,\beta)$ is called *self paired*.

The idea of the suborbital graphs of a permutation group G acting on a set Δ was introduced by Sims in [19]. Also, in [6], [15] and [20] it was applied to the finite groups.

We now take $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ and $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$ instead of G and Ω respectively. Since $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ acts transitively on $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$, each suborbital contains a pair $\big(\infty,\frac{u}{p^2}\big)$ for some $\frac{u}{p^2}\in\hat{\mathbb{Q}}(2\cdot 3^2p^2)$, where $(u,p^2)=1$. We denote the suborbital by $O\big(\infty,\frac{u}{p^2}\big)$ and the corresponding suborbital graph by G_{u,p^2}.

Definition 3.1. *By a directed circuit, we mean that a sequence* $v_1,v_2,\dots,v_m$ *of different vertices such that* $v_1\longrightarrow v_2\longrightarrow\cdots\longrightarrow v_m\longrightarrow v_1$, *where* $m\geq 3$.

If $m=3, m=4$ *and* $m=6$, *then the circuit, directed or not, is called a triangle, a quadrilateral, and a hexagon, respectively.*

If $m=2$, *then we will call the configuration* $v_1\longrightarrow v_2\longrightarrow v_1$ *a self paired.*

We call a graph a forest *if it does not contain any circuits.*

Theorem 3.2. *There is an edge* $\frac{r}{s}\to\frac{x}{y}$ *in* G_{u,p^2} *if and only if*

(i) *If* $2\cdot 3^2p^2\|s$, *then* $x\equiv\pm ur\pmod{p^2}, y\equiv\pm us\pmod{p^2}$, $ry-sx=\pm p^2$,
(ii) *If* $3^2p^2\|s$, *then* $x\equiv\pm 2ur\pmod{p^2}, y\equiv\pm 2us\pmod{p^2}$, $ry-sx=\pm p^2$,
(iii) *If* $2\cdot 3p^2\|s$, *then* $x\equiv\pm 3ur\pmod{p^2}, y\equiv\pm 3us\pmod{p^2}$, $ry-sx=\pm 3p^2$,
(iv) *If* $3p^2\|s$, *then* $x\equiv\pm 6ur\pmod{p^2}, y\equiv\pm 6us\pmod{p^2}$, $ry-sx=\pm 3p^2$,
(v) *If* $2p^2\|s$, *then* $x\equiv\pm 9ur\pmod{p^2}, y\equiv\pm 9us\pmod{p^2}$, $ry-sx=\pm p^2$,
(vi) *If* $p^2\|s$, *then* $x\equiv\pm 18ur\pmod{p^2}, y\equiv\pm 18us\pmod{p^2}$, $ry-sx=\pm p^2$.

Proof. Let $\frac{r}{s}\to\frac{x}{y}$ be an edge in G_{u,p^2}. Then $\big(\frac{r}{s},\frac{x}{y}\big)\in O\big(\infty,\frac{u}{p^2}\big)$. So there is an element $T=\begin{pmatrix} ae & \frac{b}{3}\\ 2\cdot 3p^2c & de\end{pmatrix}\in N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ such that $T(\infty)=\frac{r}{s}$ and $T\big(\frac{u}{p^2}\big)=\frac{x}{y}$.

We now assume that $2\cdot 3^2p^2\|s$. Therefore T must be of the form $T_1=\begin{pmatrix} a & \frac{b}{3}\\ 2\cdot 3p^2c & d\end{pmatrix}\in\Gamma_C\big(2\cdot 3^2p^2\big)$, $ad-2bcp^2=1$. If $3\mid b, 3\mid c$ and $3\nmid a, 3\nmid d$, then, for $i=0,1,$

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$$T_1(\infty)=\begin{pmatrix} a & b_0 \\ 2\cdot 3^2p^2c_0 & d\end{pmatrix}\begin{pmatrix}\frac{1}{0}\end{pmatrix}=\frac{a}{2\cdot 3^2p^2c_0}=\frac{(-1)^ir}{(-1)^is}.$$

