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ENGEL GROUPS IN BATH – TEN YEARS LATER

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Dedicated to our friend Prof. Pavel Shumyatsky on the occasion of his 60th birthday

Abstract. The eighth edition of the international series of Groups St Andrews conferences was held at the University of Bath in 2009 and one of the theme days was dedicated to Engel groups. Since then much attention has been devoted to a verbal generalization of Engel groups. In this paper we will survey the development of this investigation during the last decade.

1. Group-words

Let F be the free group on free generators $x_1, \ldots, x_d$, for some $d > 1$. A group-word w is any nontrivial element of F, that is, a product of finitely many x_i's and their inverses. The word w is called a commutator word or a non-commutator word, depending on whether w does or does not lie in the commutator subgroup of F. In particular, a non-commutator word is a word such that the sum of the exponents of some variable involved in it is nonzero.

Following [24], we say that the commutator word

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}] = [\ldots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_k}]$$

is a simple commutator word if $k \geq 2, i_1 \neq i_2$ and $i_j \in \{1, \ldots, d\}$ for any $j \in \{1, \ldots, k\}$. Examples of simple commutator words are the lower central words $\gamma_k = [x_1, \ldots, x_k]$, with $k \geq 2$ different variables, and the n-Engel words $[x,_n y]$, with $n \geq 1$. Here, as usual, the commutator $[x,_n y]$ is defined inductively by the rules

$$[x,_1 y] = [x, y] \quad \text{and} \quad [x,_n y] = [[x,_{n-1} y], y] \quad \text{for } n \geq 2.$$

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Let $w = w(x_1, \ldots, x_d)$ be a group-word in the variables $x_1, \ldots, x_d$. For any group G and arbitrary $g_1, \ldots, g_d \in G$, the elements of the form $w(g_1, \ldots, g_d)$ are called the w-values in G. We denote by G_w the set of all w-values in G. The verbal subgroup of G corresponding to w is the subgroup $w(G)$ of G generated by G_w. If $w(G) = \{1\}$, then G is said to satisfy the identity $w \equiv 1$. The class of all groups satisfying the identity $w \equiv 1$ is the variety determined by w. More generally, a variety is a class of groups defined by equations. By a well-known theorem of Birkhoff (see, for instance, [26, 2.3.5]), varieties are precisely classes of groups closed with respect to taking subgroups, quotients and Cartesian products of their members.

Most of the words considered in this survey are multilinear commutator words, also known under the name of outer commutator words. These are words that have a form of a multilinear Lie monomial, i.e., they are constructed by nesting commutators but using always different variables. Thus $[x, [y_1, y_2], [z_1, z_2, z_3]]$ is a multilinear commutator word while the n-Engel word is not, for any $n \geq 2$. Formally, multilinear commutator words are recursively defined as follows: the variable x is a multilinear commutator word of weight 1; if $u = u(x_1, \ldots, x_i)$ and $v = v(y_1, \ldots, y_j)$ are multilinear commutator words involving different variables of weights i and j respectively, then $[u, v]$ is a multilinear commutator word of weight $i + j$.

An important family of multilinear commutator words consists of the lower central words γ_k. The corresponding verbal subgroups $\gamma_k(G)$ are the terms of the lower central series of G. Another distinguished sequence of outer commutator words are the derived words δ_k, on 2^k variables, which are defined by

$$\delta_0 = x_1 \quad \text{and} \quad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})], \quad \text{for } k \geq 1.$$

The verbal subgroup that corresponds to the word δ_k is the familiar kth derived subgroup $G^{(k)}$ of G.

The following lemma shows the close connection between multilinear commutator words and derived words.

Lemma 1.1. [31, Lemma 4.1] *Let v a multilinear commutator word of weight $k \geq 1$. Then, for any group G, every δ_k-value in G is a v-value.*

We also mention a property of the set of v-values, when v is a multilinear commutator word.

Lemma 1.2. [12, Lemma 2.2] *Let v a multilinear commutator word. Then, for any group G, the set G_v is symmetric, i.e., $x \in G_v$ implies that $x^{-1} \in G_v$.*

2. Groups satisfying an Engel type identity

An element g of a group G is called a left Engel element if for any $x \in G$ there exists $n = n(g, x) \geq 1$ such that $[x,_n g] = 1$. If n can be chosen independently of x, then g is a left n-Engel element or simply a left bounded Engel element. Right Engel elements are defined in a similar way, assuming that the variable x appears on the right.

The following result is well-known and is due to Heineken (see, for instance, [26, 12.3.1]).

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Lemma 2.1. *In any group G the inverse of a right Engel element is a left Engel element, and the inverse of a right n-Engel element is a left $(n+1)$-Engel element.*

Clearly, given a group-word w, a group in which all w-values are left (or right) n-Engel elements satisfies an identity.

A group is called an Engel group (or an n-Engel group, resp.) if its elements are both left and right Engel (or left and right n-Engel, resp.). It is a long-standing problem whether any n-Engel group G is locally nilpotent. This is the analogue of the Burnside problem in the realm of Engel groups (see, for instance, [36]). Following Zelmanov's solution of the restricted Burnside problem [43, 44], Wilson proved that this is true if G is residually finite ([38], see also [36]). As a consequence, the class of locally nilpotent n-Engel groups turns out to be a variety [45].

In [18], Kim and Rhemtulla extended Wilson's theorem by showing that any locally graded n-Engel group is locally nilpotent. Recall that a group is locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index. Later, in [30] (see also [29]), Shumyatsky proved that if all commutators $[x_1, \dots, x_k]$ of a residually finite group G are left n-Engel elements, then the subgroup $\gamma_k(G)$ is locally nilpotent. This, together with the aforementioned results, leads to formulate the following conjectures.

Conjecture. *Let n be a positive integer and w an arbitrary group-word.*

- $\mathbf{L_1}$. *If G is a locally graded group in which all w-values are left n-Engel elements, then the verbal subgroup $w(G)$ is locally nilpotent (see [4]).*
- $\mathbf{L_2}$. *The class of all groups G in which the w-values are left n-Engel elements and the verbal subgroup $w(G)$ is locally nilpotent is a variety.*

The class of locally graded groups is rather wide, since it includes locally (soluble-by-finite) groups and residually finite groups. A quotient of a locally graded group need not be locally graded. Nevertheless, we have the following sufficient condition for a quotient to be locally graded.

Proposition 2.2. [21] *Let G be a locally graded group and let N be a normal locally nilpotent subgroup of G. Then G/N is locally graded.*

An answer to Conjectures L_1 and L_2 is provided when w is a power of a multilinear commutator word.

Theorem 2.3. *Let m, n be positive integers, v a multilinear commutator word and $w = v^m$.*

- (i) *If G is a locally graded group in which all w-values are left n-Engel elements, then the verbal subgroup $w(G)$ is locally nilpotent* ([34, Theorem 1.1], see also [4, Theorem A]).
- (ii) *The class of all groups G in which the w-values are left n-Engel elements and the verbal subgroup $w(G)$ is locally nilpotent is a variety* [35, Theorem A].

We also remark that in the residually finite case a quantitative version of Theorem 2.3 (i) can be deduced from Proposition 1 of [10].

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Corollary 2.4. *Let d, m, n be positive integers, v a multilinear commutator word and $w = v^m$. Let G be a residually finite group in which all w-values are left n-Engel elements, and suppose that $w(G)$ is generated by d elements. Then $w(G)$ is nilpotent of class $c = c(d, m, n, v)$.*

A subset X of a group is commutator closed if $[x, y] \in X$ for any $x, y \in X$. Of course, in any group all δ_k-values form a (normal) commutator closed set. Usually the restriction to multilinear commutator words is motivated by Lemma 1.1, which allows to deal with δ_k-values, for some $k \geq 1$. Thus, when $m = 1$, part (i) of Theorem 2.3 can be seen as a consequence of the next theorem (applied to $w(G)$ and taking into account that in a group having an ascending normal series with locally soluble factors the set of left Engel elements is a locally nilpotent subgroup [4, Proposition 7]).

Theorem 2.5. *Let G be a group satisfying an identity and suppose that G is generated by a commutator closed set X of bounded left Engel elements.*

(i) If G is residually finite, then G is locally nilpotent [3, Theorem A].

(ii) If G is locally graded and X is a normal set, then G is locally nilpotent [3, Theorem 3.4].

The proof of Theorem 2.5 (i) depends on Lie-theoretic techniques created by Zelmanov in his solution of the restricted Burnside problem.