So we get $r=(-1)^ia$ and $s=(-1)^i2\cdot 3^2p^2c_0$. Furthermore, for $j=0,1$

$$T_1\Big(\frac{u}{p^2}\Big)=\begin{pmatrix} a & b_0 \\ 2\cdot 3^2p^2c_0 & d\end{pmatrix}\begin{pmatrix}\frac{u}{p^2}\end{pmatrix}=\frac{au+b_0p^2}{2\cdot 3^2p^2c_0u+dp^2}=\frac{(-1)^jx}{(-1)^jy}.$$

As $det\ T_1=1$, we have $\big(au+b_0p^2, 2\cdot 3^2p^2c_0u+dp^2\big)=1$. Hence we obtain $x=(-1)^j(au+b_0p^2)$ and $y=(-1)^j(2^23p^2c_0u+dp^2)$, that is,

$$x\equiv(-1)^{i+j}ur \pmod{p^2}\ \ and\ \ y\equiv(-1)^{i+j}us \pmod{p^2}.$$

Since

$$\begin{pmatrix} a & b_0 \\ 2^23p^2c_0 & d\end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2\end{pmatrix}=\begin{pmatrix} a & au+b_0p^2 \\ 2^23p^2c_0 & 2^23p^2c_0u+dp^2\end{pmatrix}=\begin{pmatrix} (-1)^ir & (-1)^jx \\ (-1)^is & (-1)^jy\end{pmatrix},$$

for $i,j=0,1$, we have $ry-sx=\pm p^2$. This proves (i).

Now, let $3p^2\|s$ and we take the element $T_2=\left(\begin{smallmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 2d\end{smallmatrix}\right)$, $2ad-bcp^2=1$. If $3\nmid a, 3\nmid b,\ 3\nmid c$, then for $i=0,1$

$$T_2(\infty)=\begin{pmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 3d\end{pmatrix}\begin{pmatrix}\frac{1}{0}\end{pmatrix}=\frac{a}{3p^2c}=\frac{(-1)^ir}{(-1)^is},$$

which gives that $r=(-1)^ia$ and $s=(-1)^i3p^2c$. Furthermore, for $j=0,1$, we have

$$T_2\Big(\frac{u}{p^2}\Big)=\begin{pmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 2d\end{pmatrix}\begin{pmatrix}\frac{u}{p^2}\end{pmatrix}=\frac{6au+bp^2}{6(3p^2cu+dp^2)}=\frac{(-1)^jx}{(-1)^jy}.$$

As the matrix $\left(\begin{smallmatrix} 2a & \frac{b}{3} \\ 3p^2c & d\end{smallmatrix}\right)$ has determinant 1 and $(u,p^2)=1$, we obtain

$$\big(6au+bp^2, 3(3p^2cu+dp^2)\big)=1.$$

Therefore $\big(6au+bp^2, 6(3p^2cu+dp^2)\big)=1$. Thus we get

$$x\equiv(-1)^{i+j}\ 6ur \pmod{p^2}\ \ and\ \ y\equiv(-1)^{i+j}\ 6us \pmod{p^2}.$$

Since

$$\begin{pmatrix} 2a & \frac{b}{3} \\ 2\cdot 3p^2c & 2d\end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2\end{pmatrix}=\begin{pmatrix} a & 6au+bp^2 \\ 3p^2c & 6(2p^2cu+dp^2)\end{pmatrix}=\begin{pmatrix} (-1)^ir & (-1)^jx \\ (-1)^is & (-1)^jy\end{pmatrix},$$

we have $ry-sx=\pm 3p^2(2ad-bcp^2)=\pm 3p^2$, where $i=0,1$. This proves (iv).

If the following conditions hold for T_1 , then (iii) and (v) are obtained;

- $3\nmid a, 3\nmid b$ *and* $3\nmid c$
- $3\mid a$ and $3\nmid b, 3\nmid c$ respectively.

Also, if the conditions

- $3\nmid a$ and $3\mid b, 3\mid c$ and

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- $3 \mid a$ and $3 \nmid b, 3 \nmid c$

hold for T_2, we get (ii) and (vi) respectively.