Let L be a Lie algebra over a field. An element $b \in L$ is called ad-nilpotent if there exists a positive integer n such that

$$[a,_n b] = [[\dots[[a, \underbrace{b], b], \dots], b}_{n \text{ times}}] = 0$$

for all $a \in L$. Following [39], we say that a subset X of L is a Lie set if $[a, b] \in X$ for any $a, b \in X$. We denote by $S\langle X\rangle$ the Lie set generated by X, namely the smallest Lie set containing X.

Let F be the free Lie algebra over the same field as L on the generators $x_1, \dots, x_d$. For a nonzero element $f = f(x_1, \dots, x_d)$ of F, the Lie algebra L is said to satisfy the polynomial identity $f \equiv 0$ if $f(a_1, \dots, a_d) = 0$ for all $a_1, \dots, a_d \in L$.

In this context, the following theorem is crucial.

Theorem 2.6. ([39, Theorem 1.1], see also [40, 41, 42]) *Let L be a Lie algebra satisfying a polynomial identity and generated by elements $a_1, \dots, a_d$. If every element $b \in S\langle a_1, \dots, a_d\rangle$ is ad-nilpotent, then L is nilpotent.*

Next, using Theorem 2.3 (i), we prove Conjecture L_1 for non-commutator words (see also [4, Theorem C]).

Proposition 2.7. *Let n be a positive integer and w a non-commutator word. If G is a locally graded group in which all w-values are left n-Engel elements, then the verbal subgroup $w(G)$ is locally nilpotent.*

Indeed, let $w = w(x_1, \dots, x_k)$ be a non-commutator word and let G be a locally graded group in which all w-values are left n-Engel elements. Assume that the sum of the exponents of some x_i is $m \neq 0$. Substituting an arbitrary element $g \in G$ for x_i and 1 for the other variables, we see that g^m is a w-value for any $g \in G$. Hence every mth power is a left n-Engel element in G and so, by Theorem 2.3

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(i), the subgroup G^m is locally nilpotent. Applying Proposition 2.2, it follows that G/G^m is a locally graded group of finite exponent. Thus G/G^m is locally finite, by the positive solution of the restricted Burnside problem (see, for instance, [22, Theorem 1]). Since finite and soluble groups generated by finitely many left Engel elements are nilpotent (see, for instance, [26, 12.3.3, 12.3.7]), we deduce that $w(G)$ is locally nilpotent.

Another standard argument yields a positive answer to Conjecture L_2 for non-commutator words (see [35, Theorem A]).

Theorem 2.8. *Let n be a positive integer and w a non-commutator word. Then the class of all groups G in which the w-values are left n-Engel elements and the verbal subgroup $w(G)$ is locally nilpotent is a variety.*

A group G is orderable if there exists a full order relation on the set G such that $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in G$, i.e., the order on G is compatible with the product of G. Any orderable group is torsion-free and, conversely, any torsion-free nilpotent group is orderable (see [5]). According to a result of Kim and Rhemtulla, every orderable n-Engel group is nilpotent ([19], see also [20]).

In the same spirit of Theorem 2.3 (i), we have the following result concerning orderable groups.

Theorem 2.9. [32, Theorem 1.1] *Let m, n be positive integers, v a multilinear commutator word and $w = v^m$. If G is an orderable group in which all w-values are left n-Engel elements, then the verbal subgroup $v(G)$ is locally nilpotent.*

Notice that unlike the situation with the locally graded groups where the result is that $w(G)$ is locally nilpotent, Theorem 2.9 states that so is $v(G)$. Of course, in general $w(G) \leq v(G)$. Furthermore $v(G) = G$ is nilpotent, in the particular case where $v = x$.

Theorem 2.10. [32, Theorem 1.2] *Let m, n be positive integers and let G be an orderable group. If x^m is a left n-Engel element, for any $x \in G$, then G is nilpotent.*

There still remains the question whether, under the hypotheses of Theorem 2.9, the subgroup $w(G)$ is actually nilpotent.

We now reformulate Conjectures L_1 and L_2 in terms of right Engel elements.

Conjecture. *Let n be a positive integer and w an arbitrary group-word.*

- $\mathbf{R_1}$. *If G is a locally graded group in which all w-values are right n-Engel elements, then the verbal subgroup $w(G)$ is locally nilpotent.*
- $\mathbf{R_2}$. *The class of all groups G in which the w-values are right n-Engel elements and the verbal subgroup $w(G)$ is locally nilpotent is a variety.*

Despite to the "left conjectures", we are able to say that Conjecture R_1, for residually finite groups, and Conjecture R_2 are right!

Theorem 2.11. *Let n be a positive integer and w an arbitrary group-word.*

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(i) *If G is a residually finite group satisfying an identity, then the set of right Engel elements of G is contained in the Hirsch-Plotkin radical of G* [33, Theorem A].

(ii) *The class of all groups in which the w-values are right n-Engel elements and $w(G)$ is locally nilpotent is a variety* [33, Theorem B].

Recall that the Hirsch-Plotkin radical of a group G is the unique maximal normal locally nilpotent subgroup of G, and it contains all normal locally nilpotent subgroups of G (see [26, 12.1.3]).

The proof of Thorem 2.11 (i) is based on the following corollary of Theorem 2.6 (see also [23, Lemma 5]).

Corollary 2.12. [33, Corollary 2.2] *Let p be a prime and let L be a Lie algebra over the field with p elements satisfying a polynomial identity. Suppose that L is generated by elements $a_1, \dots, a_d$ such that $[a_{i,p^k} b] = 0$, for some $k \geq 1$ and any $b \in S\langle a_1, \dots, a_d\rangle$. Then L is nilpotent.*

Let p be a prime. Using Theorem 2.6, Zelmanov proved that any residually finite p-group which satisfies an identity is locally finite [39, Theorem 1.2]. It is still unknown whether a similar result holds for periodic groups. However, this suggested the following conjecture [3].

Conjecture. *Let G be a residually finite Engel group satisfying an identity. Then G is locally nilpotent.*

Part (i) of Theorem 2.11 confirms the previous conjecture. More generally, by a folklore argument, the result can be extended to locally graded groups (see, for instance, [3, Theorem C]).

Corollary 2.13. *Every locally graded Engel group satisfying an identity is locally nilpotent.*

Following [3], we say that a group G is a nil group if all elements of G are bounded Engel elements, i.e., for any $g \in G$ there is $n = n(g) \geq 1$ such that $[x,_n g] = 1$ for all $x \in G$. A nil group satisfying an identity might not be locally nilpotent, as announced by E. Rips in a series of lectures on n-Engel groups. On the other hand, the following question remains open.

Question. *Is any locally graded nil group necessarily locally nilpotent?*

For the sake of completeness we point out that, applying Lemmas 1.2 and 2.1, Theorem 2.9 is still true for right Engel elements.

Theorem 2.14. *Let m, n be positive integers, v a multilinear commutator word and $w = v^m$. If G is an orderable group in which all w-values are right n-Engel elements, then the verbal subgroup $v(G)$ is locally nilpotent.*

3. Conciseness of some Engel type words

A group-word w is said to be concise in a class of groups $\mathcal{X}$ if $w(G)$ is finite whenever G_w is finite for a group $G \in \mathcal{X}$. In addition, w is boundedly concise in $\mathcal{X}$ if for any positive integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that $|w(G)| \leq \nu$ whenever $|G_w| \leq m$ for a group $G \in \mathcal{X}$. When $\mathcal{X}$ is the class of all groups we speak of concise and boundedly concise words, respectively. Notice that every word which is concise in the class of all groups is also boundedly concise.

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Theorem 3.1. [12, Theorem A.1] *Let w be a concise word and let G be a group in which w takes $m \geq 1$ values. Then the order of $w(G)$ is bounded by a function depending on w and m.*

In the sixties P. Hall conjectured that every word is concise, but his conjecture was refuted in 1989 by Ivanov.

Example 3.2. [16] *For any odd integer $n > 10^{10}$ and any prime number $p > 5000$, there exists a 2-generator torsion-free group A with cyclic centre such that the word*

$$v(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$$

takes only two values in A and the nontrivial value is a generator of $Z(A)$. Hence, $v(A)$ is infinite.