In the opposite direction, we show only (v) and the plus sign. The others are similar. Suppose that $2p^2 \| s$, $x \equiv 9ur \pmod p$, $y \equiv 9us \pmod p$ and $ry - sx = p^2$. So there exist some integers k and ℓ such that $x = 9ur + kp^2$ and $y = 9us + \ell p^2$. Since $ry - sx = p^2$, we have $r\ell - ks = 1$, or $r\ell - \frac{3ks}{3} = 1$. Therefore we obtain the element $T = \begin{pmatrix} r & \frac{k}{3} \\ 3s & \ell \end{pmatrix}$ with determinant 1.

Since $2p^2 \| s$, $T \in \Gamma_C(2 \cdot 3^2 p^2)$. Also $T(\infty) = \frac{r}{s}$ and $T\left(\frac{u}{p^2}\right) = \frac{x}{y}$. Thus $\frac{r}{s} \to \frac{x}{y}$ is an edge in G_{u,p^2}. □

Theorem 3.3. *G_{u,p^2} is self-paired if and only if* $18u^2 \equiv -1 \pmod{p^2}$.

Proof. First we suppose that G_{u,p^2} is self-paired. Then $O\left(\infty, \frac{u}{p^2}\right) = O\left(\frac{u}{p^2}, \infty\right)$. Therefore there is an element $T = \begin{pmatrix} ae & \frac{b}{3} \\ 2 \cdot 3p^2 c & de \end{pmatrix} \in N\left(\Gamma_0(2 \cdot 3^2 p^2)\right)$ such that $T(\infty) = \frac{u}{p^2}$ and $T\left(\frac{u}{p^2}\right) = \infty$. As

$$T(\infty) = \frac{ae}{2 \cdot 3p^2 c} = \frac{u}{p^2},$$

we get $3|a, e = 2$ *and* $c = 1$. Since $3|a$, there exist an integer a_0 such that $a = 3a_0$. Hence, we obtain $a_0 = u$ by above equation. Also, from

$$T\left(\frac{u}{p^2}\right) = \frac{3aue + bp^2}{3p^2(6cu + de)} = \frac{1}{0},$$

we have $6cu + de = 0$. Since $e = 2$ *and* $c = 1$, we determine $d = -3u$. As *det* $T = 2ad - bcp^2 = 1$, we obtain $6a_0 d - bcp^2 = 1$, that is, $18u^2 \equiv -1 \pmod{p^2}$.

Now let $18u^2 \equiv -1 \pmod{p^2}$. Then there exists $k \in \mathbb{Z}$ such that $18u^2 = -1 - kp^2$. Hence $-36u^2 - 2kp^2 = 1$. So $T = \begin{pmatrix} 6u & \frac{k}{3} \\ 2 \cdot 3p^2 & -6u \end{pmatrix} \in N\left(\Gamma_0(2 \cdot 3^2 p^2)\right)$, and $T(\infty) = \frac{u}{p^2}$, $T\left(\frac{u}{p^2}\right) = \frac{-1}{0} = \infty$. Thus $O\left(\infty, \frac{u}{p^2}\right) = O\left(\frac{u}{p^2}, \infty\right)$. This shows that G_{u,p^2} is self-paired. □

4. Subgraph F_{u,p^2}

Since $N\left(\Gamma_0(2 \cdot 3^2 p^2)\right)$ acts transitively on $\hat{\mathbb{Q}}(2 \cdot 3^2 p^2)$, $N\left(\Gamma_0(2 \cdot 3^2 p^2)\right)$ permutes the blocks transitively; so the subgraphs are all isomorphic. Thus it is sufficient to study with only one block. We denote by F_{u,p^2} the subgraph of G_{u,p^2} whose vertices are in the block

$$[\infty] := \binom{1}{p^2} \cup \binom{1}{2p^2} \cup \binom{1}{3p^2} \cup \binom{2}{3p^2} \cup \binom{1}{2 \cdot 3p^2} \cup \binom{11}{2 \cdot 3p^2} \cup \binom{1}{3^2 p^2} \cup \binom{1}{2 \cdot 3^2 p^2},$$

so that G_{u,p^2} consists of two disjoint copies F_{u,p^2}.