Further examples of words which are not concise were produced in [25, p. 439] and [6, Proposition 4.1]. However, many words of common use are known to be concise. For example, every non-commutator word is concise [27, Part 1, Lemma 4.27] and similarly for the commutator of two non-commutator words [8, Theorem 1.1]. By a result of J. C. R. Wilson, multilinear commutator words are also concise [37] (see also [12] for another proof). Nevertheless, it is an open question whether every n-Engel word is concise. This was proved only for $n \leq 4$ in [1] and, independently, in [13] by using the locally nilpotency of n-Engel groups for these particular values of n.

Let w be any group-word and suppose that G_w is finite for a group G. It is well-known that the derived subgroup $w(G)'$ of $w(G)$ is always finite and its order is bounded by a function depending only on the order of G_w (see, for instance, [13, Proposition 1]). The word w is called semiconcise if the finiteness of G_w for a group G always implies the finiteness of the subgroup $[w(G), G]$ (see [7]).

Obviously, concise words are semiconcise. More interesting examples are Engel words.

Theorem 3.3. [13, Proposition 4] *For any positive integer n, the n-Engel word $[x,_n y]$ is semiconcise.*

By Proposition 4.2 of [7], if w is a semiconcise word and z is any variable not appearing in w, then the word $[w, z]$ is semiconcise. Thus, we have:

Corollary 3.4. [7, Corollary 4.3.] *For any positive integers m, n, the word $[x,_n y, z_1, \ldots, z_m]$ is semiconcise.*

The next result is analogous to Theorem 3.1 and, similarly to this latter, its proof needs the ultraproduct construction of groups.

Proposition 3.5. [7, Proposition 3.3] *Let w be a semiconcise word and let G be a group in which w takes $m \geq 1$ values. Then the order of $[w(G), G]$ is bounded by a function depending only on m.*

So far, it is unknow whether there exists a semiconcise word which is not concise. Anyway, a suitable modification of Ivanov's example, given in [6], shows that there exists a word which is not semiconcise.

Proposition 3.6. [7, Proposition 4.4.] *Take A and v as in Example 3.2, and set $w(x, y) = v(x^2, y^2)$. Let $G = A \,\mathrm{wr}\, B$ where B is a cyclic group of order 2. Then $|G_w| \leq 4$ and $[w(G), G]$ is infinite. In particular, w is not semiconcise.*

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Recently, the conciseness of words in residually finite groups has been investigated. In fact, it is unclear whether every group-word is concise in the class of residually finite groups ([28, p. 15], [17]). We also mention the conjecture that any group-word which is concise in the class of residually finite groups is boundedly concise [14].

Following [10], we say that a word w implies virtual nilpotency if every finitely generated metabelian group satisfying the identity $w \equiv 1$ has a nilpotent subgroup of finite index. Since soluble n-Engel groups are locally nilpotent (see [26, 12.3.3]), the Engel words imply virtual nilpotency. More generally, this is true for simple commutator words.

Proposition 3.7. [24, Corollary 3.2] *Every simple commutator word implies virtual nilpotency.*

According to Theorem 1.2 of [10], words implying virtual nilpotency are boundedly concise in residually finite groups. This, together with Proposition 3.7, yields the following corollary.

Corollary 3.8. [24, Corollary 3.3] *Every simple commutator word is boundedly concise in residually finite groups.*

In [14] (see also [2]) it has been shown that if v is a multilinear commutator word and q is a prime-power, then the word v^q is boundedly concise in the class of residually finite groups. Applying Corollary 2.4, this has been extended as follows.

Theorem 3.9. *Let n be a positive integer, $v = v(x_1, \dots, x_k)$ a multilinear commutator word and q a prime-power. Then:*

- (i) *The word $[v,_n y]$ is boundedly concise in residually finite groups* ([11, Theorem 1.1], see also [10, Theorem 1]).
- (ii) *The word $[v^q,_n y]$ is concise in residually finite groups* [11, Theorem 1.2].

Also, much more can be said dealing with weakly rational words. Recall that a group-word w is weakly rational if for any finite group G and any integer e relatively prime to $|G|$, the set G_w is closed under taking eth powers of its elements. Given an integer $m > 1$, it is easy to see that w^m is weakly rational whenever w is. Actually, the only known examples of weakly rational words are the words $[\dots[x_1^{n_1}, x_2]^{n_2}, \dots, x_k]^{n_k}$, for any positive integers $n_1, \dots, n_k$ [15, Theorem 3]. In particular, the lower central words are weakly rational.

Theorem 3.10. *Let k, m, n be positive integers and $v = v(x_1, \dots, x_k)$ a weakly rational word. Then:*

- (i) *The word $[v,_n y]$ is boundedly concise in residually finite groups* [9, Theorem 1.2].
- (ii) *Assuming $v = [x_1, \dots, x_k]$, the word $[y,_n v^m]$ is boundedly concise in residually finite groups* [10, Theorem 3].

We conclude with the conjecture that even the word $[y,_n v]$ is boundedly concise in residually finite groups, whenever v is weakly rational.

Conjecture. *Let k, n be positive integers and $v = v(x_1, \dots, x_k)$ a weakly rational word. Then the word $[y,_n v]$ is boundedly concise in residually finite groups.*

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References

[1] A. Abdollahi and F. Russo, On a problem of P. Hall for Engel words, *Arch. Math. (Basel)*, **97** no. 5 (2011) 407–412.

[2] C. Acciarri and P. Shumyatsky, On words that are concise in residually finite groups, *J. Pure Appl. Algebra*, **218** no. 1 (2014) 130–134.

[3] R. Bastos, N. Mansuroğlu, A. Tortora and M. Tota, Bounded Engel elements in groups satisfying an identity, *Arch. Math. (Basel)*, **110** no. 4 (2018) 311–318.

[4] R. Bastos, P. Shumyatsky, A. Tortora and M. Tota, On groups admitting a word whose values are Engel, *Internat. J. Algebra Comput.*, **23** no. 1 (2013) 81–89.

[5] R. Botto Mura and A. H. Rhemtulla, *Orderable groups, Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York - Basel, 1977.

[6] S. Brazil, A. Krasilnikov and P. Shumyatsky, Groups with bounded verbal conjugacy classes, *J. Group Theory*, **9** no. 1 (2006) 127–137.

[7] C. Delizia, P. Shumyatsky and A. Tortora, *On semiconcise words*, *J. Group Theory*, **23** (2020) 629–639.

[8] C. Delizia, P. Shumyatsky, A. Tortora and M. Tota, On conciseness of some commutator words, *Arch. Math. (Basel)*, **112** no. 1 (2019) 27–32.

[9] E. Detomi, M. Morigi and P. Shumyatsky, On bounded conciseness of Engel-like words in residually finite groups, *J. Algebra*, **521** (2019) 1–15.

[10] E. Detomi, M. Morigi and P. Shumyatsky, Words of Engel type are concise in residually finite groups, *Bull. Math. Sci.*, **9** no. 2 (2019) 19 pp.

[11] E. Detomi, M. Morigi and P. Shumyatsky, Words of Engel type are concise in residually finite groups, Part II, *Group Gem. Dyn*, to appear.

[12] G. A. Fernández-Alcober and M. Morigi, Outer commutator words are uniformly concise, *J. Lond. Math. Soc. (2)*, **82** no. 3 (2010) 581–595.

[13] G. A. Fernández-Alcober, M. Morigi and G. Traustason, A note on conciseness of Engel words, *Comm. Algebra*, **40** no. 7 (2012) 2570–2576.

[14] G. A. Fernández-Alcober and P. Shumyatsky, On bounded conciseness of words in residually finite groups, *J. Algebra*, **500** (2018) 19–29.

[15] R. Guralnick and P. Shumyatsky, On rational and concise words, *J. Algebra*, **429** (2015) 213–217.

[16] S. V. Ivanov, P. Hall's conjecture on the finiteness of verbal subgroups, *Soviet Math. (Iz. VUZ)*, **33** no. 6 (1989) 59–70.

[17] A. Jaikin-Zapirain, On the verbal width of finitely generated pro-p groups, *Rev. Mat. Iberoam.*, **24** no. 2 (2008) 617–630.

[18] Y. Kim and A. H. Rhemtulla, *On locally graded groups*, Groups-Korea '94 (Pusan), de Gruyter, Berlin, 1995 189–197

[19] Y. Kim and A. H. Rhemtulla, *Groups with ordered structures*, Groups-Korea '94 (Pusan), de Gruyter, Berlin, 1995 199–210.