Theorem 4.1. *$N_0\left(2 \cdot 3^2 p^2\right)$ permutes the vertices and the edges of F_{u,p^2} transitively.*

Proof. We show that $N_0\big(2\cdot 3^2p^2\big)$ permutes the vertices of F_{u,p^2}. The proof for the edges of F_{u,p^2} is similar. Let $u,w\in[\infty]$ are vertices of F_{u,p^2}. Since $N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ acts transitively on $\hat{\mathbb{Q}}(2\cdot 3^2p^2)$, there is an element $T=\begin{pmatrix} ae & \frac{b}{3}\\ 2\cdot 3p^2c & de\end{pmatrix}\in N\big(\Gamma_0(2\cdot 3^2p^2)\big)$ such that $T(v)=w$. As $u,w\in[\infty]$, $v=\frac{x}{yp^2}$ and $v=\frac{k}{\ell p^2}$. Therefore

$$\begin{pmatrix} ae & \frac{b}{3}\\ 2\cdot 3p^2c & de\end{pmatrix}\left(\frac{x}{yp^2}\right)=\frac{3aex+byp^2}{3p^2(6cx+dey)}=\frac{k}{\ell p^2}.$$

So e must be 1 or 2. Thus $T\in N_0\big(2\cdot 3^2p^2\big)$. Consequently, the action of $N_0\big(2\cdot 3^2p^2\big)$ on the block $[\infty]$ is transitive.

With similar calculation, it can shown that $N_0\big(2\cdot 3^2p^2\big)$ permutes the edges of F_{u,p^2}. □

Theorem 4.2. *No edges of F_{u,p^2} cross in $\mathbb{H}$.*

Proof. We assume that the edges $\infty\rightarrow\frac{u}{p^2}$ and $\frac{x_1}{y_1p^2}\overset{<}{\rightarrow}\frac{x_2}{y_2p^2}$ cross in $\mathbb{H}$, where all letters are positive integers. So $\frac{x_1}{y_1p^2}<\frac{u}{p^2}<\frac{x_2}{y_2p^2}$. By Theorem 3.2, we obtain $x_1y_2-x_2y_1=-1$ or $x_1y_2-x_2y_1=-3$. If $x_1y_2-x_2y_1=-3$, we obtain $y_1=3$ or $y_1=6$ by the edge conditions (iii) and (iv) in Theorem 3.2. Since $\frac{x_1}{y_1}<u<\frac{x_2}{y_2}$, we get $x_1<uy_1$ and $uy_2<x_2$. Therefore

$$3=x_2y_1-x_1y_2>x_2y_1-uy_1y_2=y_1(x_2-uy_2).$$

Since $u,x_2,y_2\in\mathbb{Z}$, then $x_2-uy_2\geq 1$. Thus we obtain a contradiction $y_1<3$. By a similar calculation for $x_1y_2-x_2y_1=-1$, it is seen that $y_1<1$, which is impossible. □

Corollary 4.3. *F_{u,p^2} has a self paired edge iff $18u^2\equiv -1 \pmod{p^2}$.*

Theorem 4.4. *F_{u,p^2} contains no triangle.*

Proof. Suppose on the contrary that F_{u,p^2} contains a triangle. Because of transitive action, we can take this triangle of the form $\frac{1}{0}\rightarrow\frac{u}{p^2}\rightarrow\frac{k}{\ell p^2}\rightarrow\frac{1}{0}$. By Theorem 3.2, $\ell=1$ or $\ell=3$.

Let $\ell=3$. Then, for the edge $\frac{u}{p^2}\rightarrow\frac{k}{3p^2}$, we have $3u-k=\pm 1$ and $3u\mp 1\equiv\pm 18u^2 \pmod{p^2}$, that is,

$$18u^2\mp 3u+1\equiv 0 \pmod{p^2}.$$

Furthermore, since $1\equiv -6uk \pmod{p^2}$ for the third edge,

$$18u^2\mp 6u+1\equiv 0 \pmod{p^2}$$

is obtained. Thus, $3u\equiv 0 \pmod{p^2}$. Therefore we get the contradiction $u\equiv 0 \pmod{p^2}$.