[20] P. Longobardi and M. Maj, On some classes of orderable groups, *Rend. Sem. Mat. Fis. Milano*, **68** (1998) 203–216.

[21] P. Longobardi, M. Maj and H. Smith, A note on locally graded groups, *Rend. Sem. Mat. Univ. Padova*, **94** (1995) 275–277.

DOI: http://dx.doi.org/10.22108/ijgt.2020.120132.1584

[22] O. Macedoṅska, On difficult problems and locally graded groups, *J. Math. Sci. (N. Y.)*, **142** no. 2 (2007) 1949–1953.

[23] C. Martínez and E. Zelmanov, On Lie rings of torsion groups, *Bull. Math. Sci.*, **6** no. 3 (2016) 371–377.

[24] C. Monetta and A. Tortora, A nilpotency criterion for some verbal subgroups, *Bull. Aust. Math. Soc.*, **100** no. 2 (2019) 281–289.

[25] A. Yu. Ol'shanskiĭ, *Geometry of defining relations in groups*, Kluwer Academic Publishers Group, Dordrecht, 1991.

[26] D. J. S. Robinson, *A course in the theory of groups*, 2nd edition, Springer-Verlag, New York, 1996.

[27] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1 and Part 2, Springer-Verlag, New York-Berlin, 1972.

[28] D. Segal, Words: notes on verbal width in groups, *London Math. Soc. Lecture Note Ser.*, **361**, Cambridge University Press, Cambridge, 2009.

[29] P. Shumyatsky, *Applications of Lie ring methods to group theory*, Nonassociative algebra and its applications, Lecture Notes in Pure and Appl. Math., **211**, Dekker, New York, 2000 373–395.

[30] P. Shumyatsky, On residually finite groups in which commutators are Engel, *Comm. Algebra*, **27** no. 4 (1999) 1937–1940.

[31] P. Shumyatsky, Verbal subgroups in residually finite groups, *Q. J. Math.*, **51** (2000) 523–528.

[32] P. Shumyatsky, A. Tortora and M. Tota, An Engel condition for orderable groups, *Bull. Braz. Math. Soc. (N.S.)*, **46** no. 3 (2015) 461–468.

[33] P. Shumyatsky, A. Tortora and M. Tota, Engel groups with an identity, *Internat. J. Algebra Comput.*, **29** no. 1 (2019) 1–7.

[34] P. Shumyatsky, A. Tortora and M. Tota, On locally graded groups with a word whose values are Engel, *Proc. Edinburgh Math. Soc.*, **59** no. 2 (2016) 533–539.

[35] P. Shumyatsky, A. Tortora and M. Tota, On varieties of groups satisfying an Engel type identity, *J. Algebra*, **447** (2016) 479–489.

[36] G. Traustason, *Engel groups*, Groups St Andrews 2009 in Bath, **2**, Cambridge Univ. Press, Cambridge, 2011 520–550.

[37] J. C. R. Wilson, On outer-commutator words, *Canad. J. Math.*, **26** (1974) 608–620.

[38] J. S. Wilson, Two-generator conditions for residually finite groups, *Bull. London Math. Soc.*, **23** no. 3 (1991) 239–248.

[39] E. I. Zelmanov, Lie algebras and torsion groups with identity, *J. Comb. Algebra*, **1** no. 3 (2017) 289–340.

[40] E. I. Zelmanov, *Lie methods in the theory of nilpotent groups*, Groups '93 Galway/St Andrews, **2**, Cambridge University Press, Cambridge, 1995 567–585.

[41] E. I. Zelmanov, Nil rings and periodic groups, *KMS Lecture Notes Math.*, Korean Mathematical Society, Seoul, 1992.

[42] E. I. Zelmanov, *On the restricted Burnside problem*, Proceedings of the International Congress of Mathematicians, **I, II**, (Kyoto, 1990), 395–402, Math. Soc. Japan, Tokyo, 1991.

[43] E. I. Zelmanov, Solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR-Izv.*, **36** (1991) 41–60.

[44] E. I. Zelmanov, Solution of the restricted Burnside problem for 2-groups, *Math. USSR-Sb.*, **72** (1992) 543–565.

[45] E. I. Zelmanov, Some problems in the theory of groups and Lie algebras, *Math. USSR-Sb.*, **66** no. 1 (1990) 159–168.

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