For $\ell=1$, the case is similar. Consequently, there is no triangle in F_{u,p^2}. □

Theorem 4.5. *F_{u,p^2} contains a quadrilateral if and only if $18u^2\pm 6u+1\equiv 0 \pmod{p^2}$.*

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Proof. Firstly, we assume that the subgraph F_{u,p^2} has the quadrilateral

$$\frac{x_1}{y_1p^2} \xrightarrow{<} \frac{x_2}{y_2p^2} \xrightarrow{<} \frac{x_3}{y_3p^2} \xrightarrow{<} \frac{x_4}{y_4p^2} \xrightarrow{<} \frac{x_1}{y_1p^2}.$$

Since $N_0(2\cdot 3^2p^2)$ permutes the vertices and the edges of F_{u,p^2} by Theorem 4.1, this quadrilateral can be take of the form $\frac{1}{0} \rightarrow \frac{u}{p^2} \xrightarrow{<} \frac{x}{yp^2} \xrightarrow{<} \frac{k}{\ell p^2} \rightarrow \frac{1}{0}$.

Using Theorem 3.2 for the second edge, $x = uy+1$ and $x \equiv -18u^2 \pmod{p^2}$ are obtained. Therefore

$$18u^2 + uy + 1 \equiv 0 \pmod{p^2}. \tag{4.1}$$

For the last edge, ℓ must be 1 or 3.

Now let $\ell = 1$. So from the edge $\frac{x}{yp^2} \xrightarrow{<} \frac{k}{\ell p^2}$, we get $x - ky = -1$ or $x - ky = -3$. If $x - ky = -1$, then from $x = uy + 1$, we have $y(u - k) = -2$. So $y = 1$ or $y = 2$. By the edge conditions, we obtain $k = u+2$ and $18u^2+u+1 \equiv 0 \pmod{p^2}$ for $y = 1$, and $k = u+1$ and $18u^2+2u+1 \equiv 0 \pmod{p^2}$ for $y = 2$. From here and the last edge, we get $u \equiv 0 \pmod{p^2}$. This contradicts the fact that $(u, p^2) = 1$.

Let $x-ky = -3$. Then, by Theorem 3.2, y must be 3 or 6. Since $x = uy+1$, we obtain $y(u-k) = -4$. Therefore the contradiction $3|4$ or $6|4$ is obtained.

Now we suppose that $\ell = 3$. So $1 \equiv -6uk \pmod{p^2}$ for the edge $\frac{k}{3p^2} \rightarrow \frac{1}{0}$. From Eq.(4.1), $k \equiv 3u + \frac{y}{6} \pmod{p^2}$. Hence y must be 6 or 18. If $y = 18$, we get $k = 3u + 3$, which contradicts $(k, 3p^2) = 1$.

Consequently, we have $y = 6$ and $\ell = 3$. Hence by Eq.(4.1), $18u^2 + 6u + 1 \equiv 0 \pmod{p^2}$. Similarly, for the quadrilateral $\frac{1}{0} \rightarrow \frac{u}{p^2} \xrightarrow{>} \frac{x}{yp^2} \xrightarrow{>} \frac{k}{\ell p^2} \rightarrow \frac{1}{0}$, $18u^2 - 6u + 1 \equiv 0 \pmod{p^2}$ is obtained.

Conversely, suppose that $12u^2 \pm 6u + 1 \equiv 0 \pmod{p^2}$. Using the edge conditions in Theorem 3.2, it is easily seen that

$$\frac{1}{0} \rightarrow \frac{u}{p^2} \rightarrow \frac{6u \pm 1}{6p^2} \rightarrow \frac{3u \pm 1}{3p^2} \rightarrow \frac{1}{0}$$

is a quadrilateral in F_{u,p^2}. □

Example 1. $\frac{1}{3\cdot 25} \underset{(1)}{\xrightarrow{<}} \frac{19}{54\cdot 25} \underset{(2)}{\xrightarrow{<}} \frac{6}{17\cdot 25} \underset{(3)}{\xrightarrow{<}} \frac{17}{48\cdot 25} \underset{(4)}{\xrightarrow{>}} \frac{1}{3\cdot 25}$ *is a quadrilateral in* $F_{1,25}$.

According to the edge conditions (iv), (i), (vi) and (iii) in Theorem 3.2, we get

- ***(1)*** $3p^2\|s$, $19 \equiv -6\cdot 1\cdot 1 \pmod{25}$ *and* $ry - sx = -3\cdot 25$
- ***(2)*** $18p^2\|s$, $6 \equiv -1\cdot 1\cdot 19 \pmod{25}$ *and* $ry - sx = -25$
- ***(3)*** $p^2\|s$, $17 \equiv -18\cdot 6\cdot 1 \pmod{25}$ *and* $ry - sx = -25$
- ***(4)*** $2\cdot 3p^2\|s$, $1 \equiv 3\cdot 17\cdot 1 \pmod{25}$ *and* $ry - sx = 3\cdot 25$ *respectively.*

Also $\frac{1}{0} \xrightarrow{>} \frac{1}{25} \xrightarrow{<} \frac{7}{6\cdot 25} \xrightarrow{<} \frac{4}{3\cdot 25} \xrightarrow{<} \frac{1}{0}$ *in* $F_{1,25}$, *and the element* $T = \begin{pmatrix} -24 & \frac{1}{3} \\ -6\cdot 25 & 2 \end{pmatrix} \in N_0(2\cdot 3^2p^2)$ *sends*

$$\frac{1}{3\cdot 25} \rightarrow \frac{19}{54\cdot 25} \rightarrow \frac{6}{17\cdot 25} \rightarrow \frac{17}{48\cdot 25} \rightarrow \frac{1}{3\cdot 25} \text{ to } \frac{1}{0} \rightarrow \frac{1}{25} \rightarrow \frac{7}{6\cdot 25} \rightarrow \frac{4}{3\cdot 25} \rightarrow \frac{1}{0}$$

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which are given in Figure 1.

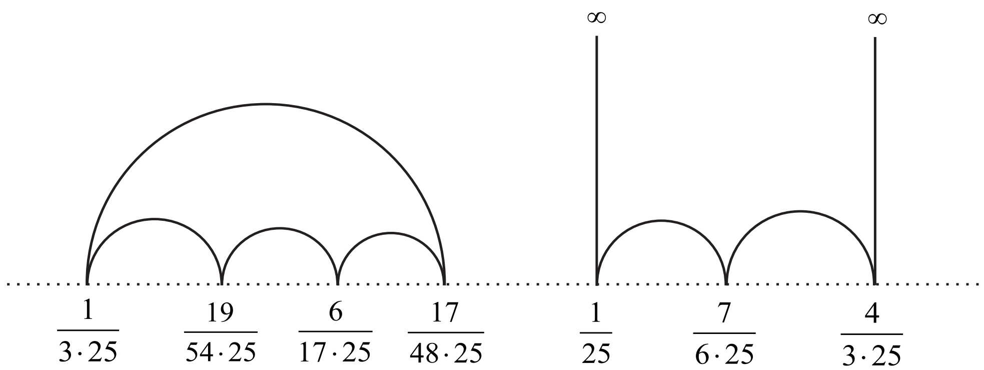

FIGURE 1. The quadrilaterals in $F_{1,25}$

Furthermore by similar calculations, we obtain the quadrilaterals

$$\frac{1}{0} \to \frac{7}{25} \to \frac{43}{6\cdot 25} \to \frac{22}{3\cdot 25} \to \frac{1}{0} \; and \; \frac{2}{9\cdot 25} \to \frac{3}{14\cdot 25} \to \frac{11}{51\cdot 25} \to \frac{13}{60\cdot 25} \to \frac{2}{9\cdot 25} \; in F_{7,25},$$
$$\frac{1}{0} \to \frac{40}{169} \to \frac{239}{6\cdot 169} \to \frac{119}{3\cdot 169} \to \frac{1}{0} \; in \;\; F_{40,169},$$
$$\frac{1}{0} \to \frac{42}{289} \to \frac{251}{6\cdot 289} \to \frac{125}{3\cdot 289} \to \frac{1}{0} \; in \;\; F_{42,289}.$$

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